Probability Axioms

Reading: Chapter 1.3, 1.4, and 1.5
Homework: 1.3.4, 1.3.5, 1.4.3, 1.5.4, 1.5.5

Axioms of Probability

**Probability measure** is a function, P[.], that maps events in the sample space, S, to real numbers such that:

Axiom 1: \( P[A] \geq 0 \) for any event A.
Axiom 2: \( P[S] = 1 \).
Axiom 3: \( P[A_1 \cup A_2 \cup ...] = P[A_1]+P[A_2]+... \) for any countable collection of mutually exclusive events \( A_1, A_2, ... \).

**Countable sets:** all finite sets, set of all integers, fractions, rational numbers.
Corollaries of Axiom 3

- **Theorem 1.3**: For mutually exclusive events $A$ and $B$, \[ P[A \cup B] = P[A] + P[B] \]
  - Two events $A$ and $B$ are countable.

- **Theorem 1.4**: For mutually exclusive events $A_1, A_2, \ldots, A_n$,
  \[ P[A_1 \cup A_2 \cup \ldots \cup A_n] = P[A_1] + P[A_2] + \ldots + P[A_n] \]
  - $n$ events are countable.

- **Theorem 1.5**: For event $A = \{s_1, s_2, \ldots, s_n\}$ of $n$ outcomes, \[ P[A] = P[s_1] + P[s_2] + \ldots + P[s_n] \]

Equally Likely Outcomes

- **Theorem 1.6**: If each outcome $s_i$ in the sample space $S = \{s_1, s_2, \ldots, s_n\}$ is equally likely, $P[s_i] = 1/n$.
  - Proof: $P[s_1] = P[s_2] = \ldots = P[s_n] \Rightarrow P[S] = n \times P[s_i] = 1$

- **Roll a die**
  - $P[4] = 1/6$

- **Draw one card from a whole deck of cards**
  - $P[\text{ace of spade}] = 1/52$
  - $P[\text{a club}] = 1/4$
  - $P[\text{a red card}] = 1/2$
More (Trivial) Facts

- **Theorem 1.7:**
  - \( P[\emptyset] = 0 \)
  - \( P[A^c] = 1 - P[A] \)
  - \( P[A \cup B] = P[A] + P[B] - P[A \cap B] \)
  - \( P[A] \leq P[B] \) if \( A \subseteq B \)

- **Theorem 1.8:** For an event \( A \) and an event space \( \{B_1, B_2, \ldots, B_m\} \), \( P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots + P[A \cap B_m] \)

Example 1.14

- **Procedure:** Monitor incoming telephone calls
- **Observation:**
  - Type of the call: voice (v), data (d), fax (f);
  - Length of the call: long (l) if longer than 3 minutes, brief (b).
- **Sample space:** \( \{lv, ld, lf, bv, bd, bf\} \)
- **Probabilities of outcomes:**
  - \( P[lv] = 0.3, P[lid] = 0.12, P[lf] = 0.15, P[bv] = 0.2, P[bd] = 0.08, P[bf] = 0.15 \)
- **Events:** \( L, B, V, D, F \)
  - \( P[L] = P[lv] + P[lid] + P[lf] = 0.57 \)
  - \( P[D] = P[lid] + P[bd] = 0.20 \)
- **Event Spaces:** \( \{L, B\}, \{V, D, F\} \)
Conditional Probability

- Conditional probability measures the probability of an event given certain conditions (e.g. the occurrence of another event).

- Conditional probability of the event $A$ given the occurrence of the event $B$ is

$$P[A|B] = \frac{P[AB]}{P[B]} \quad (AB: \text{short for } A \cap B)$$

- Theorem 1.9: the conditional probability measure, $P[.|B]$, defined above satisfies the 3 probability axioms.

Examples 1.15 and 1.16

- Procedure: test two ICs from the same wafer.
- Observation: whether the ICs are functionally correct.
- Model: two decisions on reject (r) or accept (a).
- Sample space: $S=\{rr, ra, ar, aa\}$
- (Priori) probabilities of outcomes:
  - $P[rr] = 0.01$, $P[ra] = 0.01$, $P[ar] = 0.01$, $P[aa] = 0.97$
Examples 1.15 and 1.16

- Events:
  - \( A = \{ \text{the second IC is rejected} \} = \{ rr, ar \} \)
  - \( B = \{ \text{the first IC is rejected} \} = \{ rr, ra \} \)
  - \( P[A] = P[rr] + P[ar] = 0.02 \)
  - \( P[B] = P[rr] + P[ra] = 0.02 \)
  - \( P[B^c] = 1 - P[B] = 0.98 \) \( (B^c: \text{first IC is accepted}) \)
  - \( P[AB] = P[rr] = 0.01 \)

- Conditional probability
  - \( P[A|B] = \frac{P[AB]}{P[B]} = \frac{0.01}{0.02} = \frac{1}{2} \)
  - \( P[A|B^c] = \frac{P[AB^c]}{P[B^c]} = \frac{0.01}{0.98} = \frac{1}{98} \)

Example 1.18

- Procedure: roll two fair 4-sided dice
- Observation: the numbers, \( x \) and \( y \), on each die
- Model: both \( x = 1/2/3/4 \) and \( y = 1/2/3/4 \) with equal likelihood
- Sample space: \( S = \{ 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44 \} \), 16 outcomes.
- Events:
  - \( A = \{ xy | x > 1 \} = \{ 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44 \} \)
  - \( B = \{ xy | y > x \} = \{ 12, 13, 14, 23, 24, 34 \} \)
  - \( P[A|B] = \frac{P[AB]}{P[B]} = \frac{3/16}{6/16} = \frac{1}{2} < P[A] = \frac{12}{16} \)
Problem 1.5.6

Experiment: a deer tick carries L disease and/or H disease.
Sample space: S={l, h, lh, none}
Events:
- L = {carries L} = {l, lh}
- H = {carries H} = {h, lh}
- LH = {carries both L and H} = {lh}
Priori probabilities:
P[L] = 0.16

Question 1: what is P[LH]?
- \( P[L \cup H] = P[L] + P[H] - P[LH] = 0.26 - P[LH] \)
- \( P[LH] = 0.10 \times P[L \cup H] = 0.10 \times (0.26 - P[LH]) \)
- \( P[LH] = 0.26/11 \approx 0.0236 \)

Question 2: what is P[H|L]?
- \( P[H|L] = P[LH]/P[L] \approx 0.0236/0.16 \approx 0.1475 \)

Question 3: what is P[L|H]?
- \( P[L|H] = P[LH]/P[H] \approx 0.0236/0.10 = 0.236 \)