Auction Mechanism for Spectrum Allocation and Profit Sharing

Sung Hyun Chun and Richard J. La

***Abstract*— We examine the problem of designing an auction mechanism for dynamic spectrum sharing when there are multiple sellers and multiple buyers. First, we study the interaction among homogeneous buyers of spectrum as a noncooperative game and show the existence of a symmetric mixed-strategy Nash equilibrium (SMSNE). Second, we prove that there exists an incentive for risk neutral sellers of the spectrum to cooperate to maximize their expected profits at the SMSNEs of buyers' noncooperative game. Finally, we model the interaction among the sellers as a cooperative game and demonstrate that the core of the cooperative game is nonempty. This indicates that there exists a way for the sellers to share the profits in a such manner that no subset of sellers will deviate from cooperating with the remaining sellers.**

I. INTRODUCTION

A. Background

A conventional way of managing available frequency spectrum is a static allocation to a set of users, where each user receives dedicated spectrum. In many countries, a government agency (e.g., the Federal Communications Commission (FCC) in the U.S. [1]) bears the responsibility to plan, allocate, and manage the spectrum. Unfortunately, this static assignment of available spectrum leads to several drawbacks. First, it hampers the entrance of a new service provider. Secondly, recent studies [2], [8], [9], [13], [20] suggest that much of the assigned spectrum is under-utilized in many places. Thus, a natural question that arises is: "How can we increase the frequency usage efficiency?"

There are several new approaches put forth to address this issue. One approach to increasing the spectrum utilization in cellular frequency bands introduces a new class of service providers called Mobile Virtual Network Operators (MVNOs). An MVNO is an operator that provides mobile communication services without its own *licensed* spectrum and necessary infrastructure. In order to provide the services, they have business agreements with Mobile Network Operators (MNOs) to use the frequency spectrum and some of infrastructure owned by the MNOs. In the U.S., Virgin Mobile has successfully launched its service with Sprint Nextel as its MNO.

Another approach to more flexible use of spectrum is based on Cognitive Radio (CR) [14], which is being considered as a candidate for a new frequency management scheme by the FCC [3]. The CR is based on software-defined radio technology; it allows a CR user to switch its radio access (RA) technology based on the availability and/or performance of available networks. As a result, in principle a CR user can utilize any frequency band by adopting a suitable RA technology. CR users, however, should not interfere with *licensed* users, also called *primary* users, that paid for the spectrum.

There are several proposed solutions to ensuring that CR users do not interfere with licensed users: Under a spectrum rental protocol [15] the owner advertises the frequency bands for rent, and a renter (i.e., a CR user) may express interest. Another solution is spectrum sensing; CR users continually scan the spectrum to find an idle frequency band, called *spectrum hole*. The CR users can utilize the idle frequency band until an activity by a primary user is detected, at which point the CR user must relinquish the band. A third approach is based on an interference metric called *interference temperature* [2]. Under the proposed solution, a CR user can make use of a frequency band as long as the interference level at *every* primary user's receiver remains below a certain threshold.

There are several existing studies on dynamic spectrum sharing between primary users and unlicensed secondary users: Mutlu et al. [16] investigated an efficient pricing policy of an MNO for secondary spectrum usage of MVNOs in the presence of both primary and secondary users. Wang et al. [21] proposed a novel joint power/channel allocation scheme to improve the network's performance by modeling the spectrum allocation problem as a noncooperative game among the CR users. Etkin et al. suggested a repeated game approach to enforce an efficient and fair outcome and incentive compatible spectrum sharing [10]. In [18] the channel allocation problem in a CR network was formulated as a potential game that has provable convergence to a Nash equilibrium. The interference temperature model is applied in an auction-based spectrum sharing mechanism in [12]. Bae et al. [4] proposed a sequential auction mechanism for sharing spectrum and power among competing transmitters.

B. Motivation

These recently proposed solutions have the potential to improve the spectrum usage by filling spectrum holes without interfering with the services of primary users. However, they also suffer from several drawbacks that have not been addressed effectively. First, since the MVNOs share the infrastructure with the MNOs, they are often forced to employ the same RA technologies. This subordinate relationship limits the set of services the MVNOs can provide to their customers.

This work was supported in part by the National Science Foundation under Grant ANI02-37997.

Authors are with the Department of Electrical & Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742. E-mail:{shchun, hyongla}@umd.edu

Second, most of existing studies on CR focus primarily on the resource allocation among the secondary users and often assume that the secondary users can use the spectrum free of charge. This may be reasonable if the owner or licensee is a government agency that is interested in maximizing social welfare or if the spectrum is set aside for research purposes. However, in many cases, the frequency spectrum is allocated for commercial use and primary service providers (PSPs) have paid for the exclusive right. In such a scenario, it may be unrealistic to assume that the primary service providers will share their spectrum without charging for the use, even when the secondary users do not interfere with the services to their customers.

Third, when there is no centralized authority, individual unlicensed users may access under-utilized frequency bands in a distributed, unorganized manner. The gain in spectrum utilization from such unorganized access, however, may be limited. We suspect that introducing *secondary service providers* (SSPs) that can grant access to under-utilized spectrum in a more organized manner, by leveraging, for instance, CR users, may present a better recourse.

A well designed spectrum sharing and pricing scheme between PSPs and SSPs will encourage and facilitate sharing of spectrum in a more dynamic and flexible fashion. This is the scenario we consider in this paper. We assume that there are (i) SSPs whose infrastructure and customers' equipments have the capability for dynamic spectrum access (e.g., CR) and (ii) PSPs that wish to lend their surplus frequency spectrum according to a contract with the SSPs. This is shown in Fig. 1. Our setting is also applicable to the frequency spectrum trading between PSPs (e.g., [6]).

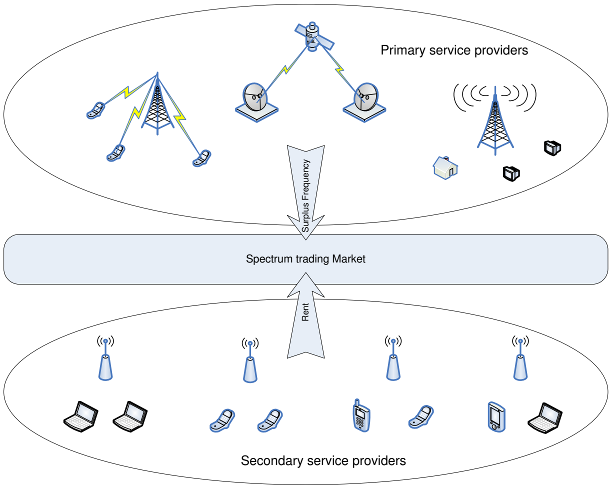

Fig. 1. Dynamic spectrum sharing market.

Realizing dynamic sharing of under-utilized spectrum between PSPs and SSPs calls for a new spectrum trading mechanism. In this paper we propose an auction-based framework for devising such a mechanism. An auction mechanism offers a natural tool for the problem: It defines the strategies of participating players, means for exchange of information, and allocation and payment schemes. Moreover, a well designed auction mechanism can have desirable properties, such as efficiency and incentive compatibility.

C. *Summary of results*

We study the setting where the PSPs, which are the *sellers* in the market, can form arbitrary coalitions (i.e., subsets of sellers). A coalition of sellers acts as if it were one seller and holds one auction to sell the spectrum made available by all the members of the coalition. We assume that the probability that a given coalition will form is known to the buyers (i.e., the SSPs).

The values a buyer has for the frequency bands it wins are modeled using a random variable called the *type* of the buyer. We assume that the buyers are homogeneous and independent (i.e., their types are independent and identically distributed (i.i.d.)). Each buyer, at the beginning, selects a seller whose auction it will participate in. First, we model the interaction among the buyers as a noncooperative game and show that there exists a *symmetric* mixed-strategy Nash equilibrium (SMSNE). The Nash equilibrium is, however, not necessarily unique. We assume that the buyers can reach one of the SMSNEs when there exist more than one SMSNE.

Second, we demonstrate that, if C_1 and C_2 are two disjoint coalitions of the sellers, then the sum of the expected profits of these two coalitions is not larger than the expected profit of the coalition $C_1 \cup C_2$. This implies that risk neutral sellers will have an incentive to cooperate and form one coalition that includes all the sellers, in order to maximize their expected profit, assuming that they can find a suitable way of sharing the profit.

Third, we model the interaction among the sellers as a cooperative game and prove that its *core* is not empty. This tells us that there exists a way for the sellers to share the profit so that no subset of sellers will have either the power or an incentive to deviate from the coalition including all the sellers and increase its expected profit.

The rest of the paper is organized as follows: Section II introduces the model and the optimal mechanism we assume the sellers adopt for allocating and pricing the spectrum bands. The noncooperative game among the buyers is studied in Section III. Section IV demonstrates the existence of an incentive for cooperation among the sellers, followed by our results on the cooperative game among the sellers in Section V.

II. SETUP

We are interested in designing a new spectrum trading mechanism for PSPs and SSPs. We assume that spectrum trading is performed periodically, for instance, by an electronic system with participating service providers. The PSPs are the sellers interested in lending frequency bands, which are the goods or items to be sold, and the SSPs are the buyers or bidders interested in purchasing the goods. In order to make progress, we assume that the frequency spectrum is traded in an agreed unit (e.g., 100 kHz). For example, a seller that wants to sell 1 MHz spectrum will have 10 units of 100 kHz frequency bands.

A. Model

Let $\mathcal{P} = \{1, 2, \ldots, M\}$ be the set of sellers and $\mathcal{S} = \{1, 2, \ldots, N\}$ the set of buyers. The sellers are assumed risk neutral and interested in maximizing their expected profit. The spectrum is divided into a set of frequency bands, denoted by $\mathcal{F}$. In a general setting, the area over which a seller operates (e.g., the United States) is partitioned into regions or markets (e.g., Washington D.C. metropolitan area). This partition is given by $\mathcal{R}$. In this paper we consider a simpler setting with only one region.

1) Sellers: Each seller owns a set of frequency bands. We denote the set of frequency bands owned by a seller $i \in \mathcal{P}$ by $\mathcal{F}^i$, and the set of frequency bands assigned to the sellers is given by $\cup_{i \in \mathcal{P}} \mathcal{F}^i \subset \mathcal{F}$. Moreover, we assume that a frequency band $f \in \mathcal{F}$ is owned by at most one seller, i.e., $\mathcal{F}^i \cap \mathcal{F}^{\tilde{i}} = \emptyset$ for all $i, \tilde{i} \in \mathcal{P}$ $(i \neq \tilde{i})$.

Sellers with under-utilized or extra frequency band(s) may participate in spectrum trading. When a seller partakes in the trading, it provides a list of frequency bands it wishes to lend to buyers (over an agreed period). Let K^i be the number of frequency bands seller i wants to sell, and $K_T = \sum_{i \in \mathcal{P}} K^i$ the total number of frequency bands available for lease.

Since the sellers have paid a price for the right to the frequency bands they own, they may have nonzero values for the frequency bands they wish to lend. Here, the values of the sellers may vary, depending on, for instance, the prices they paid for the spectrum. We denote seller i's value for the ℓ-th item it wants to sell by V^i_ℓ, $\ell = 1, \ldots, K^i$. In other words, seller i would prefer not to sell the ℓ-th frequency band if it cannot receive at least V^i_ℓ for it. Without loss of generality, we assume that the seller's items are ordered by increasing value, i.e., $V^i_1 \leq \cdots \leq V^i_{K^i}$. Let $\mathcal{V} := \{V^i_\ell;\ i \in \mathcal{P}$ and $\ell \in \{1, \ldots, K^i\}\}$.

2) Buyers: Each buyer $j \in \mathcal{S}$ has private information, namely its *type*, which is denoted by T_j. We assume that $T_j,\ j \in \mathcal{S}$, are mutually independent continuous random variables (rvs). The distribution of T_j is $\mathcal{G}_j$ with support $\mathcal{T}_j := [t_{j,\min},\ t_{j,\max}]$. Moreover, we assume that $\mathcal{G}_j$ yields a density function g_j. The value of rv T_j is revealed only to the buyer j at the beginning. Let $\mathbf{T} = (T_j;\ j \in \mathcal{S})$ be the vector of the types of the buyers and $\mathcal{T} := \prod_{j \in \mathcal{S}} \mathcal{T}_j$.

The type of a buyer determines its values for the items it wins: For each $k \in \{1, 2, \ldots, K_T\}$, let $V_{j,k} : \mathcal{T}_j \to \mathbb{R}_+ := [0, \infty)$ be the function that determines buyer j's value for the k-th item it wins,[1] i.e., $V_{j,k}(t_j)$ is the value buyer j has for the k-th item it receives when its type is t_j. The functions $V_{j,k}$ are increasing and differentiable. When the demand of buyer j, denoted by D_j, is strictly less than K_T, $V_{j,k}(t_j) = 0$ for all $t_j \in \mathcal{T}_j$ and $k = D_j + 1, \ldots, K_T$. However, we assume that $V_{j,k}(t_j) > 0$ for all $t_j \in \mathcal{T}_j$ and $k = 1, \ldots, K_T$, although they can be arbitrarily close to zero. In order to reflect the law of diminishing return, we also assume that $V_{j,1}(t_j) \geq V_{j,2}(t_j) \geq \cdots \geq V_{j,K_T}(t_j) \geq 0$ for all $t_j \in \mathcal{T}_j$.

In general, a buyer may prefer to win a block of contiguous frequency bands. However, we assume that the buyers do not differentiate the frequency bands and the total value a buyer receives from winning one or more frequency bands depends only on the total number of frequency bands it receives.

B. Optimal mechanism employed by sellers

We assume that the sellers adopt an optimal mechanism that is an extension of Branco's mechanism (BM) [5]. The BM is a generalization of well known Myerson's optimal mechanism [17] that aims to maximize the *expected* revenue of the seller that has a single item for sale. The BM is an optimal mechanism devised for *multiple homogeneous* items with one seller that has zero value for the items. Since we allow nonzero values for the sellers, the original BM is not suitable for our problem and we need to modify it to deal with nonnegative values of the sellers for the items.

1) Generalized Branco's mechanism (GBM): Let $\mathcal{P}^* = \{1, 2, \ldots, M^*\}$ and $\mathcal{S}^* = \{1, 2, \ldots, N^*\}$ be the set of sellers and the set of buyers, respectively. In our problem, $\mathcal{P}^*$ would be a coalition of sellers that are interested in selling their frequency bands together. Assume that a total of m items are available for sale from the sellers. Without loss of generality, we assume that the m items are ordered by increasing value of the seller of the item, i.e., $0 \leq V_0^{(1)} \leq V_0^{(2)} \leq \cdots \leq V_0^{(m)}$, where $V_0^{(k)}$ is the value of the seller of the k-th item.

For each buyer $j \in \mathcal{S}^*$, the functions $V_{j,k},\ k \in \{1, 2, \ldots, m\}$ are as defined earlier. Define $\mathcal{T}^* := \prod_{j \in \mathcal{S}^*} \mathcal{T}_j$. Sellers know the distributions $\mathcal{G}_j,\ j \in \mathcal{S}^*$, of buyers' types (but, not their realizations).

In the GBM, each buyer reports its type t^*_j to the sellers. The reported type t^*_j is not necessarily its true type T_j. Given the reported types of the buyers $\mathbf{t}^* = (t^*_j; j \in \mathcal{S}^*)$, the sellers compute what are called *contributions* of the buyers: The contribution of buyer j for the k-th item ($k = 1, \ldots, m$) is a mapping $\pi_{j,k} : \mathcal{T}_j \to \mathbb{R}$, where

$$\pi_{j,k}(t^*_j) = V_{j,k}(t^*_j) - \left. \frac{\partial V_{j,k}(t_j)}{\partial t_j} \right|_{t_j = t^*_j} \frac{1 - \mathcal{G}_j(t^*_j)}{g_j(t^*_j)}.$$

We order the contributions of all buyers by decreasing value and denote the ℓ-th highest contribution ($\ell = 1, \ldots, N^* \cdot m$) by $\pi_{(\ell)}(\mathbf{t}^*)$.[2]

We assume that the following holds: For all $j \in \mathcal{S}^*$ and $k = 1, 2, \ldots, m$,

(i) $(t_j - \tilde{t}_j)(\pi_{j,k}(t_j) - \pi_{j,k}(\tilde{t}_j)) \geq 0$ for all $t_j, \tilde{t}_j \in \mathcal{T}_j$, and

(ii) if $\pi_{j,k+1}(t_j) \geq 0$, then $\pi_{j,k}(t_j) \geq \pi_{j,k+1}(t_j)$ for all $t_j \in \mathcal{T}_j$.

When these conditions are satisfied, the problem is said to be *regular* [5].

In order to determine the winners and the prices they pay, sellers first compute the following quantities: For each $\ell = 1, 2, \ldots, m$,

$$\eta_\ell(\mathbf{t}) := \max\{V_0^{(\ell)}, \pi_{(\ell+1)}(\mathbf{t})\}.$$

---

[1] In general, the values of a buyer may depend on the types of other buyers as well. However, in this paper we assume that the values of a buyer depend only on its own type, but not on those of other buyers.

[2] In the event of measure zero that there are ties in the contributions, we break the ties randomly.

For each $j \in \mathcal{S}$ and $k = 1, 2, \ldots, m$, define

$$\begin{aligned}\varsigma_{j,k}(\mathbf{t}^*_{-j}) := \inf\{\hat{t}_j \in \mathcal{T}_j \mid \pi_{j,k}(\hat{t}_j) \\ \geq \min\{\eta_\ell(\hat{t}_j, \mathbf{t}^*_{-j});\ \ell = 1, 2, \ldots, m\}\},\end{aligned}$$

where $\mathbf{t}^*_{-j} = \{t^*_{\tilde{j}};\ \tilde{j} \in \mathcal{S}^* \setminus \{j\}\}$.

Under the regularity assumption on the functions $V_{j,k}$, the proposed allocation rule is given by

$$p_{j,k}(\mathbf{t}^*) = \begin{cases} 1 & \text{if } t^*_j > \varsigma_{j,k}(\mathbf{t}^*_{-j}) \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $p_{j,k}(\mathbf{t})$ is the probability that buyer j wins at least k items when the reported types are $\mathbf{t}$. The price buyer j pays for the k-th item it wins equals

$$\hat{c}_{j,k}(\mathbf{t}^*) = \begin{cases} V_{j,k}(\varsigma_{j,k}(\mathbf{t}^*_{-j})) & \text{if } p_{j,k}(\mathbf{t}^*) = 1\ , \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

From the allocation and payment rules of our GBM, it is clear that $m^\star(\mathbf{t}^*)$ items are awarded to the buyers with the $m^\star(\mathbf{t}^*)$ highest contributions, where

$$m^\star(\mathbf{t}^*) := \max\{\ell \in \{1, 2, \ldots, m\} \mid \pi_{(\ell)}(\mathbf{t}^*) > V_0^{(\ell)}\}\ .$$

When the set on the right-hand side is empty, the maximum is defined to be zero. Moreover, the price buyer j pays for the k-th item it wins is equal to the smallest value for the k-th item that would win the item (eq. (2)).

It is plain that, if every buyer is truthful and reports its true type, the expected payment for buyer j of type $t_j \in \mathcal{T}_j$ is equal to

$$c_j(t_j) := \mathbf{E}_{\mathbf{T}_{-j}} \left[\sum_{k=1}^{m} V_{j,k}(\varsigma_{j,k}(\mathbf{T}_{-j}))\ p_{j,k}(t_j, \mathbf{T}_{-j}) \right]\ ,$$

where $\mathbf{T}_{-j} = (T_{\tilde{j}};\ \tilde{j} \in \mathcal{S} \setminus \{j\})$, and the expectation is taken over the types of the other buyers.

2) Properties of the generalized Branco's mechanism: When a buyer j of type $t_j \in \mathcal{T}_j$ wins $k^\star$ items and pays a total price of c, its payoff is the difference between the value it has for the $k^\star$ items minus the price, i.e., $\sum_{k=1}^{k^\star} V_{j,k}(t_j) - c$. Let $U_j(\hat{t}_j; t_j)$ be the *expected* payoff of buyer j when its reported type is $\hat{t}_j$ and its true type is t_j.

Definition 1. *(Incentive compatibility)* A direct mechanism[3] is said to be *incentive compatible* if

$$U_j(t_j; t_j) \geq U_j(\hat{t}_j; t_j) \quad \textit{for all } j \in \mathcal{S}^* \textit{ and } t_j, \hat{t}_j \in \mathcal{T}_j\ . \tag{3}$$

Definition 2. *(Individual rationality)* A direct mechanism is said to be *individually rational* if

$$U_j(t_j; t_j) \geq 0 \quad \textit{for all } j \in \mathcal{S}^* \textit{ and } t_j \in \mathcal{T}_j\ . \tag{4}$$

Lemma 1. *The generalized Branco's mechanism is both incentive compatible and individually rational.*

We define the profit of the sellers as the total payment from the buyers plus the sum of sellers' values of the unsold items (i.e., $\sum_{k=m^\star(\mathbf{t}^*)+1}^{m} V_0^{(k)}$).

Lemma 2. *The generalized Branco's mechanism is optimal in the sense that it maximizes the expected profit of the sellers.*

III. NONCOOPERATIVE GAME AMONG THE BUYERS

There are many different ways in which the sellers can sell their available frequency bands to the buyers. For example, individual sellers can hold separate individual auctions, or a group of sellers can form a *coalition* to sell their available frequency bands together. In the latter case, each coalition will hold one auction by sharing their information (e.g., the received bids, the number of frequency bands, and the reserved value for each frequency band) and the profit according to an agreement between its members.

In order for a coalition to emerge, the sellers in the coalition must find it advantageous to cooperate and a proper profit sharing scheme must be in place. In general, it would require that (i) the expected profit of the coalition from a single auction be no smaller than the total expected profit the members can achieve by forming a set of smaller coalitions and (ii) there exist a suitable profit sharing scheme that allocates the profits in a way no subset of members finds it beneficial to leave the coalition. It is obvious that the expected profit of every seller i should be at least its expected profit from holding an individual auction.

Before we can understand how the sellers would behave, we must first examine buyers' behavior. To this end we model the interaction among the buyers as a noncooperative game [11]. At the beginning of the game each buyer first chooses a seller whose auction it will participate in[4] and then reports its type to the selected seller. We assume that a buyer's selection of the seller takes place before the type is revealed to the buyer or the selection does not depend on the revealed type.

Sellers are free to form any coalition(s) among themselves. Sellers do not announce the coalitions they form to the buyers before the buyers select sellers. In other words, buyers choose the sellers without the knowledge of the coalitions formed by the sellers; instead they only know the *probabilities* that different coalitions will form. Sellers in a coalition share the reported types of the buyers that choose a member of the coalition and decide on the set of frequency bands to be allocated and the prices to be charged according to the GBM. The buyers are then informed of the number of frequency bands they have won and the prices to pay.

Let us first examine the actions to be taken by the buyers. As mentioned earlier, each buyer must first choose one of the M sellers and report its type to the seller. However, since the GBM is incentive compatible, the optimal strategy of a buyer in the GBM is to bid its true type, and the only action required of a buyer is the selection of a seller. We formulate this problem as a noncooperative game among the buyers.

Let $\Omega_{\mathcal{P}}$ be the set of all possible partitions of the set of sellers $\mathcal{P}$ and μ a distribution over the set $\Omega_{\mathcal{P}}$. The

---

[3] In a direct mechanism the only action of a player is to report its type.

[4] Here, we assume that each buyer joins only one auction. However, we will show later that this does not impose any restrictions on our findings.

probability that coalitions in a partition $\omega \in \Omega_{\mathcal{P}}$ will emerge is given by $\mu(\omega)$. For example, suppose that $\mathcal{P} = \{1, 2\}$ and $\Omega_{\mathcal{P}} = \{\omega_1, \omega_2\} = \{\{\{1\}, \{2\}\}, \{\{1, 2\}\}\}$. Then, $\mu(\omega_1)$ is the probability that the coalitions $\{1\}$ and $\{2\}$ will form (i.e., two sellers do not cooperate) and $\mu(\omega_2)$ is the probability that coalition $\{1, 2\}$ will form (i.e., they will cooperate with each other). We assume that the distribution μ is common knowledge, i.e., buyers know the probability a coalition $C \subset \mathcal{P}$ will form, which is given by

$$\mathbf{Pr}\left[\text{coalition } C \text{ forms}\right] = \sum_{\omega \in \Omega_{\mathcal{P}} : C \in \omega} \mu(\omega).$$

Since each buyer must choose a seller, the pure strategy space Σ_j of buyer $j \in \mathcal{S}$ is given by the set of sellers $\mathcal{P}$. The (expected) payoff of buyer j given a strategy profile $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$, where $\sigma_j \in \Sigma_j$ for all $j \in \mathcal{S}$, is given by $u_j(\sigma)$. Then, the noncooperative game among the buyers is presented by $\Gamma = \{\mathcal{S}, (\Sigma_j;\ j \in \mathcal{S}), (u_j;\ j \in \mathcal{S})\}$. The goal of each buyer is to maximize its expected payoff.

A mixed strategy of a buyer j is simply a distribution ξ over $\Sigma_j = \mathcal{P}$, where $\xi(i), i \in \mathcal{P}$, is the probability that buyer j will choose seller i. A mixed-strategy Nash equilibrium (MSNE), $\Xi = (\xi^1, \xi^2, \ldots, \xi^N)$, is a set of mixed strategies, one for each buyer, such that no buyer can increase its expected payoff by unilaterally deviating from the equilibrium strategy. An MSNE, Ξ, is called a *symmetric* MSNE if $\xi^1 = \xi^2 = \cdots = \xi^N$.

In the rest of the paper we consider independent homogeneous buyers: The types of the buyers $t_j, j \in \mathcal{S}$, are i.i.d., and the value functions $V_{j,k}$ are identical for all $j \in \mathcal{S}$. In addition, we assume

$$\pi_{1, \lceil K^i/2 \rceil + 1}(t_{1,\max}) > V^i_{\lceil K^i/2 \rceil + 1} \quad \text{for all } i \in \mathcal{P}. \tag{5}$$

Theorem 1. *There always exists a symmetric mixed strategy Nash equilibrium in our game* Γ.

Proof. Since the common pure strategy space of the buyers is finite and the game is symmetric, the existence follows from the extended version of Nash equilibrium theory [7]. ☐

Unfortunately, in general a symmetric MSNE is not guaranteed to be unique. However, when no seller cooperates with any other seller(s) with probability 1 (w.p. 1), i.e., $\mu(\{\{1\}, \{2\}, \{3\}, \ldots, \{M\}\}) = 1$, the symmetric MSNE is unique.

Theorem 2. *When no coalition with more than one seller forms w.p. 1, there is a unique symmetric mixed-strategy Nash equilibrium.*

Before we prove the theorem, we introduce a lemma that will be used in the proof of the theorem. We denote the set of buyers that choose seller i by $\mathcal{S}_i$. Let $\tilde{U}_j^{(i)}(n)$ be the conditional expected payoff of buyer j given that (i) buyer j chooses seller $i \in \mathcal{P}$ and (ii) $|\mathcal{S}_i| = n + 1$, i.e., exactly n other buyers choose seller i as well.

Lemma 3. *Suppose that buyer* j *chooses seller* i*. Then,* $\tilde{U}_j^{(i)}(n-1) > \tilde{U}_j^{(i)}(n)$ *for all* $n \in \{1, 2, ..., N-1\}$.

Proof. Since the buyers are homogeneous, without loss of generality, assume that $j = 1$ and $\mathcal{S}_i = \{1, 2, \ldots, n+1\}$. Define $\mathbf{T}_{-1}^{(n)} = (T_l; l = 2, \ldots, n+1)$ and $\mathbf{t}_{-1}^{(n)} = (t_l; l = 2, \ldots, n+1)$. From the allocation rule in (1), for fixed $\mathbf{t}_{-1}^{(n)}$, the probability $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)})$ is nondecreasing in t_1.[5] In particular, $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}) = 0$ if $t_{1,\min} \leq t_1 \leq \varsigma_{1,k}(\mathbf{t}_{-1}^{(n)})$ and $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}) = 1$ if $\varsigma_{1,k}(\mathbf{t}_{-1}^{(n)}) < t_1$.

From the payment rule in (2), we can show that the expected payoff of buyer 1 that participates in seller i's auction when there are n other buyers is given by

$$\begin{aligned} &\tilde{U}_1^{(i)}(n) \\ &= \mathbf{E}_{\mathbf{T}(i)}\left[\sum_{k=1}^{K^i}\left(V_{1,k}(T_1)\ p_{1,k}(T_1, \mathbf{T}_{-1}^{(n)}) - \hat{c}_{1,k}(T_1, \mathbf{T}_{-1}^{(n)})\right)\right] \\ &= \mathbf{E}_{\mathbf{T}(i)}\left[\sum_{k=1}^{K^i}\left(\int_{t_{1,\min}}^{T_1} V'_{1,k}(x)\ p_{1,k}(x, \mathbf{T}_{-1}^{(n)})\ dx\right)\right], \end{aligned} \tag{6}$$

where $\mathbf{T}(i) = \{T_j; j \in \mathcal{S}_i\}$, and the expectation is taken over the types $\mathbf{T}(i)$. From the allocation rule (1), for any t_1 and $\mathbf{t}_{-1}$, we have

$$p_{1,k}(t_1, \mathbf{t}_{-1}^{(n-1)}) \geq p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}). \tag{7}$$

The lemma now follows from (5) - (7). ☐

Proof of Theorem 2. As mentioned earlier, the existence of a symmetric MSNE follows from the extended version of Nash equilibrium theory [7]. Suppose that there are two symmetric MSNEs, $\Xi^1 = (\xi^1, \ldots, \xi^1)$ and $\Xi^2 = (\xi^2, \ldots, \xi^2)$, where $\xi^k = (\xi_1^k, \ldots, \xi_M^k)$, $k = 1, 2$, such that $\xi^1 \neq \xi^2$. We will show that this leads to a contradiction, thus proving the uniqueness of a symmetric MSNE.

Let $U_j^{(i)}(\xi)$ denote the conditional expected payoff of buyer j, given that buyer j selects seller i, when all buyers employ the same mixed strategy ξ. The buyer j's expected payoff is equal to

$$U_j(\xi) = \sum_{i \in \mathcal{P}} \xi_i \cdot U_j^{(i)}(\xi)\ .$$

One can easily show that, at any symmetric MSNE $\Xi^\star = (\xi^\star, \ldots, \xi^\star)$, we must have

$$U_j^{(1)}(\xi^\star) = \cdots = U_j^{(M)}(\xi^\star) \qquad \text{for all } j \in \mathcal{S}. \tag{8}$$

Since buyers are assumed to select sellers independently of each other, for each $i \in \mathcal{P}$,

$$U_j^{(i)}(\xi^\star) = \sum_{n=0}^{N-1} \binom{N-1}{n}\ \xi_i^{\star n}\ (1 - \xi_i^\star)^{N-1-n}\ \tilde{U}_j^{(i)}(n).$$

We can compute the derivatives of $U_j^{(i)}(\xi)$ and show, using Lemma 3, that $\partial U_j^{(i)}(\xi) / \partial \xi_i < 0$. If $\xi^1 \neq \xi^2$, there must exist i^+ and i^* such that (i) $\xi_{i^+}^1 < \xi_{i^+}^2$ and (ii) $\xi_{i^*}^1 > \xi_{i^*}^2$.

---

[5] Here, for different values of n, we have different auctions. However, with a little abuse of notation, we do not explicitly indicate the dependence of $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)})$, $\varsigma_{1,k}(\mathbf{t}_{-1}^{(n)})$, and $\hat{c}_{1,k}(T_1, \mathbf{T}_{-1}^{(n)})$ on n (or more precisely, on the set $\mathcal{S}_i$).

Our finding that $U_j^{(i)}(\xi)$ is strictly decreasing in ξ_i implies that $U_j^{(i^+)}(\xi^1) > U_j^{(i^+)}(\xi^2) = U_j^{(i^*)}(\xi^2) > U_j^{(i^*)}(\xi^1)$, which contradicts (8). $\square$

Consider the following example: Suppose that $M \geq 3$ and $\mu(\{\{1,2\},\{3\},\{4\},...,\{M\}\}) = 1$. From Theorem 2, there is a unique symmetric MSNE, $\Xi^\star = (\xi^\star, \xi^\star, \ldots, \xi^\star)$, of a new game $\Gamma^\star$ with $M-1$ sellers, where the new seller 1 combines both sellers 1 and 2 in the original game Γ. One can easily verify that a strategy profile $\Xi^\dagger = (\xi^\dagger, \ldots, \xi^\dagger)$, where $\xi_1^\star = \xi_1^\dagger + \xi_2^\dagger$ and $\xi_l^\dagger = \xi_l^\star$ for all $l = 3, \ldots, M$, is a symmetric MSNE of Γ. Thus, if $\xi_1^\star > 0$, there are uncountably many symmetric MSNEs of Γ.

When there are more than one symmetric MSNEs, we assume that the buyers can reach a symmetric MSNE (for example, according to lexicographic order).

IV. EXISTENCE OF AN INCENTIVE FOR COOPERATION AMONG THE SELLERS

As mentioned earlier, a coalition of sellers will emerge only if its members find it beneficial to cooperate in that they can earn higher expected profits. In order to examine the existence of such an incentive for some or all of the sellers to cooperate, we compare the expected profits of different coalitions at a symmetric MSNE of the noncooperative game Γ.

Consider one auction with n buyers and m items to be sold. The seller in the auction can be either an individual seller or a coalition of sellers. The seller's payoff from the auction is given by the payment it receives for the items sold plus the values of the unsold frequency bands. Hence, its expected payoff is

$$\begin{aligned} U_0(p,c) &= \sum_{j=1}^{n} \mathrm{E}_{T_j}\left[c_j(T_j)\right] + \mathrm{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right] \\ &= \sum_{j=1}^{n} \mathrm{E}_{\mathbf{T}}\left[\sum_{k=1}^{m} \pi_{j,k}(\mathbf{T})\ p_{j,k}(\mathbf{T})\right] \\ &\quad + \mathrm{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right], \end{aligned} \tag{9}$$

where $\mathbf{T}$ is the random vector of buyers' types, $m^\star(\mathbf{T})$ is the number of items sold, and the expectations are taken over $\mathbf{T}$.

Define a random vector $\mathbf{B} := (B^j; j \in \mathcal{S})$, where B^j is the seller chosen by buyer j (using the selected symmetric MSNE strategy), and $\mathcal{S}_i(B) = \{j \in \mathcal{S} \mid B^j = i\} \subset \mathcal{S}$. Given fixed types of the buyers, $\mathbf{t} \in \mathcal{T}$,

1) $\pi_{j\cdot}(t_j) = \{\pi_{j,k}(t_j); k = 1, 2, \ldots, K_T\}$ is the set of the contributions of buyer j,
2) $\tilde{\Pi}_{\mathbf{t}} := \{\pi_{j\cdot}(\mathbf{t}); j \in \mathcal{S}\}$,
3) $\Pi_{\mathbf{t}} := (\pi_{(k)}(\mathbf{t}); k = 1, 2, \ldots, D_T)$ is the vector of the contributions in $\tilde{\Pi}_{\mathbf{t}}$ ordered by decreasing value, where $D_T = N \cdot K_T$,
4) for every $i \in \mathcal{P}$, $\tilde{\Pi}_{\mathbf{t}}^i(\mathbf{B}) := \{\pi_{j\cdot}(\mathbf{t}); j \in \mathcal{S}_i(\mathbf{B})\}$, and
5) $\Pi_{\mathbf{t}}^i(\mathbf{B})$, $i \in \mathcal{P}$, is the order statistics of $\tilde{\Pi}_{\mathbf{t}}^i(\mathbf{B})$.

For every $\mathbf{t} \in \mathcal{T}$, define a mapping $\overline{\Pi}_{\mathbf{t}} : \mathcal{B} \to \mathcal{H}(\mathbf{t})$, where $\mathcal{B} = \mathcal{P}^N$, $\overline{\Pi}_{\mathbf{t}}(\mathbf{b}) = \{\Pi_{\mathbf{t}}^i(\mathbf{b}); i \in \mathcal{P}\}$, and $\mathcal{H}(\mathbf{t}) := \{\overline{\Pi}_{\mathbf{t}}(\mathbf{b});\ \mathbf{b} \in \mathcal{B}\}$. For each $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, π^i denotes the ordered contributions of the buyers that choose seller i when the types of the buyers are given by $\mathbf{t}$.

Let $\mathbf{b}_{\mathbf{t}} : \mathcal{H}(\mathbf{t}) \to \mathcal{B}$, where $\mathbf{b}_{\mathbf{t}}(\overline{\pi})$, $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, is the vector that tells us the selected sellers of the buyers under $\overline{\pi}$. Suppose that $\nu_{\mathbf{t}}$ is a distribution over the set $\mathcal{H}(\mathbf{t})$, where $\nu_{\mathbf{t}}(\overline{\pi})$, $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, is the probability $\mathbf{Pr}\left[\mathbf{B} = \mathbf{b}_{\mathbf{t}}(\overline{\pi})\right]$ determined by the symmetric MSNE.

For a given $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, denote the set of winning *contributions* in a coalition $C \subset \mathcal{P}$ by $\Psi_{\overline{\pi}}(C) \subset \tilde{\Pi}_{\mathbf{t}}$ and the sum of the winning *contributions* of coalition C by $\zeta(C, \overline{\pi})$. Similarly, define $\Phi_{\overline{\pi}}(C)$ to be the set of sellers' values of the unsold frequency bands in the coalition C, and $\lambda(C, \overline{\pi}) := \sum_{x \in \Phi_{\overline{\pi}}(C)} x$ the total value of the unsold items in coalition C.

For example, suppose that there are two sellers with a unit supply (i.e., one frequency band to sell) and that $\{\pi_{(1)}(\mathbf{t}), \pi_{(2)}(\mathbf{t}), \pi_{(3)}(\mathbf{t})\} \subset \tilde{\Pi}_{\mathbf{t}}^1(\mathbf{B})$, while $\{\pi_{(4)}(\mathbf{t}), \pi_{(5)}(\mathbf{t})\} \subset \tilde{\Pi}_{\mathbf{t}}^2(\mathbf{B})$, where $\pi_{(4)}(\mathbf{t}) < \pi_{(3)}(\mathbf{t})$. Also, sellers' values satisfy $V_0^{(1)} \leq V_0^{(2)} \leq \pi_{(5)}(\mathbf{t})$. Then, the winning *contributions* of the coalition $C = \{1, 2\}$ are $\pi_{(1)}(\mathbf{t})$ and $\pi_{(2)}(\mathbf{t})$, whereas the winning contribution of $C_1 = \{1\}$ and $C_2 = \{2\}$ is $\pi_{(1)}(\mathbf{t})$ and $\pi_{(4)}(\mathbf{t})$, respectively.

Let $m_C^\star = |\Psi_{\overline{\pi}}(C)|$ be the number of items sold by the coalition C and $p^{(C)}(t)$ the allocation rule of the coalition C according to the GBM. Then, we have

$$\begin{aligned} \zeta(C, \overline{\pi}) &= \sum_{k=1}^{K(C)} \Big(\sum_{j \in \mathcal{S}:\ b_{\mathbf{t},j}(\overline{\pi}) \in C} \pi_{j,k}(\mathbf{t})\ p_{j,k}^{(C)}(\mathbf{t}) \Big) \\ &= \sum_{\kappa \in\ \Psi_{\overline{\pi}}(C)} \kappa\ , \end{aligned} \tag{10}$$

where $K(C) = \sum_{i \in C} K^i$, and

$$p_{j,k}^{(C)}(\mathbf{t}) = \begin{cases} 1 & \text{if buyer } j \text{ is awarded at least } k \text{ items} \\ 0 & \text{otherwise.} \end{cases}$$

When each coalition holds a separate auction using the GBM, from the allocation rule (1), the coalition C awards $m_C^\star$ items to the buyers with the $m_C^\star$ highest *contributions* in $\bigcup_{i \in C} \overline{\pi}^i$. Further, each unsold item's value is larger than that of any allocated item and any losing contribution. Therefore, it is clear that, for every disjoint coalitions C_1, $C_2 \subset \mathcal{P}$,

$$\begin{aligned} &\zeta(C_1, \overline{\pi}) + \lambda(C_1, \overline{\pi}) + \zeta(C_2, \overline{\pi}) + \lambda(C_2, \overline{\pi}) \\ &\leq \zeta(C_1 \cup C_2, \overline{\pi}) + \lambda(C_1 \cup C_2, \overline{\pi}). \end{aligned} \tag{11}$$

A strict inequality holds (i) if the smallest winning contribution in coalition C_1 is less than the largest losing contribution in coalition C_2 or vice versa or (ii) if the smallest value of unsold items in coalition C_1 is less than the largest losing contribution in coalition C_2 or vice versa.

Let us first define, for each $\mathbf{t} \in \mathcal{T}$,

$$v(C; \mathbf{t}) := \sum_{\overline{\pi} \in \mathcal{H}(\mathbf{t})} \left(\zeta(C, \overline{\pi}) + \lambda(C, \overline{\pi})\right)\ \nu_{\mathbf{t}}(\overline{\pi}). \tag{12}$$

Then, the expected payoff of a coalition C is given by $\mathbf{E}\left[v(C;\mathbf{T})\right]$. We can show the following theorem from (11) and (12).

Theorem 3. *For every two disjoint coalitions C_1 and C_2,*

$$v(C_1) + v(C_2) \leq v(C_1 \cup C_2). \tag{13}$$

Theorem 3 tells us that the expected profit function v satisfies the *superadditivity* property. In addition, it implies that risk neutral sellers will have an incentive to cooperate among themselves in order to increase their expected profits, assuming they can find an equitable way of sharing the profit.

V. PROFIT SHARING AND A COOPERATIVE GAME AMONG THE SELLERS

Our finding in the previous section indicates that the sellers will find it advantageous to cooperate with each other and form a grand coalition that includes all sellers if they want to maximize their expected profits. However, in order for the sellers to maintain such cooperation, they must be able to find an equitable way of sharing the profits. In light of this, a natural question that arises is how the sellers should share the profit among themselves when they decide to cooperate. In order to answer this question we turn to cooperative game theory, and we model the interaction between the sellers as a cooperative game.

A cooperative game is often given by a *characteristic function* $v : 2^{\mathcal{P}} \rightarrow \mathbb{R}$. The characteristic function v assigns to each coalition $C \subset \mathcal{P}$ a value that is the total payoff of the members in the coalition they can guarantee themselves against the other players. The characteristic function of the cooperative game among the sellers in our problem is defined through the expected profit of the coalitions at the assumed symmetric MSNE of the noncooperative game among the buyers. In other words, for every $C \subset \mathcal{P}$, $v(C)$ denotes the expected profit the sellers in the coalition C can achieve without the help of the remaining sellers.

We first introduce following definitions.

Definition 3. *An imputation for an M-player cooperative game is a vector $x = (x_1, ..., x_M)$ that satisfies*

$$(1) \quad \sum_{i \in \mathcal{P}} x_i = v(\mathcal{P}), \text{ and}$$
$$(2) \quad x_i \geq v(\{i\}) \text{ for all } i \in \mathcal{P} .$$

Definition 4. *Let x and y be two imputations. (i) Let $C \subset \mathcal{P}$ be a coalition. We say that x dominates y through C if*

$$(1) \quad x_i > y_i \text{ for all } i \in C, \text{ and}$$
$$(2) \quad \sum_{i \in C} x_i \leq v(C) .$$

(ii) We say that x dominates y if there exists some coalition $C^\star \subset \mathcal{P}$ such that x dominates y through $C^\star$.

Definition 5. *The set of all undominated imputations is called the core of the cooperative game.*

The following theorem gives an alternate characterization of the core of a cooperative game and a means of finding it.

Theorem 4. *[19, p.219] The core is the set of all M-vectors x satisfying*

$$(1) \quad \sum_{i \in C} x_i \geq v(C) \text{ for all } C \subset \mathcal{P}, \text{ and}$$
$$(2) \quad \sum_{i \in \mathcal{P}} x_i = v(\mathcal{P}) .$$

The conditions in Theorem 4 imply that no subset of the sellers (i.e., a coalition) has the power to increase its expected profit by deviating from the grand coalition. Therefore, a payoff vector in the core can be viewed as a stable equilibrium and a candidate for equitable sharing of the profits among the sellers.

Unfortunately, the core of a cooperative game is in general not guaranteed to be nonempty, and proving the existence of a nonempty core can be nontrivial. However, we can show that the core of the cooperative game among the sellers under consideration is nonempty. This implies that indeed there exists a way for the sellers to share the profits in such a way that no subset of the sellers will be able to leave the grand coalition and increase their expected profits.

Theorem 5. *The cooperative game v among the sellers has a non-empty core.*

Proof. A proof of the theorem is provided in the appendix. □

VI. CONCLUSION

We investigated the problem of designing a suitable trading mechanism for dynamic spectrum sharing with multiple sellers and multiple buyers. We first modeled the interaction between selfish buyers interested in leasing spectrum as a noncooperative game. We showed that, when the buyers are homogeneous, there exists a symmetric mixed-strategy Nash equilibrium. Then, we demonstrated that risk neutral sellers have an incentive to cooperate with each other in order to maximize their expected profits when the buyers behave according to a symmetric mixed-strategy Nash equilibrium. Finally, we formulated the interaction among the sellers as a cooperative game and proved that its core is nonempty. This finding suggests that there exists a profit sharing scheme under which all sellers will find it beneficial to cooperate.

APPENDIX

In order to prove the theorem, we use the following well known result for the existence of a nonempty core: Let $\mathbf{y} = (y_C; C \subset \mathcal{P})$ be a nonnegative vector that satisfies the condition

$$\sum_{C \subset \mathcal{P} : i \in C} y_C = 1 \quad \text{for all} \quad i \in \mathcal{P} . \tag{14}$$

Theorem 6. *[19, p.225] A necessary and sufficient condition for the game to have a nonempty core is that, for every nonnegative vector $(y_C; C \subset \mathcal{P})$ satisfying (14), we have*

$$\sum_{C \subset \mathcal{P}} y_C \cdot v(C) \leq v(\mathcal{P}) .$$

We first introduce following notation. Suppose that $\mathbf{y} = (y_C; C \subset \mathcal{P})$ is a nonnegative vector that satisfies (14). Then, for every $\mathbf{t} \in \mathcal{T}$ and $\mathbf{b} \in \mathcal{B}$, we have

$$\sum_{C \subset \mathcal{P} : \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C)} \left(y_C \cdot \pi_{(k)}(\mathbf{t}) \right) \leq \pi_{(k)}(\mathbf{t}) \ , \tag{15}$$

where $\overline{\pi} = \overline{\Pi}_{\mathbf{t}}(\mathbf{b})$. Define $i_k(\mathbf{t}, \mathbf{b})$ to be the seller i whose $\Pi^i_{\mathbf{t}}(\mathbf{b})$ contains $\pi_{(k)}(\mathbf{t})$, i.e., $\pi_{(k)}(\mathbf{t}) \in \Pi^{i_k(\mathbf{t},\mathbf{b})}_{\mathbf{t}}(\mathbf{b})$. The equality in (15) holds if and only if $\pi_{(k)}(\mathbf{t})$ is a winning contribution in every coalition that contains the seller $i_k(\mathbf{t}, \mathbf{b})$. With a little abuse of notation, for each $C \subset \mathcal{P}$, define

$$a^{(k)}_C = \begin{cases} y_C & \text{if } \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\Phi_{\overline{\pi}}(C)$ is the set of sellers' values of the unsold frequency bands in the coalition C. It is clear

$$\sum_{C \subset \mathcal{P} : V^{(k)}_0 \in \ \Phi_{\overline{\pi}}(C)} \left(y_C \cdot V^{(k)}_0 \right) \leq V^{(k)}_0. \tag{16}$$

Let $\bar{i}^v_k$ be the seller i that has the value $V^{(k)}_0$ for one of its frequency bands. The equality in (16) holds if and only if the frequency band with the k-th smallest value is unsold in every coalition that includes the seller $\bar{i}^v_k$. The left-hand side of (16) is equal to zero if the frequency band is allocated in every coalition C. For each $C \subset \mathcal{P}$, define

$$b^{(k)}_C = \begin{cases} y_C & \text{if } V^{(k)}_0 \in \Phi_{\overline{\pi}}(C) \\ 0 & \text{otherwise.} \end{cases}$$

We denote the number of items sold when all sellers cooperate by

$$k^\star := \max\{\ell \in \{1, 2, \ldots, K_T\} \mid \pi_{(\ell)}(\mathbf{t}) > V^{(\ell)}_0\}.$$

The maximum is equal to zero if the set on the right-hand side is empty. Let $\mathcal{K}^\star := \{1, 2, \ldots, k^\star\}$.

We partition the set of items available for sale as follows:

$$\begin{aligned}
\Theta^1 &:= \{k \in \mathcal{K}^\star \mid \text{equality in (15) holds}\} \\
&= \{k \in \mathcal{K}^\star \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C) \text{ for all } C \subset \mathcal{P} \\
&\qquad\qquad\qquad \text{such that } i_k(\mathbf{t}, \mathbf{b}) \in C\} \\
\Theta^2 &:= \mathcal{K}^\star \setminus \Theta^1 \\
\Theta^3 &:= \{k \in \{k^\star + 1, \ldots, D_T\} \mid \exists\ C \subset \mathcal{P} \text{ such that} \\
&\qquad\qquad\qquad \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C)\} \\
\Theta^4 &:= \{k \in \Theta^2 \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(\{i_k(\mathbf{t}, \mathbf{b})\})\} \\
\Theta^5 &:= \{k \in \Theta^3 \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(\{i_k(\mathbf{t}, \mathbf{b})\})\} \\
\Theta^6 &:= \Theta^2 \setminus \Theta^4 \\
\Theta^7 &:= \Theta^3 \setminus \Theta^5 \\
\Theta^8 &:= \{k \in \{k^\star + 1, \ldots, K_T\} \mid \text{equality in (16) holds}\} \\
\Theta^9 &:= \{k \in \mathcal{K}^\star \mid \text{strict inequality in (16) holds }\} \\
\Theta^{10} &:= \{k \in \{k^\star + 1, \ldots, K_T\} \mid \exists\ C \subset \mathcal{P} \text{ such that} \\
&\qquad\qquad\qquad V^{(k)}_0 \notin \Phi_{\overline{\pi}}(C)\} \\
\Theta^{11} &:= \{k \in \Theta^9 \mid V^{(k)}_0 \in \Phi_{\overline{\pi}}(\{\bar{i}^v_k\})\} \\
\Theta^{12} &:= \{k \in \Theta^{10} \mid V^{(k)}_0 \in \Phi_{\overline{\pi}}(\{\bar{i}^v_k\})\} \\
\Theta^{13} &:= \Theta^9 \setminus \Theta^{11} \\
\Theta^{14} &:= \Theta^{10} \setminus \Theta^{12}
\end{aligned}$$

Note from the definition of the sets

$$\begin{gathered}
\Theta^2 = \Theta^4 \cup \Theta^6, \ \Theta^3 = \Theta^5 \cup \Theta^7, \\
\Theta^9 = \Theta^{11} \cup \Theta^{13}, \text{ and } \Theta^{10} = \Theta^{12} \cup \Theta^{14}.
\end{gathered} \tag{17}$$

Lemma 4.

$$\begin{aligned}
&\sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^2} a^{(k)}_C + \sum_{k \in \Theta^3} a^{(k)}_C \right) \\
&\qquad + \sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^9} b^{(k)}_C + \sum_{k \in \Theta^{10}} b^{(k)}_C \right) \\
&= |\Theta^2| + |\Theta^{10}|.
\end{aligned} \tag{18}$$

Proof. For any given $C \subset \mathcal{P}$, define

$$\begin{aligned}
\Theta^{L_c}_C &:= \{k \in \Theta^4 \cup \Theta^5 \mid i_k(\mathbf{t}, \mathbf{b}) \in C \text{ and } a^{(k)}_C = 0\}, \\
\Theta^{W_c}_C &:= \{k \in \Theta^6 \cup \Theta^7 \mid i_k(\mathbf{t}, \mathbf{b}) \in C \text{ and } a^{(k)}_C = y_C\}, \\
\Theta^{L_v}_C &:= \{k \in \Theta^{11} \cup \Theta^{12} \mid \bar{i}^v_k \in C \text{ and } b^{(k)}_C = 0\}, \text{ and} \\
\Theta^{W_v}_C &:= \{k \in \Theta^{13} \cup \Theta^{14} \mid \bar{i}^v_k \in C \text{ and } b^{(k)}_C = y_C\}.
\end{aligned}$$

From the definition of the sets Θ^n, $n = 1, \ldots, 14$, the following observations can be made.

O1. Suppose that there exist $k_1 \in \Theta^6$ (resp. $k_1 \in \Theta^7$) and a coalition $C \subset \mathcal{P}$ such that $\pi_{(k_1)}(\mathbf{t})$ is a winning contribution in the coalition C. This implies that (i) there exists $k_2 \in \Theta^4 \cup \Theta^5$ (resp. $k_2 \in \Theta^5$), where $k_2 > k_1$, such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$ and $a^{(k_2)}_C = 0$, or (ii) there exists $k_2 \in \Theta^{11} \cup \Theta^{12}$ such that the seller $\bar{i}^v_{k_2} \in C$, $\pi_{(k_1)}(\mathbf{t}) > V^{(k_2)}_0$ and $b^{(k_2)}_C = 0$.
O2. Suppose that the item with seller's value $V^{(k_1)}_0$ is unsold in a coalition $C \in \mathcal{P}$ for some $k_1 \in \Theta^{13}$ (resp. $k_1 \in \Theta^{14}$). Then, (i) there exists $k_2 \in \Theta^5$ (resp. $k_2 \in \Theta^4 \cup \Theta^5$) such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$, $V^{(k_1)}_0 > \pi_{(k_2)}(\mathbf{t})$ and $a^{(k_2)}_C = 0$, or (ii) there exists $k_2 \in \Theta^{11}$ (resp. $k_2 \in \Theta^{11} \cup \Theta^{12}$), $k_1 > k_2$, such that the seller $\bar{i}^v_{k_2} \in C$ and $b^{(k_2)}_C = 0$.
O3. From observations **O1** and **O2**, $|\Theta^{L_c}_C| + |\Theta^{L_v}_C| = |\Theta^{W_c}_C| + |\Theta^{W_v}_C|$.
O4. One can show that $\Theta^1 \cup \Theta^4 \cup \Theta^5$ is the set of winning contributions and $\Theta^8 \cup \Theta^{11} \cup \Theta^{12}$ is the set of unsold items when sellers hold separate individual auctions. Hence, the cardinality of their union is the number of available items K_T. Further, it is clear from their definitions that $\Theta^1 \cup \Theta^2 = \mathcal{K}^\star$ and $\Theta^8 \cup \Theta^{10} = \{k^\star + 1, \ldots, K_T\}$. Thus, we have $|\Theta^4| + |\Theta^5| + |\Theta^{11}| + |\Theta^{12}| = |\Theta^2| + |\Theta^{10}|$.

From (17), we get

$$\begin{aligned}
&\sum_{C\subset\mathcal{P}}\left(\sum_{k\in\Theta^2} a_C^{(k)} + \sum_{k\in\Theta^3} a_C^{(k)} + \sum_{k\in\Theta^9} b_C^{(k)} + \sum_{k\in\Theta^{10}} b_C^{(k)}\right)\\
&= \sum_{C\subset\mathcal{P}}\left(\sum_{k\in\Theta^4\cup\Theta^5} a_C^{(k)} + \sum_{k\in\Theta^6\cup\Theta^7} a_C^{(k)}\right)\\
&\quad + \sum_{C\subset\mathcal{P}}\left(\sum_{k\in\Theta^{11}\cup\Theta^{12}} b_C^{(k)} + \sum_{k\in\Theta^{13}\cup\Theta^{14}} b_C^{(k)}\right) \qquad (19)
\end{aligned}$$

Using the definitions of $a_C^{(k)}$ and $b_C^{(k)}$, we can rewrite terms in (19).

$$\begin{aligned}
&(19)\\
&= \overbrace{\sum_{k\in\Theta^4\cup\Theta^5}\Big(\sum_{C\subset\mathcal{P}:\ i_k(\mathbf{t},\mathbf{b})\in C} a_C^{(k)}\Big)}^{=\Upsilon_1} + \overbrace{\sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^6\cup\Theta^7} a_C^{(k)}\Big)}^{=\Upsilon_2}\\
&+ \underbrace{\sum_{k\in\Theta^{11}\cup\Theta^{12}}\Big(\sum_{C\subset\mathcal{P}:\ \bar{i}_k^v\in C} b_C^{(k)}\Big)}_{=\Upsilon_3} + \underbrace{\sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^{13}\cup\Theta^{14}} b_C^{(k)}\Big)}_{=\Upsilon_4}.
\end{aligned}$$

From observations **O1 - O3**, for every $k \in \Theta^6 \cup \Theta^7$ and $C \subset \mathcal{P}$ such that $a_C^{(k)} = y_C$, we can find either (i) $k_1 \in \Theta^4 \cup \Theta^5$ such that $i_{k_1}(\mathbf{t},\mathbf{b}) \in C$ and $a_C^{(k_1)} = 0$ or (ii) $k_2 \in \Theta^{11} \cup \Theta^{12}$ such that $\bar{i}_{k_2}^v \in C$ and $b_C^{(k_2)} = 0$. Similarly, for every $\tilde{k} \in \Theta^{13} \cup \Theta^{14}$ and $C \subset \mathcal{P}$ such that $b_C^{(\tilde{k})} = y_C$, we can find either (i) $k_3 \in \Theta^4 \cup \Theta^5$ such that $i_{k_3}(\mathbf{t},\mathbf{b}) \in C$ and $a_C^{(k_3)} = 0$ or (ii) $k_4 \in \Theta^{11} \cup \Theta^{12}$ such that $\bar{i}_{k_4}^v \in C$ and $b_C^{(k_4)} = 0$. Therefore, we can swap the nonnegative $a_C^{(k)}$ or $b_C^{(k)}$ in Υ_2 and Υ_4, respectively, with the zero terms in Υ_1 and Υ_3. This swapping of the terms gives us

$$\begin{aligned}
&\sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} a_C^{(k)} + \sum_{k\in\Theta^3} a_C^{(k)} + \sum_{k\in\Theta^9} b_C^{(k)} + \sum_{k\in\Theta^{10}} b_C^{(k)}\Big)\\
&= \sum_{k\in\Theta^4}\Big(\sum_{C\subset\mathcal{P}:\ i_k(\mathbf{t},\mathbf{b})\in C} y_C\Big) + \sum_{k\in\Theta^5}\Big(\sum_{C\subset\mathcal{P}:\ i_k(\mathbf{t},\mathbf{b})\in C} y_C\Big)\\
&\quad + \sum_{k\in\Theta^{11}}\Big(\sum_{C\subset\mathcal{P}:\ \bar{i}_k^v\in C} y_C\Big) + \sum_{k\in\Theta^{12}}\Big(\sum_{C\subset\mathcal{P}:\ \bar{i}_k^v\in C} y_C\Big)\\
&= |\Theta^4| + |\Theta^5| + |\Theta^{11}| + |\Theta^{12}|\\
&= |\Theta^2| + |\Theta^{10}|, \qquad (20)
\end{aligned}$$

where the last equality follows from observation **O4**. This proves the lemma. □

Proof of Theorem 5. First, define $\varphi_{(2)}^{(k)} := 1 - \sum_{C\subset\mathcal{P}} a_C^{(k)}$ for $k \in \Theta^2$ and $\varphi_{(10)}^{(k)} := 1 - \sum_{C\subset\mathcal{P}} b_C^{(k)}$ for $k \in \Theta^{10}$. From (20) we obtain

$$\begin{aligned}
&\sum_{k\in\Theta^2} \varphi_{(2)}^{(k)} + \sum_{k\in\Theta^{10}} \varphi_{(10)}^{(k)}\\
&= |\Theta^2| - \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} a_C^{(k)}\Big) + |\Theta^{10}| - \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^{10}} b_C^{(k)}\Big)\\
&= \sum_{C\subset\mathcal{P}}\left(\sum_{k\in\Theta^3} a_C^{(k)} + \sum_{k\in\Theta^9} b_C^{(k)}\right) \qquad (21)
\end{aligned}$$

Let $\pi_\star := \inf\{\pi_{(k)}(\mathbf{t});\ k \in \Theta^2\}$ and $V_\star := \inf\{V_0^{(k)};\ k \in \Theta^{10}\}$. Then, we have the following inequality.

$$\begin{aligned}
&\sum_{k\in\Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^{10}} V_0^{(k)}\\
&= \sum_{k\in\Theta^2} \pi_{(k)}(\mathbf{t})\Big(\varphi_{(2)}^{(k)} + \sum_{C\subset\mathcal{P}} a_C^{(k)}\Big)\\
&\quad + \sum_{k\in\Theta^{10}} V_0^{(k)}\Big(\varphi_{(10)}^{(k)} + \sum_{C\subset\mathcal{P}} b_C^{(k)}\Big)\\
&\geq \sum_{k\in\Theta^2}\Big(\pi_\star\ \varphi_{(2)}^{(k)} + \sum_{C\subset\mathcal{P}} \pi_{(k)}(\mathbf{t})\ a_C^{(k)}\Big)\\
&\quad + \sum_{k\in\Theta^{10}}\Big(V_\star\ \varphi_{(10)}^{(k)} + \sum_{C\subset\mathcal{P}} V_0^{(k)} b_C^{(k)}\Big)\\
&\geq \sum_{k\in\Theta^2}\Big(\sum_{C\subset\mathcal{P}} \pi_{(k)}(\mathbf{t})\ a_C^{(k)}\Big) + \sum_{k\in\Theta^{10}}\Big(\sum_{C\subset\mathcal{P}} V_0^{(k)} b_C^{(k)}\Big)\\
&\quad + \min\{\pi_\star, V_\star\}\Big(\sum_{k\in\Theta^2} \varphi_{(2)}^{(k)} + \sum_{k\in\Theta^{10}} \varphi_{(10)}^{(k)}\Big) \qquad (22)
\end{aligned}$$

By interchanging the order of summations and from (21)

$$\begin{aligned}
(22) &= \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} \pi_{(k)}(\mathbf{t})\ a_C^{(k)} + \sum_{k\in\Theta^{10}} V_0^{(k)}\ b_C^{(k)}\Big)\\
&\quad + \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^3} \min\ \{\pi_\star, V_\star\} a_C^{(k)}\Big)\\
&\quad + \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^9} \min\ \{\pi_\star, V_\star\} b_C^{(k)}\Big)\ . \qquad (23)
\end{aligned}$$

Note that $\pi_{(k)}(\mathbf{t}) \leq \min\ \{\pi_\star, V_\star\}$ for all $k \in \Theta^3$ and $V_0^{(k)} \leq \min\{\pi_\star, V_\star\}$ for all $k \in \Theta^9$. Thus, from (22) - (23) and these inequalities, we get

$$\begin{aligned}
&\sum_{k\in\Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^{10}} V_0^{(k)}\\
&\geq \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} \pi_{(k)}(\mathbf{t})\ a_C^{(k)} + \sum_{k\in\Theta^3} \pi_{(k)}(\mathbf{t})\ a_C^{(k)}\Big)\\
&\quad + \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^9} V_0^{(k)}\ b_C^{(k)} + \sum_{k\in\Theta^{10}} V_0^{(k)}\ b_C^{(k)}\Big). \qquad (24)
\end{aligned}$$

Finally, from (24) and the definition of Θ^1 and Θ^8,

$$\begin{aligned}
&\sum_{C\subset\mathcal{P}} \Big(\sum_{k\in\Theta^1} a_C^{(k)}\ \pi_{(k)}(\mathbf{t}) \\
&\qquad + \sum_{k\in\Theta^2} a_C^{(k)}\ \pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^3} a_C^{(k)}\ \pi_{(k)}(\mathbf{t}) \Big) \\
&\quad + \sum_{C\subset\mathcal{P}} \Big(\sum_{k\in\Theta^9} b_C^{(k)}\ V_0^{(k)} + \sum_{k\in\Theta^{10}} b_C^{(k)}\ V_0^{(k)} \Big) \\
&\quad + \sum_{k\in\Theta^8} V_0^{(k)} \\
&= \sum_{C\subset\mathcal{P}} \big(y_C\ \ (\zeta(C,\overline{\pi}) + \lambda(C,\overline{\pi})) \big) \\
&\le \sum_{k\in\Theta^1\cup\Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^{10}} V_0^{(k)} + \sum_{k\in\Theta^8} V_0^{(k)} .
\end{aligned}$$

Since $\sum_{k\in\Theta^1\cup\Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^8\cup\Theta^{10}} V_0^{(k)} = \zeta(\mathcal{P},\overline{\pi}) + \lambda(\mathcal{P},\overline{\pi})$, we can conclude

$$\sum_{C\subset\mathcal{P}} y_C\ v(C) \le v(\mathcal{P})\ . \tag{25}$$

□

REFERENCES

[1] http://www.fcc.gov/oet/info/database/spectrum, http://www.ntia.doc.gov/osmhome/Allochrt.html.

[2] Spectrum policy task force report. Technical Report 02-135, Federal Communications Commission, November 2002.

[3] *Notice of proposed rule making and order*, ET Docket No. 03-322, Federal Communications Commision, December 2003.

[4] J. Bae, E. Beigman, R.A. Berry, M.L. Honig, and R. Vohra, “Sequential Bandwidth and Power Auctions for Distributed Spectrum Sharing,” to appear IEEE Journal on Selected Areas in Communications special issue on Game Theory in Communication Systems.

[5] F. Branco, “Multiple unit auctions of an indivisible good,” *Economic Theory*, vol. 8, pp. 77-101, 1996.

[6] M. M. Buddhikot, P. Kolodzy, S. Miller, K. Ryan, and J. Evans, “DIMSUMNet:New Directions in Wireless Networking Using Coordinated Dynamic Spectrum Access,” in *Proc. IEEE International Symposium on a World of Wireless Mobile and Multimedia Networks (WoWMoM 2005)*, pp. 78- 85, June 2005.

[7] Shih-Fen Cheng, D.M. Reeves, Y. Vorobeychik, and M.P. Wellman, “Notes on Equilibria in Symmetric Games,” In *AAMAS-04 Workshop on Game Theory and Decision Theory*, pp. 23-28, August 2004.

[8] T. Erpek, K. Steadman, and D. Jones, “Dublin Ireland Spectrum Occupancy Measurements Collected On April 16-18, 2007,” The Shared Spectrum Company, November 2007.

[9] T. Erpek, M. Lofquist, and K. Patton, “Spectrum Occupancy Measurements Loring Commerce Centre Limestone, Maine September 18-20, 2007,” The Shared Spectrum Company, 2007.

[10] R. Etkin, A. Parekh, and D. Tse, “Spectrum Sharing for Unlicensed Bands,” *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 3, pp. 517-528, April 2007.

[11] R. Gibbons, “Game Theory for Applied Economists,” Princeton University Press, 1992.

[12] J. Huang, R.A. Berry, and M.L. Honig, “Auction-based spectrum sharing,” *ACM/Springer Mobile Networks and Applications Journal (MONET)*, vol. 11, no. 3, pp. 405-418, June 2006.

[13] M. McHenry, D. McCloskey, D. Roberson, and J. MacDonald, “Spectrum Occupancy Measurements: Chicago, Illinois, November 16-18, 2005,” The Shared Spectrum Company, Dec 2005.

[14] J. Mitola, *Cognitive Radio : An Integrated Agent Architecture for Software Defined Radio*, Ph.D Thesis, Royal Institute of Technology(KTH), Stockholm, Sweden, 2000.

[15] J. Mitola, “Cognitive Radio for Flexible Mobile Multimedia communications,” *Mobile Networks and Applications*, vol. 6, pp. 435-441, 2001.

[16] H. Mutlu, M. Alanyali, and D. Starobinski, “Spot Pricing of Secondary Spectrum Usage in Wireless Cellular Networks,” in *Proc. IEEE International Conference on Computer Communications, INFOCOM 2008*, pp. 682-690, April 2008.

[17] R. B. Myerson, “Optimal Auction Design,” *Mathematics of Operations Research*, vol. 6, no. 1, pp. 58-73, February 1981.

[18] Nie Nie, and Cristina Comaniciu, “Adaptive channel allocation spectrum etiquette for cognitive radio networks,” *IEEE International Symposium on New Frontiers in Dynamic Spectrum Access Networks (DySPAN 2005)*, pp. 269-278, November 2005.

[19] G. Owen, *Game Theory*. Academic Press 1995.

[20] M. Vilimpoc, and M. McHenry, “Dupont Circle Spectrum Utilization During Peak Hours,” The New America Foundation and the Shared Spectrum Company, June 2003.

[21] F. Wang, M. Krunz, and S. Cui, “Spectrum Sharing in Cognitive Radio Networks,” in *Proc. IEEE International Conference on Computer Communications, INFOCOM 2008*, pp. 1885-1893, April 2008.