ENEE 381 Problem Set #1
9/9/04 - due 9/16/04

(1) (380 Review) The current density in a certain region is

\[ J = 0.1e^{-10^6t} \frac{\dot{r}}{r} \]

in spherical coordinates. At t=1μs how much current is crossing the surface r=5?

(2) (380 review) A current density \( \mathbf{J} = 5\mathbf{j} \text{A/m}^2 \) exists wherever \( |z| < 2 \text{m} \). (a) Find \( \mathbf{H} \) for \( |z| < 2 \) and \( |z| > 2 \). Find the magnetic vector potential \( \mathbf{A} \) for \( |z| < 2 \) if \( \mathbf{A} = 0 \) at the origin.

(3) A circular coil of 100 turns of radius 50mm, total resistance 1 ohm, and no self inductance is rotated about a vertical diameter with uniform angular velocity 100 rad/s in a horizontal magnetic flux of 0.2 Tesla. Calculate the average power needed to keep the coil in motion.

The mean power required to keep the coil in motion is

\[ W = \pi^2 \pi^2 a^4 b^2 \omega^2 / (2R) \]

What is the ohmic power dissipated in the coil?

(4) A small magnetic needle, which is free to turn slowly in a horizontal plane, is placed at the center of the coil in question (3). Calculate the angle with respect to \( \mathbf{B} \) at which it reaches equilibrium.

Show that it will set at an angle \( \phi \) to \( \mathbf{B} \) where

\[ \cot \phi = 4R / (\pi n^2 \mu \omega a) \]

(5) A charged particle starts from rest at the origin of coordinates in a region where there is a uniform electric field \( \mathbf{E} \) parallel to the \( x \)-axis, and a uniform magnetic flux density \( \mathbf{B} \) parallel to the \( z \)-axis. Show that the coordinates of the particle at a time \( t \) later will be

\[ x = \left( \frac{E}{\omega B} \right)(1 - \cos(\omega t)), \]
\[ y = -\left( \frac{E}{\omega B} \right)(\omega t - \sin(\omega t)), \]
\[ z = 0, \]

where \( \omega = eB/m \). \( (E = |\mathbf{E}|, B = |\mathbf{B}|) \) (This path is called a cycloid.)

(6) Electrons are liberated with zero velocity from the negative plate of a parallel plate capacitor, to which is applied a magnetic flux density \( \mathbf{B} \) parallel to the plates. Prove that these electrons will not reach the positive plate if the plate separation \( d \) is greater than \( 2mE/eB^2 \), where \( E = |\mathbf{E}| \) is the field between the plates.

(7) A plane circular disk of radius \( \alpha \) rotates at a speed of \( 2\pi f \text{ rad/s} \) about an axis through the center of the disk perpendicular to the plane of the disk. There is a uniform magnetic flux \( \mathbf{B} \) parallel to the axis of rotation. Prove that the \( emf \) between the center of the disk and its rim is of magnitude \( V = fB\pi a^2 \). \( (B = |\mathbf{B}|) \)
Solutions

1) \( \mathbf{J} = \frac{0.1 e^{-1}}{r} \)

At \( t = 1 \mu s \)
\[ \mathbf{J} = \frac{0.1 e^{-1}}{r} \]

\( \mathbf{J} \) is independent of \( \theta \) or \( \phi \). Therefore at \( r = 5 \)
\[ \mathbf{J} = 4\pi r^2 \left( \frac{0.1 e^{-1}}{r} \right) = 4\pi r \times 0.1 e^{-1} = 2.311 A \]

2) \( \mathbf{J} = \frac{5}{2} \hat{A} / m^2 \) for \( |z| < 2 \)

(a) The field from a surface current \( \mathbf{K} = \mathbf{H} = \frac{1}{2} \mathbf{k} \times \hat{\mathbf{n}} \),
where \( \hat{\mathbf{n}} \) is a surface normal.

For \( |z| > 2 \) the observation point is on one side or the other of the region of current flow. Therefore the effective
\( \mathbf{K} = 20 \frac{j}{2} \hat{A} / m \)

For \( z > 2 \) \( \mathbf{H} = 10 \hat{j} \times \hat{k} = 10 \hat{z} \)
For \( z < 2 \) \( \mathbf{H} = 10 \hat{j} \times (-\hat{k}) = -10 \hat{z} \)
If observation point has \( |z| < 2 \), then it is inside the current region.

At point \((x, y, z)\) the surface current above the point is \(5(2-z) A/m\), the surface current below is \(5(2+z) A/m\). The overall field is \(\frac{5}{2}(2-z)^2 + \frac{5}{2}(2+z)^2 = 5z^2\).

\[\text{For } |z| < 2 \quad H = \frac{5z}{2} \hat{z} \quad B = 5\mu z \hat{z}\]

\[B = \text{curl } A \quad \Rightarrow \quad B_{\hat{z}} = (\text{curl } A)_{\hat{z}}\]

Therefore \(\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right) \hat{z} = 5\mu z \hat{z}\)

Because \(\mathbf{j}\) is in the y direction, \(A_x\) is only in the y direction, and only \(A_y\) varies with \(z\).

\[\frac{\partial A_y}{\partial z} = -5\mu z\]

\[A_y = -\frac{5\mu z^2}{2}\]
the angle $\theta$ with $B$ is

$$\Phi = \frac{\pi a^2 NB \sin \theta}{\text{turns}}$$

$$\frac{d\Phi}{dt} = -\pi a^2 NB \cos \theta \frac{d\theta}{dt} = -\pi a^2 NB w \cos \theta$$

Now $\frac{d\Phi}{dt} = -\oint E \cdot dL = V$ potential difference across coil

Therefore $V = \pi a^2 NB w \cos \theta$

Current in coil $I = \frac{V}{R} = \frac{\pi a^2 NB w \cos \theta}{R}$

Ohmic heating $= VI = \frac{N^2 \pi^2 a^4 B^2 w^2 \cos \theta}{R}$

$$= \frac{N^2 \pi^2 a^4 B^2 w^2 (1 + \cos^2 \theta)}{2R}$$

Average ohmic heating $= \frac{N^2 \pi^2 a^4 B^2 w^2}{2R}$
The current in the coil produces a magnetic dipole in that so oriented as shown — according to Lenz's law it acts to oppose the change in external flux:

\[ M = \mu I N A \]

\[ \text{area of coil} \]

The torque that acts is:

\[ \Gamma = M \times H = M \times \frac{B}{\mu} \]

Therefore:

\[ \Gamma = \mu \pi a^2 N. \frac{\pi a^2 NB \cos \theta}{R} \frac{R}{\mu} \sin (90 - \theta) \]

\[ = \frac{\pi a^4 N^2 B^2 w \cos \theta}{R} \]

The average work done against this torque in one revolution is:

\[ \bar{\Gamma} = \frac{2\pi \Gamma}{2\pi} \]

Work done per second is:

\[ 2\pi \bar{\Gamma} \cdot \frac{w}{2\pi} \frac{\text{W}}{\text{R}} \frac{\text{No of revolutions}}{\text{per second}} \]

\[ \text{Power} = \frac{\pi a^4 N^2 B^2 w}{2R} \]

For \( N = 100 \), \( a = 50 \text{ mm} \), \( B = 0.2 \text{ T} \), \( w = 100 \text{ rad/s} \), \( R = 1 \)

\[ \text{Power} = \text{ohmic heating} = 2467 \text{ Watts} \]
Two magnetic fields act on the small magnetic needle - the external field $H$ and the field that results from the current induced in the coil, $H_i$.

$$H_i = \frac{NI}{2a} = \frac{\pi a N B \cos \theta}{2R}$$

(field at center of circular coil)

$$B_i = \mu H_i = \frac{\mu \pi a N^2 \omega B \cos \theta}{2R} = k B \cos \theta$$

($k = \frac{\mu \pi a N \omega}{2R}$)

Two torque act on the magnetic needle.

In equilibrium

$$\frac{M \times H_i}{M B \sin (90 - (\phi - \theta))} = \frac{M B \sin \phi}{\mu}$$

Therefore

$$k \cos \theta (\cos \phi \cos \theta + \sin \phi \sin \theta) = \sin \phi$$
\[ k \cos \phi (\cos \theta) + k \sin \phi \sin \theta \cos \theta = \sin \phi \]

Given \( \frac{k \cos \phi}{z} = \sin \phi \)

\[ \cot \phi = \frac{z}{k}, \quad \tan \phi = \frac{k}{z} = \frac{\pi N \mu w a}{4R} \]

with \( N = 100, \quad \omega = 100 \text{ rad/s}, \quad R = 1, \quad a = 50 \text{ mm}, \quad \mu = 9 \pi \times 10^{-7} \)

\[ \phi = 2.83^\circ \]
charged particle with charge 
\[ q \] starts from origin

\[ \mathbf{B} = B_\perp \rightarrow \mathbf{E} = E_x \]

Force on particle = \[ F = e \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \]

\[ F = e \left( E_x \hat{x} + v_y B_\perp \right) = e \left( E_x \hat{x} + v_x \frac{\mathbf{a}_x \times \mathbf{B}}{v_y} + v_y \frac{\mathbf{a}_y \times \mathbf{B}}{v_y} \right) \]

\[ F = e \left( E_x \hat{x} - v_x B_\perp \hat{y} + v_y B_\perp \hat{y} \right) \]

\[ F_x = e \left( E_x + v_y B \right) = e \left( E_x + \frac{dy_B}{dt} \right) \]

\[ F_y = -e v_x B = -e B \frac{dx}{dt} \]

\[ F_z = 0 \]

If particle mass is \[ m \] then we have

\[ m \frac{d^2 x}{dt^2} = e \left( E_x + \frac{dy_B}{dt} \right) \]

\[ m \frac{d^2 v_x}{dt^2} = e \left( E_x + \frac{v_y B}{dt} \right) \]  \( \text{(1)} \)

\[ m \frac{d^2 y}{dt^2} = -e B \frac{dx}{dt} \]

\[ m \frac{d^2 v_y}{dt^2} = -e B v_x \] \( \text{(2)} \)

From \( \text{(1)} \)

\[ m \frac{d^2 v_x}{dt^2} = e B \frac{dv_y}{dt} = -\left( \frac{eB}{m} \right) v_x \]

Therefore

\[ \frac{d^2 v_x}{dt^2} = -\left( \frac{eB}{m} \right) v_x = -\omega^2 v_x \quad \omega = \frac{eB}{m} \]
Therefore \( V_x = A \sin \omega t + B \cos \omega t \)

At \( t = 0, \ V_x = 0 \) therefore \( B = 0 \)

\( V_x = A \sin \omega t \Rightarrow \frac{dx}{dt} = A \sin \omega t \quad x = -A \cos \omega t + \text{constant} \)

At \( t = 0, \ x = 0 \), therefore \( x = A(1 - \cos \omega t) \)

\[
\frac{d^2x}{dt^2} = +A \omega^2 \sin \omega t \quad \left( \frac{d^2x}{dt^2} \right)_{t=0} = +A \omega^2 = \frac{eE}{m}
\]

Therefore \( A = \frac{eE}{m \omega^2} \quad x = \frac{eE}{m \omega^2} \left( 1 - \cos \omega t \right) = \frac{E}{WB} (1 - \cos \omega t) \)

Therefore \( m \frac{d^2y}{dt^2} = -\frac{eE}{w} (v \sin \omega t) \)

\[
\frac{d^2y}{dt^2} = -\frac{eE}{m} \sin \omega t \quad \frac{dy}{dt} = \frac{eE}{m} \cos \omega t + D
\]

At \( t = 0, \ \frac{dy}{dt} = 0 \) therefore \( D = \frac{-eE}{mw} \)

\[
\frac{dy}{dt} = -\frac{eE}{mw} (1 - \cos \omega t) \quad y = -\frac{eE}{mw} \left( t - \frac{\sin \omega t}{\omega} \right) + E
\]

At \( t = 0, \ y = 0 \) therefore \( E = 0 \)

\[
y = \frac{eE}{mw} (wt - \sin \omega t) = \frac{-E}{WB} (wt - \sin \omega t)
\]
(6) From (5) \[ x_{\text{max}} = \frac{2E}{\omega B} \]

\[ w = \frac{eB}{m} \quad x_{\text{max}} = \frac{2mE}{eB^2} \]

Therefore, charge will not reach opposite capacitor plate if \[ d > \frac{2mE}{eB^2} \]

(7) \[ w = 2\pi f \]

Along a radius of the rotating disc the motional electric field is \( E_m \) in: \( (\mathbf{\nu} \times \mathbf{B}) = E_m \)

At radius \( x \) \[ \mathbf{\nu} = \omega x \]

\[ E_m = B \omega x \hat{\mathbf{r}} \]

\[ V = - \int E \cdot d\mathbf{x} = \int \omega \mathbf{\nu} \times d\mathbf{x} = \frac{B\omega a^2}{2} \]

\[ = \frac{B 2\pi f a^2}{2} \]

\[ V = \frac{f B \pi a^2}{2} \]