(1) For zero reflectance of the P-wave the impedance must match on both sides of the boundary

\[ Z_1 \cos(\theta_1) = Z_2 \cos(\theta_2) \]

which gives

\[ n_2 \cos(\theta_1) = n_1 \cos(\theta_2) \quad \text{and} \]

\[ \left[ \cos(\theta_2) \right]^2 = 1 - \sin(\theta_2)^2 = \left( \frac{n_2}{n_1} \right)^2 \left( 1 - \sin(\theta_1)^2 \right) \]

Substituting from Snell's law

\[ n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \]

\[ 1 - \left( \frac{n_1}{n_2} \right)^2 \sin(\theta_1)^2 = \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 \sin(\theta_1)^2 \]

which gives

\[ \sin(\theta_1)^2 = \frac{1 - \left( \frac{n_2}{n_1} \right)^2}{\left( \frac{n_1}{n_2} \right)^2 - \left( \frac{n_2}{n_1} \right)^2} \]

and

\[ \sin(\theta_1) = \frac{n_2}{\sqrt{n_1^2 + n_2^2}} \quad \text{and finally} \quad \tan(\theta_1) = \frac{n_2}{n_1} \]

Applications include Brewster windows in lasers, and stacked plate polarizers

For the prism use impedances for S-waves

\[ Z_1 = \frac{Z_0}{\cos(\theta_B)} \quad Z_2 = \frac{Z_0}{n \cos(\theta_{\text{int}})} \]

\[ \rho = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{n \cos(\theta_{\text{int}}) \cdot \cos(\theta_B)}{n \cos(\theta_{\text{int}}) + n \cos(\theta_B)} \quad \text{Find } \theta_{\text{int}} \text{ from Snell's Law} \]

n := 1.6

\[ \theta_B := \arctan(n) \quad \theta_B = 1.012 \quad \text{rad} \quad \frac{\theta_B}{\text{deg}} = 57.995 \]

\[ \theta_{\text{int}} := \arcsin \left( \frac{\sin(\theta_B)}{n} \right) \quad \theta_{\text{int}} = 0.559 \quad \text{rad} \quad \frac{\theta_{\text{int}}}{\text{deg}} = 32.005 \]
\[ \rho := \frac{\cos(\theta_B) - n \cdot \cos(\theta_{\text{int}})}{\cos(\theta_B) + n \cdot \cos(\theta_{\text{int}})} \quad \rho = -0.438 \]

\[ T_1 := 1 - (|\rho|)^2 \]

\[ T_1 = 0.808 \quad \text{Transmittance of one face, but the geometry at both faces is identical, so} \]

\[ T_S := T_1^2 \quad \text{Total S-wave transmittance} \]

\[ T_S = 0.653 \quad 65.3\% \text{ transmittance of S-wave energy} \]

\[ T_{\text{overall}} := \frac{1 + T_S}{2} \quad T_{\text{overall}} = 0.826 \quad 82.6\% \text{ of total energy through} \]

An application is in laser tuning, since only one wavelength goes through without loss (refractive index varies with wavelength.)

(2) \( R \) is the radius of curvature of the phase front, \( w \) is the spotsize, which is the \( 1/e^2 \) radius of the intensity distribution of the beam

An optical system affects the Gaussian beam according to

\[ q_{\text{out}} = \frac{A \cdot q_{\text{in}} + B}{C \cdot q_{\text{in}} + D} \]

There is a close analogy with spherical waves for which

\[ R_{\text{out}} = \frac{A \cdot R_{\text{in}} + B}{C \cdot R_{\text{in}} + D} \]

A uniform medium with \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \) gives \( q_{\text{out}} = q_{\text{in}} + d \)

which agrees with the fundamental way in which a Gaussian beam changes with distance

A thin lens does not change the spotsize, only the radius of curvature, so with

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad q_{\text{out}} = \frac{q_{\text{in}}}{1 - \frac{q_{\text{in}}}{f}} \quad \frac{1}{q_{\text{out}}} = \frac{1}{q_{\text{in}}} - \frac{1}{f} \]

This is the same as \( \frac{1}{R_{\text{out}}} = \frac{1}{R_{\text{in}}} - \frac{1}{f} \) the same result as for a spherical wave

\[ \lambda_0 := 1.3 \cdot 10^{-6} \quad w := 10^{-2} \quad R := 10 \]

\[ q_{\text{in}} := \left( \frac{1}{R} - \frac{i \cdot \lambda_0}{\pi \cdot w^2} \right)^{-1} \]

\[ q_{\text{in}} = 9.983 + 0.413i \]
Clearly the beam waist is at \( z = -9.983 \text{m} \)

At the beam waist \( q_0 = 0.413 \),

\[
q_0 = \frac{i \pi w_0^2}{\lambda_0} \quad w_0 := \sqrt{\frac{0.413 \lambda_0}{\pi}}
\]

\[
w_0 = 4.134 \times 10^{-4} \quad \text{Minimum spotsize is 0.4134mm}
\]

Since the slab of glass is after the beam waist, it does not change the location of the beam waist.

\( (3) \) If you "unfold" the system the two lenses together look like a single lens of focal length \( f/2 \), which makes the focal length of the lenses the same as that of the mirror. So this is a symmetrical resonator, the beam waist is in the center.

The distance from the beam waist to the spherical mirror is \( z := 0.5 \),

\[
\lambda_0 := 633 \times 10^{-9} \quad R := 4
\]

\[
R = z + \left( \frac{\pi}{\lambda_0} \right)^2 \frac{w_0^4}{z}
\]

\[
w_0 := \left[ \frac{\lambda_0}{\pi} \cdot z (R - z) \right]^4
\]

\[
w_0 = 5.163 \times 10^{-4} \quad \text{spotsize is 0.5163mm}
\]

The resonator becomes unstable if the minimum spotsize shrinks to zero. This occurs when \( R = z \).

Maximum stable spacing is \( d = 2z = 2R \),

Maximum stable spacing is 8m.

\( (4) \) For a cylindrically symmetric, paraxial situation you can rewrite the equation of light rays as

\[
\frac{d}{dz} \left( nd (r \cdot \rho + z \cdot k) \right) = \frac{dn}{dr} \quad \text{where } z \text{ is distance along the axis and } r \text{ is radial position, } \rho \text{ is a radial unit vector, } k \text{ is an axial unit vector}
\]

taking the derivatives gives
\[ \frac{nd^2r}{dz^2} = \frac{dn}{dr} \]

with \( n(r) = n_0 e^{\frac{-r^2}{\sigma^2}} \) this gives

\[ \frac{d^2r}{dz^2} = -2\frac{r}{\sigma^2} \]

General solution is

\[ r = A \cos\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) + B \sin\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) \]

If input ray parameters are \( r_{in}, r'_{in} \), then clearly \( A = r_{in} \)

\[ B \frac{\sqrt{2}}{\sigma} = r'_{in} \]

Ray transfer matrix is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\cos\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) & \frac{\sigma}{\sqrt{2}} \sin\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) \\
\frac{-\sqrt{2}}{\sigma} \sin\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) & \cos\left(\frac{\sqrt{2}}{\sigma} \cdot z\right)
\end{pmatrix}
\]

\[ n = n_0 e^{\frac{-r^2}{\sigma^2}} \]

\[ \ln\left(\frac{n_0}{n}\right) = \frac{2}{\sigma^2} \]

\[ r := 10^{-3} \quad n_0 := 1.46 \quad n := 1.455 \quad z := 50 \cdot 10^{-3} \]

\( \sigma := \frac{r}{\sqrt{\ln\left(\frac{n_0}{n}\right)}} \]

\[ \sigma = 0.017 \quad n_0 e^{\frac{-r^2}{\sigma^2}} = 1.455 \quad \text{Check} \]

\[ C := \frac{-\sqrt{2}}{\sigma} \sin\left(\frac{\sqrt{2}}{\sigma} \cdot z\right) \]

\[ f := -\frac{1}{C} \]

\[ f = -0.014 \quad \text{focal length is -14mm} \]

For medium to have no effect the matrix must become an identity matrix which implies

\[ \frac{\sqrt{2}}{\sigma} \cdot z = 2 \cdot \pi \]

\[ z := \frac{2 \cdot \pi \cdot \sigma}{\sqrt{2}} \]

\[ z = 0.076 \quad 76 \text{mm} \]