Bayes’ Theorem, Independence, and Tree Diagram

Reading: Chapter 1.5, 1.6, and 1.7
Homework: 1.6.2, 1.6.4, 1.6.7, 1.7.1, 1.7.4

Law of Total Probability

- Conditional probability of the event $A$ given the occurrence of the event $B$ is $P[A|B] = \frac{P[AB]}{P[B]}$
- Theorem 1.10: For an event space $\{B_1, ..., B_m\}$ with $P[B_i] > 0$, $P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + ... + P[A|B_m]P[B_m]$
- $A = AB_1 \cup AB_2 \cup ... \cup AB_m$
- $P[AB_i] = P[A|B_i]P[B_i]$
Example 1.19

Model: three machines $B_1, B_2,$ and $B_3$ are making 3000, 4000, and 3000 copies of the same product, where 80%, 90%, and 60% are acceptable respectively. Mix all the products and pick one randomly.

Event: $A = \{\text{the product is acceptable}\}$

Question: $P[A] = ?$

- Event space: $B_i = \{\text{product is from machine } B_i\}$
- $P[B_1] = 0.3, P[B_2] = 0.4, P[B_3] = 0.3$
- $P[A|B_1] = 0.8, P[A|B_2] = 0.9, P[A|B_3] = 0.6$
- $P[A] = 0.8 \times 0.3 + 0.9 \times 0.4 + 0.6 \times 0.3 = 0.78$

Example 1.20

What we know:
- $P[A|B_1] = 0.8, P[A|B_2] = 0.9, P[A|B_3] = 0.6$
- $P[B_1] = 0.3, P[B_2] = 0.4, P[B_3] = 0.3, P[A] = 0.78$

Question: $P[B_3|A] = ?$

- What does this mean?
- $P[B_3|A] = P[AB_3]/P[A]$
- $P[AB_3] = P[A|B_3] \times P[B_3] = 0.6 \times 0.3 = 0.18$
- $P[B_3|A] = 0.18/0.78 \approx 0.23$
Bayes' Formula

Theorem 1.11: \( P[B|A] = P[A|B] \cdot P[B]/P[A] \)

Proof:

It is useful to make inference on the reasons when an effect is observed.

In the previous example, it is impossible to tell which machine produces a particular copy of the product.

But we know \( P[B_i] \) and we can compute \( P[B_i|A] \)

Two Independent Events

Two events A and B are independent iff \( P[AB] = P[A] \cdot P[B] \)

- Similarly, \( P[B|A] = P[B|A^c] = P[B] \)

Test independence

- \( P[AB] = P[A] \cdot P[B] \)
- \( P[A|B] = P[A] \) or \( P[A|B^c] = P[A] \)
Problem 1.6.3


Find: \( P[A \cap B], P[A \cup B], P[A \cap B^c], P[A \cup B^c] \)

Are \( A \) and \( B \) independent?

Find: \( P[C \cap D], P[C \cap D^c], P[C^c \cap D^c] \)

Are \( C^c \) and \( D^c \) independent?

n Independent Events

\( n \) events \( A_1, A_2, ..., A_n \) are independent iff

\[
P[A_{i_1}A_{i_2}...A_{i_k}] = P[A_{i_1}]P[A_{i_2}]...P[A_{i_k}]
\]

for any \( 1 \leq i_1, i_2, ..., i_k \leq n \) and \( k = 2, 3, ..., n \).

- \( k=2 \): any pair of events are independent
- any \( k \) events are independent
- different, but equivalent to the book defn.
**Tree Diagram**

Tree diagram is a useful tool to represent experiments that consist of sequential subexperiments.

**Example 1.24 (tree diagram for 1.19):**

```
                     • B1A | 0.8  |
                     • B1N | 0.2  |
                     • B2A | 0.9  |
                     • B2N | 0.1  |
                     • B3A | 0.6  |
                     • B3N | 0.4  |
```

```
A   B1    0.24
N   B1    0.96
A   B2    0.36
N   B2    0.04
A   B3    0.18
N   B3    0.12
```

**Examples**

**Example 1.26:** Pick numbers from \{1,2,3\} until the sum is 3 or higher. Win if the total is exactly 3. What is \(P[\text{Win}]\)?

**Quiz 1.7:** Page a phone until find it. Each paging attempt succeeds with probability 0.8. What is the probability of success in 3 attempts or less?

**Problem 1.7.7:** Two biased coins, A and B, come with head with prob. \(\frac{1}{4}\) and \(\frac{3}{4}\). Event \(A_1\): flip A then B; event \(B_1\): flip B then A; event \(H_i\): the i-th flip comes with head; event \(T_i\): the i-th flip comes with tail. What is \(P[H_1H_2]\)?