CHAPTER SIXTEEN

# Optics of Gaussian Beams

# 16

# Optics of Gaussian Beams

## 16.1 Introduction

In this chapter we shall look from a wave standpoint at how narrow beams of light travel through optical systems. We shall see that special solutions to the electromagnetic wave equation exist that take the form of narrow beams – called *Gaussian beams.* These beams of light have a characteristic radial intensity profile whose width varies along the beam. Because these Gaussian beams behave somewhat like spherical waves, we can match them to the curvature of the mirror of an optical resonator to find exactly what form of beam will result from a particular resonator geometry.

## 16.2 Beam-Like Solutions of the Wave Equation

We expect intuitively that the transverse modes of a laser system will take the form of narrow beams of light which propagate between the mirrors of the laser resonator and maintain a field distribution which remains distributed around and near the axis of the system. We shall therefore need to find solutions of the wave equation which take the form of narrow beams and then see how we can make these solutions compatible with a given laser cavity.

Now, the wave equation is, for any field or potential component $U_0$ of an electromagnetic wave

$$\nabla^2 U_0 - \mu\epsilon_r\epsilon_0 \frac{\partial^2 U_0}{\partial t^2} = 0 \tag{16.1}$$

where $\epsilon_r$ is the dielectric constant, which may be a function of position. The non-plane wave solutions that we are looking for are of the form

$$U_0 = U(x, y, z)e^{i(\omega t - \mathbf{k}(\mathbf{r})\cdot\mathbf{r})} \tag{16.2}$$

We allow the wave vector $\mathbf{k}(\mathbf{r})$ to be a function of $\mathbf{r}$ to include situations where the medium has a non-uniform refractive index. From (16.1) and (16.2)

$$\nabla^2 U + \mu\ \epsilon_r\epsilon_0\omega^2 U = 0 \tag{16.3}$$

where $\epsilon_r$ and $\mu$ may be functions of $\mathbf{r}$. We have shown previously that the propagation constant in the medium is $k = \omega\sqrt{\epsilon_r\epsilon_0\mu}$ so

$$\nabla^2 U + k(\mathbf{r})^2 U = 0. \tag{16.4}$$

This is the time-independent form of the wave equation, frequently referred to as the *Helmholtz* equation.

In general, if the medium is absorbing, or exhibits gain, then its dielectric constant $\epsilon_r$ has real and imaginary parts.

$$\epsilon_r = 1 + \chi(\omega) = 1 + \chi\prime(\omega) - i\chi\prime\prime(\omega) \tag{16.5}$$

and

$$k = k_0\sqrt{\epsilon_r}, \tag{16.6}$$

where $k_0 = \omega\sqrt{\epsilon_0\mu}$

If the medium were conducting, with conductivity $\sigma$, then its complex propagation vector would obey

$$k^2 = \omega^2\mu\ \epsilon_r\epsilon_0(1 - i\frac{\sigma}{\omega\epsilon_r\epsilon_0}) \tag{16.7}$$

We know that simple solutions of the time-independent wave equation above are transverse plane waves. However, these simple solutions are not adequate to describe the field distributions of transverse modes in laser systems. Let us look for solutions to the wave equation which are related to plane waves, but whose amplitude varies in some way transverse to their direction of propagation. Such solutions will be of the form $U = \psi(x, y, z)e^{-ikz}$ for waves propagating in the positive $z$ direction. For functions $\psi(x, y, z)$ which are localized near the $z$ axis the propagating wave takes the form of a narrow beam. Further, because $\psi(x, y, z)$ is not uniform, the surfaces of constant phase in the wave will no longer necessarily be plane. If we can find solutions of the wave equation where $\psi(x, y, z)$ gives phase fronts which are spherical (or approximately so over a small region) then we can make the propagating

Fig. 16.1.

beam solution $U = \psi(x, y, z)e^{-ikz}$ satisfy the boundary conditions in a resonator with spherical reflectors, provided the mirrors are placed at the position of phase fronts whose curvature equals the mirror curvature. Thus the propagating beam solution becomes a satisfactory transverse mode of the resonator. For example in Fig. (16.1) if the propagating beam is to be a satisfactory transverse mode, then a spherical mirror of radius $R_1$ must be placed at position 1 or one of radius $R_2$ at 2, etc. Mirrors placed in this correct way lead to reflection of the wave back on itself.

Substituting $U = \psi(x, y, z)e^{-ikz}$ in (16.4) we get

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)e^{-ikz} + \left(\frac{\partial^2\psi}{\partial y^2}\right)e^{-ikz} + \left(\frac{\partial^2\psi}{\partial z^2}\right)e^{-ikz} - 2ik\frac{\partial\psi}{\partial z}e^{-ikz} - k^2\psi(x, y, z)e^{-ikz} + k^2\psi(x, y, z)e^{-ikz} = 0 \tag{16.8}$$

which reduces to

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2ik\frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{16.9}$$

If the beam-like solution we are looking for remains paraxial then $\psi$ will only vary slowly with z, so we can neglect $\partial^2\psi/\partial z^2$ and get

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2ik\frac{\partial\psi}{\partial z} = 0 \tag{16.10}$$

We try as a solution

$$\psi(x, y, z) = exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\} \tag{16.11}$$

where $r^2 = x^2 + y^2$ is the square of the distance of the point $x, y$ from the axis of propagation. $P(z)$ represents a phase shift factor and $q(z)$ is called the beam parameter.

Substituting in (16.10) and using the relations below that follow from (16.11)

$$\frac{\partial\psi}{\partial x} = -\frac{ik}{2q(z)} exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\} \cdot 2x$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{ik}{q(z)} exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\} - \frac{k^2}{4q^2(z)} exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\} \cdot 4x^2$$

$$\frac{\partial\psi}{\partial z} = exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\}\{-i\left(\frac{dP}{dz} - \frac{k}{2q^2}\frac{dq}{dz}r^2\right)\}$$

we get

$$-2k(\frac{dP}{dz} + \frac{i}{q(z)}) - (\frac{k^2}{q^2(z)} - \frac{k^2}{q^2(z)}\frac{dq}{dz})(r^2) = 0 \tag{16.12}$$

Since this equation must be true for all values of r the coefficients of different powers of r must be independently equal to zero so

$$\frac{dq}{dz} = 1 \tag{16.13}$$

and

$$\frac{dP}{dz} = \frac{-i}{q(z)} \tag{16.14}$$

The solution $\psi(x, y, z) = exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\}$ is called the *fundamental* Gaussian beam solution of the time-independent wave equation since its "intensity" as a function of x and y is

$$UU^* = \psi\psi^* = exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\}exp\{i(P^*(z) + \frac{k}{2q^*(z)}r^2)\} \tag{16.15}$$

where $P^*(z)$ and $q^*(z)$ are the complex conjugates of $P(z)$ and $q(z)$, respectively, so

$$UU^* = \exp\{-i(P(z) - P^*(z))\} \exp\{\frac{-ikr^2}{2}(\frac{1}{q(z)} - \frac{1}{q^*(z)})\} \tag{16.16}$$

For convenience we introduce 2 real beam parameters $R(z)$ and $w(z)$ that are related to $q(z)$ by

$$\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2} \tag{16.17}$$

where both $R$ and $w$ depend on $z$. It is important to note that $\lambda = \lambda_o/n$

is the wavelength *in the medium.* From (16.16) above we can see that

$$\begin{aligned} UU^* &\propto \exp \frac{-ikr^2}{2}\left(-\frac{i\lambda}{\pi w^2} - \frac{i\lambda}{\pi w^2}\right) \\ &\propto \exp \frac{-2r^2}{w^2} \end{aligned} \tag{16.18}$$

Thus, the beam intensity shows a Gaussian dependence on $r$, the physical significance of $w(z)$ is that it is the distance from the axis at point $z$ where the intensity of the beam has fallen to $1/e^2$ of its peak value on axis and its amplitude to $1/e$ of its axial value; $w(z)$ is called the *spot size.* With these parameters

$$U = \exp\big\{-i\big[(kz + P(z) + \frac{kr^2}{2}\big(\frac{1}{R} - \frac{i\lambda}{\pi w^2}\big)\big]\big\}. \tag{16.19}$$

We can integrate (16.13) and (16.14). From (16.13)

$$q = q_0 + z, \tag{16.20}$$

where $q_0$ is a constant of integration which gives the value of the beam parameter at plane $z = 0$; and from (16.14)

$$\frac{dP(z)}{dz} = \frac{-i}{q(z)} = \frac{-i}{q_0 + z} \tag{16.21}$$

so

$$P(z) = -i[\ln(z + q_0)] - (\theta + i \ln q_0)] \tag{16.22}$$

where the constant of integration, $(\theta + i \ln q_0)$, is written in this way so that substituting from (16.20) and (16.22) in (16.19) we get

$$U = \exp\{-i(kz - i\ln(1 + \frac{z}{q_0}) + \theta + \frac{kr^2}{2}[\frac{1}{R} - \frac{i\lambda}{\pi w^2}])\} \tag{16.23}$$

The factor $e^{-i\theta}$ is only a constant phase factor which we can arbitrarily set to zero and get

$$U = \exp -i(kz - i\ln(1 + \frac{z}{q_0}) + \frac{kr^2}{2}[\frac{1}{R} - \frac{i\lambda}{\pi w^2}]) \tag{16.24}$$

The radial variation in phase of this field for a particular value of $z$ is

$$\phi(z) = kz + \frac{kr^2}{2R} - Re(i\ln(1 + \frac{z}{q_o})) \tag{16.25}$$

There is no radial phase variation if $R \to \infty$. We choose the value of $z$ where this occurs as our origin $z = 0$. We call this location the *beam waist*; the complex beam parameter here has the value

$$\frac{1}{q_0} = -i\frac{\lambda}{\pi w_0^2} \tag{16.26}$$

where $w_0 = w(0)$ is called the *minimum spot size.* At an arbitrary point

Fig. 16.2.

z

$$q = \frac{i\pi w_0^2}{\lambda} + z \qquad (16.27)$$

and using $\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2}$, we have the relationship for the *spot size* $w(z)$ at position $z$

$$w^2(z) = w_0^2[1 + (\frac{\lambda z}{\pi w_0^2})^2] \qquad (16.28)$$

So $w_0$ is clearly the *minimum* spot size.

The value of $R$ at position $z$ is

$$R(z) = z[1 + (\frac{\pi w_0^2}{\lambda z})^2] \qquad (16.29)$$

We shall see shortly that $R(z)$ can be identified as the radius of curvature of the phase front at $z$. From (16.28) we can see that $w(z)$ expands with distance $z$ along the axis along a hyperbola which has asymptotes inclined to the axis at an angle $\theta_{beam} = \tan^{-1}(\lambda/\pi w_0)$. This is a small angle if the beam is to have small divergence, in which case $\theta_{beam} \simeq \lambda/\pi w_0$ where $\theta_{beam}$ is the half angle of the diverging beam shown in Fig. (16.2). From (16.24) it is clear that the surfaces of constant phase are those surfaces which satisfy

$$kz + Re[-i\ln(1 + \frac{z}{q_0})] + \frac{kr^2}{2}[\frac{1}{R}] = constant \qquad (16.30)$$

which gives

$$k[z + \frac{r^2}{2R}] = constant - Re[-i\ln(1 + \frac{z}{q_0})] \qquad (16.31)$$

Now

$$\ln(1 + \frac{z}{q_0}) = \ln(1 - \frac{iz\lambda}{\pi w_0^2}) = \ln[1 + (\frac{\lambda z}{\pi w_0^2})^2]^{1/2} - i\tan^{-1}(\frac{\lambda z}{\pi w_0^2}) \qquad (16.32)$$

where we have used the relation

$$\ln(x+iy) = \ln(x^2+y^2)^{1/2} + i\tan^{-1}((y/x) \quad (16.33)$$

so from (16.33)

$$Re[-i\ln(1+\frac{z}{q_0})] = \tan^{-1}(\frac{\lambda z}{\pi w_0^2}). \quad (16.34)$$

The term $kz = \frac{2\pi z}{\lambda}$ in (16.31) is very large, except very close to the beam waist and for long wavelengths, it generally dominates the $z$ dependence of (16.31). Also, $\tan^{-1}(\frac{\lambda z}{\pi w_0^2})$ at most approaches a value of $\pi/2$, so, we can write

$$k[z+\frac{r^2}{2R}] = constant \quad (16.35)$$

as the equation of the surface of constant phase at $z$, which can be rewritten

$$z = \frac{constant}{k} - \frac{r^2}{2R}. \quad (16.36)$$

This is a parabola, which for $r^2 \ll z^2$ is a very close approximation to a spherical surface of which $R(z)$ is the radius of curvature.

The complex phase shift a distance $z$ from the point where $q$ is purely imaginary, the point called the beam waist, is found from (16.14),

$$\frac{dP(z)}{dz} = \frac{-i}{q} = \frac{-i}{z+i\pi w_0^2/\lambda} \quad (16.37)$$

Thus,

$$P(z) = -i\ln[1-i(\lambda z/\pi w_0^2)] = -i\ln(1+(\lambda z/\pi w_0^2)^2)^{1/2} - \tan^{-1}\frac{\lambda z}{\pi w_0^2} \quad (16.38)$$

where the constant of integration has been chosen so that $P(z) = 0$ at $z = 0$.

The real part of this phase shift can be written as $\Phi = \tan^{-1}\frac{\lambda z}{\pi w_0^2}$, which marks a distinction between this Gaussian beam and a plane wave. The imaginary part is, from Eqs. (16.28) and (16.38)

$$Im[P(z)] = -i\ln(w(z)/w_o) \quad (16.39)$$

This imaginary part gives a real intensity dependence of the form $w_0^2/w^2(z)$ on axis, which we would expect because of the expansion of the beam. For example, part of the "phase factor" in Eq. (16.11) gives

$$e^{-iIm[(P(z))]} = e^{-\ln(w/w_0)} = \frac{w_0}{w}.$$

Thus, from Eqs. (16.11),(16.17),and (16.39) we can write the spatial dependence of our Gaussian beam

$$U = \psi(x,y,z)e^{-ikz}$$

as

$$U(r,z) = \frac{w_0}{w}\exp\{-i(kz-\Phi) - r^2(\frac{1}{w^2} + \frac{ik}{2R})\} \tag{16.40}$$

This field distribution is called the *fundamental* or $TEM_{00}$ Gaussian mode. Its amplitude distribution and beam contour in planes of constant $z$ and planes lying in the $z$ axis, respectively, are shown in Fig. (16.2).

In any plane the radial intensity distribution of the beam can be written as

$$I(r) = I_0 e^{-2_r^2/w^2} \tag{16.40a}$$

where $I_0$ is the axial intensity.

## 16.3 Higher Order Modes

In the preceding discussion, only one possible solution for $\psi(x,y,z)$ has been considered, namely the simplest solution where $\psi(x,y,z)$ gives a beam with a Gaussian dependence of intensity as a function of its distance from the axis: the width of this Gaussian beam changes as the beam propagates along its axis. However, higher order beam solutions of (16.10) are possible.

(a) Beam Modes with Cartesian Symmetry. For a system with cartesian symmetry we can try for a solution of the form

$$\psi(x,y,z) = g(\frac{x}{w})\cdot h(\frac{y}{w})\exp\{-i[P + \frac{k}{2q}(x^2+y^2)]\} \tag{16.41}$$

where $g$ and $h$ are functions of $x$ and $z$ and $y$ and $z$ respectively, $w(z)$ is a Gaussian beam parameter and $P(z), q(z)$ are the beam parameters used previously.

For real functions $g$ and $h$, $\psi(x,y,z)$ represents a beam whose intensity patterns scale according to a Gaussian beam parameter $w(z)$. If we substitute this higher order solution into (16.10) then we find that the solutions have $g$ and $h$ obeying the following relation.

$$g\cdot h = H_m(\sqrt{2}\frac{x}{w})H_n(\sqrt{2}\frac{y}{w}) \tag{16.42}$$

where $H_m$, $H_n$ are solutions of the differential equation

$$\frac{d^2H_m}{dx^2} - 2x\frac{dH_m}{dx} + 2mH_m = 0 \tag{16.43}$$

of which the solution is the Hermite polynomial $H_m$ of order $m$; $m$ and $n$ are called the transverse mode numbers. Some of the low order Hermite polynomials are:

$$H_0(x) = 1;\ H_1(x) = x;\ H_2(x) = 4x^2 - 2;\ H_3(x) = 8x^3 - 12x \tag{16.44}$$

Fig. 16.3a.

The overall amplitude behavior of this higher order beam is

$$U_{m,n}(r,z) = \frac{w_0}{w} H_m(\sqrt{2}\frac{x}{w}) H_m(\sqrt{2}\frac{x}{w}) \exp\{-i(kz - \Phi) - r^2(\frac{1}{w^2} + \frac{ik}{2R})\} \tag{16.45}$$

The parameters $R(z)$ and $w(z)$ are the same for all these solutions. However $\Phi$ depends on $m, n$ and $z$ as

$$\Phi(m,n,z) = (m+n+1)\tan^{-1}(\frac{\lambda z}{\pi w_0^2}) \tag{16.46}$$

Since this Gaussian beam involves a product of Hermite and Gaussian functions it is called a *Hermite-Gaussian* mode and has the familiar intensity pattern observed in the output of many lasers, as shown in Fig. (16.3a). The mode is designated a $TEM_{mm}$ $HG$ mode. For example, in the plane $z = 0$ the electric field distribution of the $TEM_{mm}$ mode is, if the wave is polarized in the $x$ direction

$$E_{m,n}(x,z) = E_0 H_m(\sqrt{2}\frac{x}{w_0}) H_n(\sqrt{2}\frac{y}{w_0}) \exp(-\frac{x^2+y^2}{w_0^2}) \tag{16.47}$$

The lateral intensity variation of the mode is

$$I(x,y) \propto |E_{m,n}(x,z)|^2 \tag{16.47$a$}$$

and in surfaces of constant phase

$$E_{m,n}(x,z) = E_0 \frac{w_0}{w} H_m(\sqrt{2}\frac{x}{w}) H_n(\sqrt{2}\frac{y}{w}) \exp(-\frac{x^2+y^2}{w^2}) \tag{16.48}$$

where it is important to remember that $w^2$ is a function of $z$. Some examples of the field and intensity variations described by Eqs. (16.47) and (16.47a) are shown in Figs. (16. )-(16. ). The phase variation

$$\Phi(m,n,z) = (m+n+1)\tan^{-1}(\frac{\lambda z}{\pi w_0^2}) \tag{16.49}$$

Fig. 16.3b.

means that the phase velocity increases with mode number so that in a resonator different transverse modes have different resonant frequencies.

(b) Cylindrically Symmetric Higher Order Beams.

In this case we try for a solution of (16.10) in the form

$$\psi = g(\frac{r}{w}) \exp\{-i(P(z) + \frac{k}{2q(z)} r^2 + \ell\phi)\} \tag{16.50}$$

and find that

$$g = (\sqrt{2}\frac{r}{w})^{\ell} \cdot L_p^{\ell}(2\frac{r^2}{w^2}) \tag{16.51}$$

where $L_p^{\ell}$ is an associated Laguerre polynomial, $p$ and $\ell$ are the radial and angular mode numbers and $P_p^{\ell}$ obeys the differential equation

$$x\frac{d^2 L_p^{\ell}}{dx^2} + (\ell + 1 - x)\frac{dL_p^{\ell}}{dx} + pL_p^{\ell} = 0 \tag{16.52}$$

Some of the associated Laguerre polynomials of low order are:

$$L_0^{\ell}(x) = 1$$

$$L_1^{\ell}(x) = \ell + 1 - x$$

$$L_2^{\ell}(x) = \frac{1}{2}(\ell + 1)(\ell + 2) - (\ell + 2)x + \frac{1}{2}x^2$$

Some examples of the field distributions for LG modes of low order are shown in Fig. (16.4). These modes are designated $\mathrm{TEM}_{\mathrm{pl}}$ modes. The $\mathrm{TEM}_{01}^*$ mode shown is frequently called the *doughnut* mode. As in the case for beams with cartesian symmetry the beam parameters $w(z)$ and $R(z)$ are the same for all cylindrical modes. The phase factor $\Phi$ for these cylindrically symmetric modes is

$$\Phi(p, \ell, z) = (2p + \ell + 1)\tan^{-1}(\frac{\lambda z}{\pi w_0^2}) \tag{16.53}$$

Fig. 16.4.

The question could be asked – why should lasers that frequently have apparent cylindrical symmetry both in construction and excitation geometry generate transverse mode with Cartesian rather than radial symmetry? The answer is that this usually results because there is indeed some feature of laser construction or method of excitation that removes an apparent equivalence of all radial directions. For example, in a laser with Brewster windows the preferred polarization orientation imposes a directional constraint that forms modes of Cartesian (HG) rather than radial (LG) symmetry. The simplest way to observe the HG modes in the laboratory is to place a thin wire inside the laser cavity. The laser will then choose to operate in a transverse mode that has a nodal line in the location of the intra-cavity obstruction. Adjustment of the laser mirrors to slightly different orientations will usually change the output transverse mode, although if the mirrors are adjusted too far from optimum the laser will go out.

## 16.4 The Transformation of a Gaussian Beam by a Lens

A lens can be used to focus a laser beam to a small spot, or systems of lenses may be used to expand the beam and recollimate it. An ideal thin lens in such an application will not change the transverse mode intensity pattern measured at the lens but it will alter the radius of curvature of the phase fronts of the beam.

We have seen that a Gaussian laser beam of whatever order is characterized by a complex beam parameter $q(z)$ which changes as the beam propagates in an isotropic, homogeneous material according to

Fig. 16.5.

$q(z) = q_0 + z$, where

$$q_0 = \frac{i\pi w_0^2}{\lambda} \quad \text{and where} \frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w^2} \tag{16.54}$$

$R(z)$ is the radius of curvature of the (approximately) spherical phase front at $z$ and is given by

$$R(z) = z\left[1 + (\frac{\pi w_0^2}{\lambda z})^2\right] \tag{16.55}$$

This Gaussian beam becomes a true spherical wave as $w \to \infty$. Now, a spherical wave changes its radius of curvature as it propagates according to $R(z) = R_0 + z$ where $R_0$ is its radius of curvature at $z = 0$. So the complex beam parameter of a Gaussian wave changes in just the same way as it propagates as does the radius of curvature of a spherical wave.

When a Gaussian beam strikes a lens the spot size, which measures the transverse width of the beam intensity distribution, is unchanged at the lens. However, the radius of curvature of its wavefront is altered in just the same way as a spherical wave. If $R_1$ and $R_2$ are the radii of curvature of the incoming and outgoing waves measured at the lens, as shown in Fig. (16.5), then as in the case of a true spherical wave

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{16.56}$$

So, for the change of the overall beam parameter, since $w$ is unchanged at the lens, we have the following relationship between the beam parameters, *measured at the lens*

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \tag{16.57}$$

If instead $q_1$ and $q_2$ are measured at distances $d_1$ and $d_2$ from the lens as shown in Fig. (16.6) then at the lens

Fig. 16.6.

$$\frac{1}{(q_2)_L} = \frac{1}{(q_1)_L} - \frac{1}{f} \tag{16.58}$$

and since $(q_1)_L = q_1 + d_1$ and $(q_2)_L = q_2 - d_2$ we have

$$\frac{1}{q_2 - d_2} = \frac{1}{q_1 + d_1} - \frac{1}{f}$$

which gives

$$q_2 = \frac{(1 - \frac{d_2}{f})q_1 + (d_1 + d_2 - \frac{d_1 d_2}{f})}{(\frac{-q_1}{f}) + (1 - \frac{d_1}{f})} \tag{16.59}$$

If the lens is placed at the beam waist of the input beam then

$$\frac{1}{(q_1)_L} = -i\frac{\lambda}{\pi w_{01}^2} \tag{16.60}$$

where $w_{01}$ is the spot size of the input beam. The beam parameter immediately after the lens is

$$\frac{1}{(q_2)_L} = -\frac{i\lambda}{\pi w_{01}^2} - \frac{1}{f} \tag{16.61}$$

which can be rewritten as

$$q_{2L} = \frac{-\pi w_{01}^2 f}{i\lambda f + \pi w_{01}^2} \tag{16.62}$$

A distance $d_2$ after the lens

$$q_2 = \frac{-\pi w_{01}^2 f}{i\lambda f + \pi w_{01}^2} + d_2 \tag{16.63}$$

which can be rearranged to give

$$\frac{1}{q_2} = \frac{(d_2 - f) + (\frac{\lambda f}{\pi w_0^2})^2 d_2 - i(\frac{\lambda f^2}{\pi w_0^2})}{(d_2 - f)^2 + (\lambda f d_2)^2} \tag{16.64}$$

The location of the new beam waist (which is where the beam will be focused to its new minimum spot size) is determined by the condition $Re(1/q_2) = 0$: namely

$$(d_2 - f) + (\frac{\lambda f}{\pi w_0^2})^2 d_2 = 0 \quad (16.65)$$

which gives

$$d_2 = \frac{f}{1 + (\frac{\lambda f}{\pi w_0^2})^2} \quad (16.66)$$

Almost always $(\frac{\lambda f}{\pi w_0^2}) \ll 1$ so the new beam waist is very close to the focal point of the lens.

Examination of the imaginary part of the RHS of Eq. (16.64) reveals that the spot size of the focused beam is

$$w_2 = \frac{f \frac{\lambda}{\pi w_0}}{[1 + (\frac{\lambda f}{\pi w_0^2})^2]^{1/2}} \approx f\theta \quad (16.67)$$

where $\theta = \frac{\lambda}{\pi w_0}$ is the half angle of divergence of the input beam.

If the lens is not placed at the beam waist of the incoming beam then the new focused spot size can be shown to be

$$w_2 \simeq f[(\frac{\lambda}{\pi w_1})^2 + \frac{w_1^2}{R_1^2}]^{1/2} \quad (16.68)$$

where $w_1$ and $R_1$, respectively, are the spot size and radius of curvature of the input beam *at the lens*. This result is exact at $d_2 = f$.

Thus, if the focusing lens is placed a great distance from, or very close to, the input beam waist then the size of the focused spot is always close to $f\theta$. If, in fact, the beam incident on the lens were a plane wave, then the finite size of the lens (radius r) would be the dominant factor in determining the size of the focused spot. We can take the "spot size" of the plane wave as approximately the radius of the lens, and from (16.67), setting $w_0 = r$, the radius of the lens we get

$$w_2 \approx \frac{\lambda f}{\pi r}. \quad (16.69)$$

This focused spot cannot be smaller than a certain size since for any lens the value of r clearly has to satisfy the condition $r \leq f$

Thus, the minimum focal spot size that can result when a plane wave is focused by a lens is

$$(w_2)_{min} \sim \frac{\lambda}{\pi} \quad (16.70)$$

We do not have an equality sign in equation (16.70) because a lens for which $r = f$ does not qualify as a thin lens, so equation (16.69) does not hold exactly.

Fig. 16.7.

We can see that from (16.67) that in order to focus a laser beam to a small spot we must either use a lens of very short focal length or a beam of small beam divergence. We cannot, however, reduce the focal length of the focusing lens indefinitely, as when the focal length does not satisfy $f_1 \ll w_0, w_1$, the lens ceases to satisfy our definition of it as a thin lens ($f \ll r$). To obtain a laser beam of small divergence we must expand and recollimate the beam; there are two simple ways of doing this:

(i) With a Galilean telecope as shown in Fig. (16.7). The expansion ratio for this arrangement is $-f_2/f_1$, where it should be noted that the focal length $f_1$ of the diverging input lens is negative. This type of arrangement has the advantage that the laser beam is not brought to a focus within the telescope, so the arrangement is very suitable for the expansion of high power laser beams. Very high power beams can cause air breakdown if brought to a focus, which considerably reduces the energy transmission through the system.

(ii) With an astronomical telecope, as shown in Fig. (16.8). The expansion ratio for this arrangement is $f_2/f_1$. The beam is brought to a focus within the telescope, which can be a disadvantage when expanding high intensity laser beams because breakdown at the common focal point can occur. (The telescope can be evacuated or filled to high pressure to help prevent such breakdown occurring). An advantage of this system is that by placing a small circular aperture at the common focal point it is possible to obtain an output beam with a smoother radial intensity profile than the input beam. The aperture should be chosen to have a radius about the same size, or slightly larger than, the spot size of the focused Gaussian beam at the focal point. This process is called spatial

Fig. 16.8.

Fig. 16.9.

filtering and is illustrated in Fig. (16.9). In both the astronomical and Galilean telescopes, spherical aberration * is reduced by the use of bispherical lenses. This distributes the focusing power over the maximum number of surfaces.

## 16.5 Transformation of Gaussian Beams by General Optical Systems

As we have seen, the complex beam parameter of a Gaussian beam is transformed by a lens in just the same way as the radius of curvature of a spherical wave. Now since the transformation of the radius of curvature of a spherical wave by an optical system with transfer matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

* The lens aberration in which rays of light travelling parallel to the axis that strike a lens at different distances from the axis are brought to different focal points.

obeys

$$R_2 = \frac{AR_1 + B}{CR_1 + D} \tag{16.71}$$

then by continuing to draw a parallel between the $q$ of a Gaussian beam and the $R$ of a spherical wave we can postulate that the transformation of the complex beam parameter obeys a similar relation, i.e.

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{16.72}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the transfer matrix for paraxial rays. We can illustrate the use of the transfer matrix in this way by following the propagation of a Gaussian beam in a lens waveguide.

## 16.6 Gaussian Beams in Lens Waveguides

In a biperiodic lens sequence containing equally spaced lenses of focal lengths $f_1$ and $f_2$ the transfer matrix for $n$ unit cells of the sequence is, from Eq. (15.3)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \frac{1}{\sin\phi} \begin{pmatrix} A\sin n\phi - \sin(n-1)\phi & B\sin n\phi \\ C\sin n\phi & D\sin n\phi - \sin(n-1)\phi \end{pmatrix} \tag{16.73}$$

so from (16.72) and writing $q_2 = q_{n+1}$, since we are interested in propagation through n unit cells of the sequence,

$$q_{n+1} = \frac{[A\sin n\phi - \sin(n-1)\phi]q_1 + B\sin n\phi}{C\sin n\phi q_1 + D\sin n\phi - \sin(n-1)\phi}. \tag{16.74}$$

The condition for stable confinement of the Gaussian beam by the lens sequence is the same as in the case of paraxial rays. This is the condition that $\phi$ remains real, i.e. $|\cos\phi| \leq 1$ where

$$\cos\phi = \tfrac{1}{2}(A + D) = \tfrac{1}{2}(1 - \frac{d}{f_1} - \frac{d}{f_2} + \frac{d^2}{2f_1f_2}) \tag{16.75}$$

which gives

$$\leq (1 - \frac{d}{2f_1})(1 - \frac{d}{2f_2}) \leq 1 \tag{16.76}$$

## 16.7 The Propagation of A Gaussian Beam in a Medium with a Quadratic Refractive Index Profile

We can most simply deduce what happens to a Gaussian beam in such a medium, whose refractive index variation is given by $n(r) = n_0 - \frac{1}{2}n_2r^2$

by using (16.72) and the transfer matrix of the medium, which from Eqs. (15.37) and (15.38) is

$$\begin{pmatrix} \cos((\frac{n_2}{n_0})^{1/2}z) & (\frac{n_0}{n_2})^{1/2}\sin((\frac{n_2}{n_0})^{1/2}z) \\ -(\frac{n_2}{n_0})^{1/2}\sin((\frac{n_2}{n_0})^{1/2}z) & \cos((\frac{n_2}{n_0})^{1/2}z) \end{pmatrix} \tag{16.77}$$

Thus if the input beam parameter is $q_0$ we have

$$q_{out}(z) = \frac{\cos(z(\frac{n_2}{n_0})^{1/2})q_{in} + (\frac{n_0}{n_2})^{1/2}\sin(z(\frac{n_2}{n_0})^{1/2})}{-(\frac{n_2}{n_0})^{1/2}\sin(z(\frac{n_2}{n_0})^{1/2})q_{in} + \cos(z(\frac{n_2}{n_0})^{1/2})} \tag{16.78}$$

where $\frac{1}{q_{in}} = \frac{1}{R_0} - \frac{i\lambda}{\pi w_0^2}$. The condition for stable propagation of this beam is that $q_{out}(z) = q_{in}$, which from (16.78) gives

$$\begin{aligned} &-(\frac{n_2}{n_0})^{1/2}\sin(z(\frac{n_2}{n_0})^{1/2})q^2 + \cos(z(\frac{n_2}{n_0})^{1/2})q = \cos(z(\frac{n_2}{n_0})^{1/2})q \\ &+(\frac{n_0}{n_2})^{1/2}\sin(z(\frac{n_2}{n_0})^{1/2}), \end{aligned} \tag{16.79}$$

which has the solution

$$q^2 = -\frac{n_0}{n_2} \text{ and } q = i(\frac{n_0}{n_2})^{1/2} \tag{16.80}$$

This implies the propagation of a Gaussian beam with planar phase fronts and constant spot size

$$w = (\frac{\lambda}{\pi})^{1/2}(\frac{n_0}{n_2})^{1/4}. \tag{16.81}$$

where, once again, we stress that $\lambda = \lambda_0/n$ is the wavelength in the medium.*

Propagation of such a Gaussian beam without spreading is clearly very desirable in the transmission of laser beams over long distances. In most modern optical communication systems laser beams are guided inside optical fibers that have a maximum refractive index on their axis. These fibers, although they may not have an exact parabolic radial index profile, achieve the same continuous refocusing effect. We shall consider the properties of such fibers in greater detail in Chapter 17.

## 16.8 The Propagation of Gaussian Beams in Media with Spatial Gain or Absorption Variations

We can describe the effect of gain or absorption on the field amplitudes

* In the relationship between beam parameter $q$ and the radius and spotsize of the Gaussian beam a uniform refactive index is assumed. In taking $n = n_o$ in a medium with a quadratic index profile we are assuming only a small variation of the index over the width of the beam. So, for example, $n_2 w^2 \ll n_o$.

of a wave propagating through an amplifying or lossy medium by using a complex propagation constant $k'$, where

$$k' = k + i\gamma/2 \tag{16.82}$$

For a wave travelling in the positive $z$ direction the variation with field amplitude with distance is $e^{-ik'z}$.

Consequently, $\gamma$ can be identified as the intensity gain coefficient of the medium and k as the effective magnitude of the wave vector, satisfying

$$k = \frac{2\pi}{\lambda} \tag{16.83}$$

where $\lambda$ is the wavelength in the medium.

An alternative way to describe the effect the medium has on the propagation of the wave is to describe the medium by a complex refractive index $n'$. This complex refractive index can be found from the relation

$$k' = n'k_0 \tag{16.84}$$

where $k_0$ is the propagation constant *in vacuo.*

Therefore,

$$n' = \frac{k}{k_0} + \frac{i\gamma}{2k_0} \tag{16.85}$$

We can recognize $k/k_0$ as a refractive index factor that describes the variation in phase velocity from one medium to another. Eq. (16.85) is then most conveniently written as

$$n' = n_0 + \frac{i\gamma}{2k_0}$$

If the Gaussian beam is sufficiently well collimated that it obeys the paraxial condition, then we can describe the effect of a length $z$ of the medium by the use of the ray transfer matrix for a length of uniform medium, namely

$$M = \begin{pmatrix} 1 & z/n' \\ 0 & 1 \end{pmatrix} \tag{16.86}$$

If the input beam parameter is $q_1$ then

$$q(z) = q_1 + z/n' \tag{16.87}$$

and if the medium begins at the beam waist then

$$q_1 = \frac{i\pi w_0^2}{\lambda} \tag{16.88}$$

After some algebra we get the following results

$$R(z) = \frac{z}{n_0}\big[1 + \big(\frac{\pi n_0 w_0^2}{\lambda z}\big)^2\big(1 - \frac{\gamma z}{n_0 w_0^2 k^2}\big)^2\big], \tag{16.89}$$

$$w^2 = w_0^2\big[1 + \big(\frac{\lambda z}{\pi n_0 w_0^2}\big)^2\big(1 - \frac{\gamma z}{n_0 w_0^2 k^2}\big)^{-1} - \frac{\gamma z}{n_0 w_0^2 k^2}\big)\big]. \tag{16.90}$$

so the radius of curvature and spot size are affected in a complex way by the presence of gain unless $\gamma z \ll n_0 w_0^2 k^2$. However, gain × length parameters as large as this are rarely, if ever, encountered in practice. For example, consider the laser with the largest measured small signal gain, $400dBm^{-1}$ at $3.5\mu m$ in the xenon laser. In this case $10log_{10}(I/I_0) = 400$ and $I/I_0 = e^{\gamma}$, giving $\gamma = 92.1m^{-1}$ For this laser a typical spot size would not be likely to be smaller than 1mm. Therefore,

$$n_0 w_0^2 k^2 = \frac{10^{-6} x 4\pi}{(3.5)^2 x 10^{-12}} \simeq 10^6$$

which is much greater than $\gamma z$ for any reasonable value of $z$.

We can conclude that the propagation of a Gaussian beam through a typical spatially uniform gain medium will not further affect the spot size or radius of curvature in a significant way. Interestingly enough, this is not true if the gain (or absorption) is spatially non-uniform. As an example, we shall consider a medium where the refractive index varies quadratically with distance from the axis.

## 16.9 Propagation in a Medium with a Parabolic Gain Profile

In this case we can write

$$k'(r) = k + \frac{1}{2} i(\gamma_0 - \gamma_2 r^2) \tag{16.91}$$

which is equivalent to including an imaginary term in the refractive index so that

$$n(r) = \frac{k + \frac{1}{2} i(\gamma_0 - \gamma_2 r^2)}{k_0} \tag{16.92}$$

giving

$$n(r) = n_0 + \frac{i\gamma_0}{2k_0} - \frac{i\gamma_2}{2k_0} r^2 \tag{16.93}$$

The transfer matrix **M** for a length z of this medium is similar to Eq. (16.77), namely

$$\mathbf{M} = \begin{pmatrix} \cos(z\sqrt{\frac{n_2}{n'}}) & \sqrt{\frac{n'}{n_2}} \sin(z\sqrt{\frac{n_2}{n'}}) \\ -\sqrt{\frac{n_2}{n'}} \sin(z\sqrt{\frac{n_2}{n'}}) & \cos(z\sqrt{\frac{n_2}{n'}}) \end{pmatrix} \tag{16.94}$$

where $n_2 = i\gamma_2/k_0, n' = n_0 + i\gamma_0/2k_0$.

The beam parameter of a Gaussian beam propagating through such a medium will be constant, as was the case for Eq. (16.79), if

$$q^2 = -n^{'}/n_2 \quad \text{and} \quad q = i\sqrt{\frac{n^{'}}{n_2}} \tag{16.95}$$

Thus

$$q = \sqrt{\frac{\gamma_2}{2n_0 k_0}}\left\{\left(1 - \frac{\gamma_0}{4k_0}\right) - i\left(1 + \frac{\gamma_0}{4k_0}\right)\right\} \tag{16.96}$$

$k_0 = 2\pi/\lambda_0$ is much larger than achievable values of $\gamma_0$, even for millimeter wave lasers, so

$$q = \sqrt{\frac{\gamma_2}{2n_0 k_0}}(1 - i) \tag{16.97}$$

which from Eq. (16.17) gives

$$R = \sqrt{\frac{2n_0 k_0}{\gamma_2}} = 2\sqrt{\frac{\pi}{\lambda\gamma_2}} \tag{16.98}$$

$$w^2 = 2\sqrt{\frac{\lambda}{\pi\gamma_2}} \tag{16.99}$$

Remember that $\lambda$ is the wavelength in the medium of refractive index $n_0$. Thus, we have a Gaussian beam which does not spread yet has spherical wavefronts.

A radial distribution of gain occurs in most gas lasers because of radial variations in the electron, or metastable density. This affects radially the rate of excitation of laser levels, either those excited directly by electrons or by transfer of energy from metastables. The radial variation is close to quadratic in lasers where the excitation is by single-collision direct-electron-impact, but where the radial electron density is parabolic as a function of radius. The radial electron density is nearly parabolic in gas discharges at moderate to high pressures where the flow of charged particles to the walls occurs by ambipolar diffusion.* Quasi-parabolic gain profiles can also result in optically-pumped cylindrical solid state lasers.

## 16.10 Gaussian Beams in Plane and Spherical Mirror Resonators

We have already mentioned that a beam-like solution of Maxwell's equations will be a satisfactory transverse mode of a plane or spherical mirror resonator provided we place the resonator mirrors at points where their

* A situation in which positive ions and electrons diffuse relative to each other at a rate set by the concentration gradient of each. The discharge usually remains electrically neutral because the charge densities of positive and negative charge are spatially equal.

Fig. 16.10.

Fig. 16.11.

radii of curvature match the radii of curvature of the phase fronts of the beam.

So for a Gaussian beam a double-concave mirror resonator matches the phase fronts as shown in Fig. (16.10) and plano-spherical and concave-convex resonators as shown in Figs. (16.11) and (16.12).

We can consider these resonators in terms of their equivalent biperiodic lens sequences as shown in Fig. (16.13). Propagation of a Gaussian beam from plane 1 to plane 3 in the biperiodic lens sequence is equivalent to one complete round trip inside the equivalent spherical mirror resonator. If the complex beam parameters at planes 1, 2 and 3 are $q_1$, $q_2$ and $q_3$, then

$$\frac{1}{q_1 + d} - \frac{1}{f_1} = \frac{1}{q_2} \tag{16.100}$$

Fig. 16.12.

Fig. 16.13.

which gives

$$q_2 = \frac{(q_1 + d)f_1}{f_1 - q_1 - d} \tag{16.101}$$

and

$$\frac{1}{q_2 + d} - \frac{1}{q_2} = \frac{1}{q_3} \tag{16.102}$$

Now if the Gaussian beam is to be a real transverse mode of the cavity then we want it to repeat itself after a complete round trip – at least as far as spot size and radius of curvature are concerned - that is we want

$$q_3 = q_1 = q. \tag{16.103}$$

## 16.11 Symmetrical Resonators

If both mirrors of the resonator have equal radii of curvature $R = 2f$ the condition that the beam be a transverse mode is

$$q_3 = q_2 = q_1 = q. \tag{16.104}$$

so from Eq. (16.100) above

$$fq - (q + d)q = (q + d)f \tag{16.105}$$

which gives

$$\frac{1}{q^2} + \frac{1}{df} + \frac{1}{fq} = 0 \tag{16.106}$$

which has roots.

$$\frac{1}{q} = \frac{-1}{2f} \pm \frac{1}{2}\sqrt{\frac{1}{f^2} - \frac{4}{fd}} = -\frac{1}{2f} \pm i\sqrt{\frac{1}{fd} - \frac{1}{4f^2}} \tag{16.107}$$

Now, $\frac{1}{q} = \frac{1}{R} - \frac{i\lambda}{\pi w^2}$, so for a real spot size, that is $w^2$ positive, we must take the negative sign above. We do not get a real spot size if the resonator is geometrically unstable ($d > 4f$).

The spot size at either of the 2 (equal curvature) mirrors is found from

$$\frac{\lambda}{\pi w^2} = \sqrt{\frac{1}{fd} - \frac{1}{4f^2}} \tag{16.108}$$

$$\frac{\lambda}{\pi w^2} = \sqrt{\frac{2}{Rd} - \frac{1}{R^2}} \tag{16.109}$$

which gives

$$w^2 = \frac{\left(\frac{\lambda R}{\pi}\right)}{\left(\sqrt{\frac{2R}{d}} - 1\right)} \tag{16.110}$$

The radius of curvature of the phase fronts at the mirrors is the same as the radii of curvature of the mirrors.

Because this resonator is symmetrical, we expect the beam waist to be in the center of the resonator, at $z = d/2$ if we measure from one of the mirrors. So since $q_0 = q + z$ and $q_0 = \frac{i\lambda}{\pi w_0^2}$ from Eq. (16.107) we get

$$w_0^2 = \frac{\lambda}{2\pi}\sqrt{d(2R - d)} \tag{16.111}$$

Note that the spot size gets larger as the mirrors become closer and closer to plane.

Now we know that a TEM mode, the $TEM_{mn}$ cartesian mode for example, propagates as

$$U_{m,n}(r,z) = \frac{w_0}{w} H_m\left(\sqrt{2}\frac{x}{w} H_m\right)\left(\sqrt{2}\frac{y}{w}\right) \exp\left\{-i(kz - \Phi) - r^2\left(\frac{1}{w^2} + \frac{ik}{2R}\right)\right\} \tag{16.112}$$

where the phase factor associated with the mode is

$$\Phi(m,n,z) = (m + n + 1)\tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right).$$

On axis, r=0, the overall phase of the mode is $(kz - \Phi)$.

In order for standing wave resonance to occur in the cavity, this phase shift from one mirror to the other must correspond to an integral number of half wavelengths, i.e.

$$kd - 2(m+n+1)\tan^{-1}(\frac{\lambda d}{2\pi w_0^2}) = (q+1)\pi \qquad (16.113)$$

where the cavity in this case holds $(q+1)$ half wavelengths. For cylindrical $TEM_{p\ell}$ modes the resonance condition would be

$$kd - 2(2p+\ell+1)\tan^{-1}(\frac{\lambda d}{2\pi w_0^2}) = (q+1)\pi \qquad (16.114)$$

If we write $\Delta\nu = c/2d$ (the frequency spacing between modes). Then since $k = 2\pi\nu/c$

$$kd = 2\pi\nu d/c = \pi\nu/\Delta\nu, \qquad (16.115)$$

we get

$$\frac{\nu}{\Delta\nu} = (q+1) + \frac{2}{\pi}(m+n+1)\tan^{-1}(\frac{1}{\sqrt{\frac{2R}{d}-1}}). \qquad (16.116)$$

By writing $2\tan^{-1}(\frac{1}{\sqrt{\frac{2R}{d}-1}}) = x$ and using $\sin^2(x/2) = \frac{1}{2}(1-\cos x)$, $\cos^2(x/2) = \frac{1}{2}(1+\cos x)$, we get the formula

$$\frac{\nu}{\Delta\nu} = (q+1) + \frac{1}{\pi}(m+n+1)\cos^{-1}(1-d/R) \qquad (16.117)$$

for the resonant frequencies of the longitudinal modes of the $TEM_{mn}$ cartesian mode in a symmetrical mirror cavity.

From inspection of the above formula we can see that the resonant frequencies of longitudinal modes of the same order (the same integer q) depend on m and n. When the cavity is plane-plane, in which case $R = \propto, \nu/\Delta\nu = (q+1)$, the familar result for a plane parallel Fabry-Perot cavity.

When d=R the cavity is said to be confocal and we have the special relations

$$w^2 = \frac{\lambda d}{\pi}, \quad w_0^2 = \frac{\lambda d}{2\pi} \qquad (16.118)$$

$$\nu/\Delta\nu = (q+1) + \frac{1}{2}(m+n+1). \qquad (16.119)$$

The beam spot size increases by a factor of $\sqrt{2}$ between the center and the mirrors.

### 16.12 An Example of Resonator Design

We consider the problem of designing a symmetrical (double concave) resonator with $R_1 = R_2 = R$ and mirror spacing $d = 1m$ for $\lambda = 488$ nm.

If the system were made confocal, $R_1 = R_2 = R = d$ then the minimum spot size would be

$$(w_0)_{confocal} = \big(\frac{\lambda d}{2\pi}\big)^{1/2} = 0.28mm$$

At the mirrors $(w_{1,2})_{confocal} = w_0\sqrt{2} \simeq 0.39mm$.

If we wanted to increase the mirror spot size to 2mm, say, we would need to use longer radius mirrors $R \gg d$ such that

$$w_{12}^2 = (\lambda R/\pi)\big(\sqrt{\frac{2R}{d} - 1}\big)^{1/2} = 4 \times 10^{-6}$$

which gives

$$\frac{w_{11}2}{(\lambda d/2\pi)^{1/2}} = (\frac{2R}{d})^{1/4}$$

and $R \simeq 346d = 346m$ (so R is $\gg d$).

Thus, it can be seen that to increase the spot size even by the small factor above we need to go to very large radius mirrors. Even in this spherical mirror resonator the $TEM_{00}$ transverse mode is still a very narrow beam.

In the general case the resonator mirrors do not have equal radii of curvature, so we need to solve (16.100) and (16.102) for $q_1 = q_3 = q$. To find the position of the mirrors relative to the position of the beam waist it is easiest to proceed directly from the equation describing the variation of the curvature of the wavefront with distance inside the resonator, i.e.

$$R(z) = z\big[1 + \big(\frac{\pi w_0^2}{\lambda z}\big)^2\big] \qquad (16.120)$$

where $z$ is measured from the beam waist. So for the general case shown in fig. (16.10)

$$R_1 = -t_1\big[1 + \big(\frac{\pi w_0^2}{\lambda t_1}\big)^2\big]$$

which gives

$$R_1 t_1 + t_1^2 + \big(\frac{\pi w_0^2}{\lambda}\big)^2 = 0 \qquad (16.121)$$

and

$$R_2 = t_2\big[1 + \big(\frac{\pi w_0^2}{\lambda t_2}\big)^2\big]$$

which gives

$$R_2 t_2 - t_2^2 - \big(\frac{\pi w_0^2}{\lambda}\big)^2 = 0 \qquad (16.122)$$

from which we get

$$t_1 = -\frac{R_1}{2} \pm \frac{1}{2}\sqrt{R_1^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2} \qquad (16.123)$$

$$t_2 = \frac{R_2}{2} \pm \frac{1}{2}\sqrt{R_2^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2}. \qquad (16.124)$$

If we choose the minimum spot size that we want in the resonator the above two equations will tell us where to place our mirrors.

If we write $d = t_2 + t_1$ then we can solve (16.123) and (16.124) for $w_0$.

If we choose the positive sign in Eqs. (16.123) and (16.124), then

$$t_1 = -\frac{R_1}{2} + \frac{1}{2}\sqrt{R_1^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2} \qquad (16.125)$$

and

$$t_2 = \frac{R_2}{2} + \frac{1}{2}\sqrt{R_2^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2} \qquad (16.126)$$

Therefore

$$d = \big(\frac{R_2 - R_1}{2}\big) + \frac{1}{2}\sqrt{R_2^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2} + \frac{1}{2}\sqrt{R_1^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2} \qquad (16.127)$$

so

$$(2d - R_2 + R_1)^2 = R_2^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2 + R_1^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2$$
$$+2\sqrt{(R_1^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2)(R_2^2 - 4\big(\frac{\pi w_0^2}{\lambda}\big)^2)} \qquad (16.128)$$

which can be solved for $w_0$ to give

$$w_0^4 = \big(\frac{\lambda}{\pi}\big)^2 \frac{d(-R_1 - d)(R_2 - d)(R_2 - R_1 - d)}{(R_2 - R_1 - 2d)^2} \qquad (16.129)$$

The spot sizes at the 2 mirrors are found from

$$w_1^4 = \big(\frac{\lambda R_1}{\pi}\big)^2 \frac{(R_2 - d)}{(-R_1 - d)} \frac{d}{(R_2 - R_1 - d)} \qquad (16.130)$$

$$w_2^4 = \big(\frac{\lambda R_2}{\pi}\big)^2 \frac{(-R_1 - d)}{(R_2 - d)} \frac{d}{(R_2 - R_1 - d)}, \qquad (16.131)$$

where we must continue to remember that for the double-concave mirror resonator we are considering, $R_1$, the radius of curvature of the Gaussian beam at mirror 1, is negative.*

* The radius of the Gaussian beam is negative for $z < 0$, positive for $z > 0$, so there is a difference in sign convention between the curvature of the Gaussian beam phase fronts and the curvature of the 2 mirrors that constitute the optical resonator.

The distances $t_1$ and $t_2$ between the beam waist and the mirrors are

$$t_1 = \frac{d(R_2 - d)}{R_2 - R_1 - 2d} \tag{16.132}$$

$$t_2 = \frac{d(R_1 + d)}{R_2 - R_1 - 2d} \tag{16.133}$$

For this resonator the resonant condition for the longitudinal modes of the $TEM_{mn}$ cartesian mode is

$$\frac{\nu}{\Delta\nu} = (q+1) + \frac{1}{\pi}(m+n+1)\cos^{-1}(\sqrt{(1+\frac{d}{R_1})(1-\frac{d}{R_2})}) \tag{16.134}$$

where the sign of the square root is taken the same as the sign of $(1 + d/R_1)$, which is the same as the sign of $(1 - d/R_2)$ for a stable resonator.

## 16.13 Diffraction Losses

In our discussions of transverse modes we have not taken into account the finite size of the resonator mirrors, so the discussion is only strictly valid for mirrors whose physical size makes them much larger than the spot size of the mode at the mirror in question. If this condition is not satisfied, some of the energy in the Gaussian beam leaks around the edge of the mirrors. This causes modes of large spot size to suffer high diffraction losses. So, since plane parallel resonators in theory have infinite spot size, these resonators suffer enormous diffraction losses.

Because higher order $TEM_{mn}$ modes have larger spot sizes than low order modes, we can prevent them from operating in a laser cavity by increasing their diffraction loss. This is done by placing an aperture between the mirrors whose size is less than the spotsize of the high order mode at the position of the aperture yet greater than the spot size of a lower order mode (usually $TEM_{00}$) which it is desired to have operate in the cavity.

The diffraction loss of a resonator is usually described in terms of its Fresnel number $a_1 a_2/\lambda d$, where $a_1, a_2$ are the radii of the mirrors and $d$ is their spacing. Very roughly the fractional diffraction loss at a mirror is $\approx w/a$. When the diffraction losses of a resonator are significant then the real field distribution of the modes can be found using Fresnel-Kirchoff diffraction theory. If the field distribution at the first mirror of the resonator is $U_{11}$, then the field $U_{21}$ on a second mirror is found from the sum of all the Huygen's secondary wavelets originating on the first mirror as:

$$U_{21}(x, y) = \frac{ik}{4\pi}\int_1 U_{11}(x, y)\frac{e^{-ikr}}{r}(1 + \cos\theta)dS \tag{16.135}$$

The integral is taken over the surface of mirror 1: r is the distance of a point on mirror 1 to the point of interest on mirror 2. $\theta$ is the angle which this direction makes with the axis of the system, $dS$ is an element of area. Having found $U_{21}(x, y)$ by the above method we can recalculate the field on the first mirror as

$$U_{12}(x, y) = \frac{ik}{4\pi}\int_2 U_{21}(x, y)\frac{e^{ikr}}{r}(1 + \cos\theta)dS \tag{16.136}$$

and if we require $U_{12}(x, y)$ to be equivalent to $U_{11}(x, y)$, apart from an arbitrary phase factor, then our transverse field distribution becomes self consistent and is a solution for the transverse mode distribution of a resonator with finite aperture mirrors. This is the approach that was first used (by Fox and Li) to analyze the transverse field distributions that would result in optical resonators.

## 16.14 Unstable Resonators

Many high power lasers have sufficiently large gain that they do not require the amount of feedback provided by a stable optical resonator with high reflectance mirrors. To achieve maximum power output from such lasers, one resonator mirror must have relatively low reflectance and the laser beam should have a large enough spot size so as to extract energy from a large volume of the medium. However, as the spot size of the oscillating transverse mode becomes larger, an increasing portion of the energy lost from the cavity "leaks" out past the edge of the mirrors. As the configuration of the cavity approaches the unstable region in Fig. (   ) the spot size grows infinitely large. At the same time, the divergence of the output beam falls, making focussing and control of the output easier. Operation of the resonator in the unstable region provides efficient energy extraction from a high gain laser, and provides a low divergence beam of generally quite high transverse mode quality.

Schawlow first suggested the use of an unstable resonator, with its associated intracavity beam defocussing properties, to counteract the tendency of some lasers to focus internally into high intensity filaments. This property of unstable resonators is valuable in reducing the occurrence of "hot spots" in the output beam. These can result if a high gain laser with spatial variations in its refractive index profile (produced,

Fig. 16.14.

for example, by nonhomogeneous pumping) is operated in a stable resonator arrangement. A simple unstable resonator configuration that accomplishes this is shown in Fig. (16.14). The output laser beam will have a characteristic "doughnut" shaped intensity pattern in the near field.

The loss per pass of an unstable resonator can be determined approximately by a simple geometric optics analysis first presented by Siegman. The geometry used in the analysis is shown in Fig. (16.15). The wave originating from mirror $M_1$ is assumed to be a spherical wave whose center of curvature is at point $P_1$ (not in general the same point as the center of curvature of mirror $M_1$). The right-travelling wave is partially reflected at mirror $M_2$, and is now assumed to be a spherical wave that has apparently originated at point $P_2$. This wave now reflects from M, again, as if it had come from point $P_1$ and another round-trip cycle commences. The positions of $P_1$ and $P_2$ must be self-consistent for this to occur, each must be the virtual image of the other in the appropriate spherical mirror. The distances of $P_1$ and $P_2$ from $M_1$ and $M_2$ are written as $r_1L, r_2L$, respectively, where L is the resonator length. $r_1$ and $r_2$ are measured in the positive sense in the direction of the arrows in Fig. (16.15). If, for example $P_2$ were to the left of $M_2$, then $r_2$ would be negative. We also use our previous sign convention that the radii of curvature of $M_1$, and $M_2$, $R_1, R_2$, respectively, are both negative.

From the fundamental lens equation we have

$$\frac{1}{r_1} - \frac{1}{r_2+1} = -\frac{2L}{R_1} = 2(g_1 - 1) \tag{16.137}$$

$$\frac{1}{r_2} - \frac{1}{r_1+1} = -\frac{2L}{R_2} = 2(g_2 - 1) \tag{16.138}$$

where $g_1$ and $g_2$ are dimensionless parameters.

Fig. 16.15.

So,

$$r_1 = \pm \frac{\sqrt{1 - 1/(g_1 g_2)} - 1 + 1/g_1}{2 - 1/g_1 - 1/g_2} \tag{16.139}$$

$$r_2 = \pm \frac{\sqrt{1 - 1/(g_1 g_2)} - 1 + 1/g_2}{2 - 1/g_1 - 1/g_2} \tag{16.140}$$

These equations determine the location of the virtual centers of the *spherical* waves in the resonator.

A wave of unit intensity that leaves mirror $M_1$ will partially spill around the edge of mirror $M_2$ – this is sometimes loosely referred to as a *diffraction loss*, but it is not. A fraction $\Gamma_1$ of the energy is reflected at $M_1$ so a fraction $(1 - \Gamma_1)$ is lost from the resonator. A fraction $\Gamma_2$ of this reflected wave reflects from mirror $M_2$. Thus, after one complete round trip our initial intensity has been reduced to $\Gamma^2 \equiv \Gamma_1 \Gamma_2$.

We can get some feeling for how the loss per round trip actually depends on the geometry of the resonator by considering the resonator in Fig. (16.15) as a purely two-dimensional one. This is equivalent to having infinitely long strip-shaped mirrors that only curve in the lateral direction.

From the geometry of Fig. (16.15) it is easy to see that

$$\Gamma_1 = \frac{r_1 a_2}{(r_1 + 1) a_1} \tag{16.141}$$

$$\Gamma_2 = \frac{r_2 a_1}{(r_2 + 1) a_2} \tag{16.142}$$

Thus,

$$\Gamma_{2D}^2 = \Gamma_1 \Gamma_2 = \pm \frac{r_1 r_2}{(r_1 + 1)(r_2 + 1)} \tag{16.143}$$

Surprisingly, the round trip energy lost is *independent* of the dimensions of the 2 mirrors.

For mirrors that curve in two dimensions the loss per round trip is $\Gamma_{3D} = \Gamma_{2D}^2$ Eq. (16.143) can be rewritten

$$\Gamma_{2D}^2 = \pm \frac{1 - \sqrt{1 - 1/(g_1 g_2)}}{1 + \sqrt{1 - 1/(g_1 g_2)}} \tag{16.144}$$

the appropriate sign is chosen so that $\Gamma_{2D}^2 > 0$. This result does not include any real diffraction effects, which can be important if one or both mirrors become sufficiently small relative to the laser wavelength that the Fresnel number, $N_F = (a_1 a_2)/L\lambda$, becomes small. For $a_1 = a_2$, Eq. (16.144) becomes unreliable for $N_F < 2$.

## 16.15 Some More Ray Theory for Cylindrical Fibers

Our analysis of cylindrical optical fibers by means of simple ray theory has so far considered only the paths of meridional rays, those rays whose trajectories lie within the plane containing the fiber axis. We have also tacitly assumed that Snell's Law strictly delineates between those rays that totally internally reflect, and are *bound* within the fiber, and those that escape that *refract* into the cladding. However, if we consider the trajectories of skew rays we shall see thatthe third class of rays, so-called *tunneling* or *leaky* rays exist. These are rays that appear at first glance to satisfy the Snell's Law requirement for total internal reflection. However, because the refraction occurs at a curved surface they can in certain circumstances leak propagating energy into the cladding. In our desription of the phenomenon we follow the essential arguments and notation given by Snyder and Love.

Fig. (16.16) shows the paths of skew ray in step-index and graded under fibers projected onto a plane perpendicular to the fiber axis. The path of a particular skew ray is characterized by the angle $\theta_z$ it makes with the $Z$-axis and the angle $\theta_\phi$ that it makes with the azimuthal direction.

In a step-index fiber these angles are the same at all points along the ray path, in a graded index fiber they vary with the radial distance of the ray from the fiber axis. The projected ray paths of skew rays are bounded, for a particular family of skew rays by the inner and outer *caustic* surfaces.

### *16.15.1 Step-Index Fibers*

In a step-index fiber the outer caustic surface is the boundary between

Fig. (16.16). (a) Projected path of a skew ray in a step-index cylindrical fiber, (b) projected path of a skew ray in a graded index cylindrical fiber, (c) projected path of a skew ray in a step-index fiber showing the radial dependence of the azimuthal angle $\theta_\phi(v)$.

core and cladding. Fig. (16.16) shows clearly that the radius of the inner caustic in a step-index fiber is

$$v_{ic} = \rho \cos \theta_\phi$$

For meridional rays $\theta_\phi = \pi/2$ and $r_{ic} = 0$.

When a skew ray strikes the boundary between core and cladding it makes an angle $\alpha$ with the surface normal where

$$\cos \alpha = \sin \theta_z \sin \theta_\phi$$

Simple application of Snell's Law predicts that total internal reflection should occur if

$$\sin \alpha > \frac{n_{\text{cladding}}}{n_{\text{core}}}$$

We introduce a new angle $\theta_c$, related to the critical angle $\alpha_c$ by $\theta_c + \alpha_c = 90^\circ$

$$\leq \overline{\beta}^2 + \overline{\ell}^2 < n_{\text{cladding}}^2 \qquad \text{Refracting Rays}$$

$$n_{\text{cladding}}^2 < \overline{\beta}^2 + \overline{\ell}^2 \leq n_{\text{core}}^2 \text{ and } 0 \leq \overline{\beta} < n_{\text{cladding}} \qquad \text{Tunneling Rays}$$

An important consequence of the ray invariants is that the length of time it takes light to travel a distance $L$ along the fiber is independent of skewness. From Fig. (16.16) it is easy to see that the distance from $P$ to $Q$ is

$$L_p = 2\rho \frac{\sin \theta_\phi}{\sin \theta_z} = \frac{2\rho n_{\text{core}}^2 - \overline{\beta}^2 - \overline{\ell}^2)^{1/2}}{n_{\text{core}}^2 - \overline{\beta}^2}$$

The corresponding distance traveled along in the axial direction is

$$L = \frac{L_p}{\cos \theta_z}$$

and the time taken is

$$\tau = \frac{L_p n_{\text{core}}}{c_0 \cos\theta_z}$$

The time taken per unit distance traveled is

$$\frac{\tau}{L} = \frac{n_{\text{core}}}{c_0 \cos\theta_z},$$

which clearly depends only on axial angle $\theta$, and does not depend on skew angle $\theta_\phi$. This has important consequences for optical pulse propagation along a fiber. In this ray model an optical pulse injected at angle $\theta_z$ will take a single ??????.

Rays can be classified according to the values of these various angles:

$$\theta \le \theta_z < \theta_c \qquad \text{Bound rays}$$

$$0 \le \alpha \le \alpha_c \qquad \text{Refracting rays}$$

$$\theta_c \le \theta_z \le 90°; \alpha_c \le \alpha \le 90° \qquad \text{Tunneling rays}$$

It is the existence of this third class of rays that is somewhat surprising[16.1] In order to explain the existence of such rays we need to consider various qualities that remain constant, or *invariant* for rays in a step-index fiber. These invariants follow for simple geometric reasons and are:

$$\overline{\beta} = n_{\text{core}} \cos\theta_z$$

$$\overline{\ell} = n_{\text{core}} \sin\theta_z \cos\theta_\phi$$

It follows also from Eq. (16. ) that

$$\overline{\beta}^2 + \overline{\ell}^2 = n_{\text{core}}^2 \sin^2\alpha$$

For meridional rays $\overline{\ell} = 0$, whereas for skew rays $\overline{\ell} > 0$. In terms of these ray invariants the classification of rays in a step-index fiber becomes:

$$n_{\text{cladding}} < \overline{\beta} \le n_{\text{core}} \qquad \text{Bound rays}$$

fixed time to travel along the fiber, independent of whether it travels as a meridional or skew ray.

### *16.15.2 Graded Index Fibers*

To discover the important ray invariants cylindrically in a symmetric graded index fiber we use Eq. (16. )

$$\frac{d}{ds}\left[n(\mathbf{r}\frac{d\mathbf{r}}{ds}\right] = \text{grad } n(\mathbf{r})$$

If we transform to cylindrical coordinates, for the case where $n$ only varies radially

$$\frac{d}{ds}\left[n(r)\frac{dr}{ds}\right] - rn(r)\left(\frac{d\phi}{ds}\right)^2 = \frac{dn(v)}{dr} \qquad (a)$$

$$\frac{d}{ds}\left[n(r)\frac{d\phi}{ds}\right]+\frac{2n(r)}{r}\frac{d\phi}{ds}\frac{dr}{ds}=0 \qquad (b)$$

$$\frac{d}{ds}\left[n(r)\frac{dz}{ds}\right]=0 \qquad (c)$$

The path of a ray in projection in this case is shown in Fig. (16.16). At each point along the ray path the tangent to the ray path makes an angle $\theta_z(v)$ with the fiber axis.

In the projected path shown in Fig. (16.16) the project tangent to the ray path makes an angle with the radius of (90-$\theta_\phi(v)$).

As before, from Eq. (16. )

$$\cos\alpha(r)=\sin\theta_z(v)\sin\theta_\phi(v).$$

Various other relations follow

$$\frac{dr}{ds}=\cos\alpha(v)$$

$$\frac{dz}{ds}=\cos\theta_z(r),$$

and because $ds^2=dv^2+r^2d\phi^2+dz^2$ it also follows that

$$\frac{d\phi}{ds}=\frac{1}{4}\sin\theta_z(v)\cos\theta_\phi(r).$$

Therefore, we can integrate Eq. (16. )(c) to five

$$\overline{\beta}=n(r)\frac{dz}{ds}=n(v)\cos\theta_z(r), \qquad (d)$$

where $\overline{\beta}$ is a ray invariant.

Integration of Eq. (16. )(b) gives

$$\overline{\ell}=\frac{r^2}{\rho}n(v)\frac{d\phi}{ds}=\frac{r}{\rho}n(v)\sin\theta_z(v)\cos\theta_\phi(v), \qquad (c)$$

where we have introduced a normalizing radius $\rho$ so that $\overline{\ell}$ remains dimensionless. This could be a reference radius, or more usually the boundary between core and cladding when only the case has a graded index.

If we eliminate $\theta_z$ from Eqs. (16. ) and (16. ) we get

$$\cos\theta_\phi(r)=\frac{\rho}{r}\frac{\overline{\ell}}{[n^2(r)-\overline{\beta}^2]^{1/2}}$$

Now, as shown schematically in Fig. (16. 16) skew rays take a helical path in the core, but not one whose cross-section is necessarily circular. A particular family of skew rays will not come closer to the axis than the inner caustic and will not go further from the axis than the outer caustic. Inspection of Fig. (16.16) reveals that on each caustic $\theta_\phi(r)=0$. Consequently, the radii of the inner and outer caustic, $r_{ic}, r_{tp}$, respectively, are the roots of

$$g(v) = n^2(r) - \overline{\beta}^2 - \overline{\ell}^2 \frac{\rho^2}{r^2} = 0$$

*16.15.3 Bound, Refracting, and Tunneling Rays*

We are now in a position to explain mathematically the distinction between these three classes of rays. From Eq. (16. )(d) we can write

$$\frac{dz}{ds} = \frac{\overline{\beta}}{n(r)}$$

and from Eq. (16. )(c)

$$\frac{d\phi}{ds} = \frac{\rho\overline{\ell}}{r^2 n(v)}$$

Subsitution from Eq. (16. ) into Eq. (16. )(a), noting that

$$\frac{d}{ds} \equiv \frac{dz}{ds}\frac{d}{dz},$$

gives

$$\overline{\beta}^2 \frac{d^2 r}{dz^2} - \overline{\ell}^2 \frac{\rho^2}{r^3} = \frac{1}{2}\frac{dn^2(r)}{dr}$$

Writing $d^2r/dz^2 = r' dv'/dv$ where $r' = dr/dz$ gives

$$\overline{\beta}^2 r' \frac{dr'}{dr} - \overline{\ell}^2 \frac{\ell^2}{r^3} = \frac{1}{2}\frac{dn^2(v)}{dr},$$

which can be integrated to give

$$\frac{1}{2}\overline{\beta}^2 (r')^2 + \frac{1}{2}\frac{\overline{\ell}^2 \rho^2}{r^2} = \frac{1}{2} n^2(v) + \text{ constant}$$

Using Eq. (16. ) allows Eq. (16. ) to be written as

$$\overline{\beta}^2 \left(\frac{dv}{dz}\right)^2 = g(v) + \overline{\beta}^2 + \text{ constant.}$$

At the outer caustic surface $(dr/dz)_{V_{tp}} = 0$, and also $g(r_{tp} = 0$, so the constant of integration in Eq. (16. ) is $-\overline{\beta}^2)$. Finally, therefore we have

$$\overline{\beta}^2 \left(\frac{dr}{dz}\right)^2 = g(v) = n^2(v) - \overline{\beta}^2 - \overline{\ell}^2 \frac{\rho^2}{r^2}$$

Ray paths can only exist if $g(v)$ is non-negative.

The roots of $g(r)$ determine the location of the inner and outer caustic surfaces. For a refracting ray there is no outer caustic surface: the ray reaches the core boundary (or the outer surface of the fiber). If we assume for simplicity that $n(v)$ increases monotonically from the axis this requires $g(\rho) > 0$. From Eq. (16. ) this requires

$$n^2_{\text{cladding}} > \overline{\beta}^2 + \overline{\ell}^2.$$

There is only an inner caustic in this case, as showin in Fig. (16.16)(c).

For a ray to be bound $g(v) < 0$ everywhere in the cladding, which requires

$$n_{\text{cladding}} < \overline{\beta},$$

which additional requires, not surprisingly, that

$$n_{\text{cladding}} < \overline{\beta} \leq n_{\text{core}}.$$

Both meridional and skew ray bound rays exist. For the meridional rays $\overline{\ell} = 0$ and $g(v)$ has only a single root, corresponding to $r_{tp}$ where

$$n(r_{tp}) = \overline{\beta}.$$

For the skew bound rays there are two roots of $g(v)$ corresponding to the inner and outer caustic surfaces, as shown in Fig. (16.16)(b).

Tunneling rays fall into a different category where

$$\overline{\beta} < n_{\text{cladding}} \text{ and } n^2_{\text{cladding}} < \overline{\beta}^2 + \overline{\ell}^2.$$

The behavior of $g(v)$ in this case is such that between $r_{tp}$ and an outer radius $r_{\text{rad}} g(r)$ is negative and no ray paths exist. However, for $r > r_{\text{rad}} g(v)$ is non-negative. The ray "tunnels" from radius $r_{tp}$ to $r_{\text{rad}}$, where it becomes a free ray. Since $g(r_{\text{rad}}) = 0$, it follows that

$$r_{\text{rad}} = \frac{\rho\overline{\ell}}{(n^2_{\text{cladding}} - \overline{\beta}^2)^{1/2}}$$

The mechanism by which energy leads from the core to the outer part of the fiber is similar to the coupling of energy between two closely sspaced surfaces, at one of which total internal reflection occurs, as discussed in Chapter 14.

Depending on the circumstances, tunneling or "leaky" rays may correspond to a very gradual loss of energy from the fiber and the fiber can guide energy in such rays over relatively long distances.

## 16.16 Problem for Chapter 16

Compute and plot the fraction of laser power in a $\text{TEM}_{00}$ mode of spot size $w_0$ that passes through an aperture of radius $a$ as a formation of the ratio $a/w_0$.

# References

A.W. Snyder, “Leaky-ray theory of optical waveguides of cricular cross-section,” Appl. Phys., 4, 2730298, 1974.