Set Theory, Sample Space, and Event Space

Reading: Chapter 1.1 and 1.2
Homework: 1.1.1, 1.2.2, 1.2.3, 1.2.4

Set and Element

**A set** is a collection of things, objects, or elements.

**Examples:**
- A={1,2,3,4,5,6}
- B={January, February, May, July}
- C={all months that end with a 'y'}
- D={x|x is a multiple of 7 and x > 0}
- E={x|x is the square of some integer}

- x Є A if x is an element of set A. 7 Є D, 7 Є A
Cardinality

- The cardinality of a set $A$, $|A|$, is the number of elements in the set.
- A finite set has finite number of elements. (example: sets $A$, $B$, and $C$)
- An infinite set has infinite number of elements. (example: sets $D$ and $E$)
- The set that has zero element is called empty set, or null set, and denoted by $\emptyset$.

Subset and Universal Set

- Set $A$ is a subset of set $B$, $A \subseteq B$, if every element of $A$ is also an element of $B$.
  - If $A \subseteq B$, then $|A| \leq |B|
  - $\emptyset \subseteq A$
  - If $A \subseteq B$ and $B \subseteq A$, then $A = B$
- The universal set, $S$, consists of all the possible elements within a given context.
  - $S=\{0,1,2,3,\ldots\}$ when we consider the number of students in a class.
  - $S=\{x|0 \leq x \leq 100, \ x=0.5*k \text{ for some } k = 0,1,2,3,\ldots\}$ when we consider the scores of an exam where no fractions less than 0.5 are given.
  - Any set within the same context is a subset of $S$. 
Set Algebra

- **Union**: \( A \cup B = \{x | x \in A \text{ or } x \in B\} \)
- **Intersection**: \( A \cap B = \{x | x \in A \text{ and } x \in B\} \)
- **Complement**: \( A^c = \{x | x \in S \text{ but } x \notin A\} \)
- **Difference**: \( A - B = \{x | x \in A \text{ but } x \notin B\} \)

**Facts:**
- \( A \subseteq A \cup B \)
- \( A \cap B \subseteq A \)
- \( A \cup A^c = S \)
- \( A \cap A^c = \emptyset \)
- \( A - B = A \cap B^c \)
- \( A \subseteq A \cup A^c \)

Exclusive and Exhaustive

- Two sets, \( A \) and \( B \), are **disjoint** if \( A \cap B = \emptyset \).
  - \( A = \{\text{all even numbers}\} \)
  - \( B = \{\text{all odd numbers}\} \)
- A group/set of sets, \( A_1, A_2, \ldots, A_n \), are **mutually exclusive** iff for any \( i \neq j \): \( A_i \cap A_j = \emptyset \).
  - \( A_i = \{x | x = i \text{ (mod } n)\} \)
- A group/set of sets, \( A_1, A_2, \ldots, A_n \), is **collectively exhaustive** iff \( A_1 \cup A_2 \cup \ldots \cup A_n = S \).
  - \( A \cup B = S = Z \) (all integers)
  - \( \bigcup_{i=1}^{n} A_i = S = Z \)
  - \( \{1,2,3\} \cup \{2,4,6\} \cup \{2,3,5\} = S = \{\text{numbers on a die}\} \)
Theorem 1.1: \((A \cup B)^c = A^c \cap B^c\)

Proof:

Venn diagram
- \(S, A, B\)
- \(A \cap B\)
- \(A \cup B\)
- \(A^c, B^c\)
- \(A - B\)
- De Morgan’s law

Experiment and Model

A trial is the observation/result/outcome following a given procedure.
- Head/tail when flip a coin.
- Number when roll a die.
- Temperature at a given time of the day.

An experiment is a collection of trials.

A model is a more abstract representation of an experiment and captures the important part.
- Flip a coin: head/tail are equally likely each time.
Outcome and Sample Space

- An outcome of an experiment is any possible observation of the experiment.
- The sample space of an experiment is the finest-grain, mutually exclusive, and collectively exhaustive set of all outcomes.

Examples:
- Flip a coin: S={head, tail}
- Flip three coins and observe the number of heads: S={0,1,2,3}
- Flip three coins and observe the sequence of head/tail: S={hhh, hht, hth, htt, thh, tht, tth, ttt}
- Roll one die: S={1,2,3,4,5,6}
- The sum of two numbers when roll two dice: S={2,3,4,5,6,7,8,9,10,11,12}

Event and Event Space

- An event is a set of outcomes of an experiment.
- Events of rolling a die:
  - E1: an even number;
  - E2: an odd number;
  - E3: a number of 3 or larger;
  - E4: 4;

- An event space is a collectively exhaustive, mutually exclusive set of events.
- The set \{E1, E2\} is an event space.
Example

- **Procedure:** get a 4-bit message from a reliable communication channel
- **Observation:** observe the bit information of the message
- **Model:** 4 bits, each bit can be 0 or 1 with equal likelihood.
- **Sample space:** {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}
- **Events:** $E_i = \{\text{the } i\text{-th bit is 1}\}$, $F_i = \{\text{the } i\text{-th bit is 0}\}$, $G_i = \{\text{there are } i \text{ 1's}\}$
- **Event space:** $\{E_0, F_0\}, \{G_0, G_1, G_2, G_3, G_4\}$

Theorem on Event Space

- **Theorem 1.2.**
  For an event space $B = \{B_1, B_2, \ldots\}$ and any event $A$ in the sample space, let $C_i = A \cap B_i$, then events $C_i$'s are mutually exclusive and $A = \bigcup_{i=1,2} C_i$

- **Proof:**

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