ENEE 381 Solutions to First Examination Questions

(1) The normal definition \( E = \text{grad} V \) does not work when time varying fields are involved because \( \text{curl} E = \text{curl} \text{grad} V = 0 \) but Maxwell’s equations require that \( \text{curl} E = -\partial E/\partial t \).

Note that

\[
\text{curl} E = -\frac{\partial}{\partial t} \text{curl} A
\]

gives

\[
\text{curl}(E + \frac{\partial A}{\partial t}) = 0,
\]

which suggests that a new definition for \( E \) in terms of potentials is

\[
E = -\text{grad} V - \frac{\partial A}{\partial t}.
\]

Now

\[
\text{curl} \text{curl} A = \text{grad} \text{div} A - \nabla^2 A,
\]

and

\[
\text{curl} B = \mu_r \mu_0 \text{curl} H = \mu_r \mu_0 (j + \epsilon_r \epsilon_0 \frac{\partial E}{\partial t}),
\]

which gives

\[
\text{grad} \text{div} A - \nabla^2 A = \mu_r \mu_0 j + \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial}{\partial t} (-\text{grad} V - \frac{\partial A}{\partial t}).
\]

If \( j = 0 \) and

\[
\text{grad} \text{div} A = -\mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial}{\partial t} \text{grad} V,
\]

then

\[
\nabla^2 A = \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial^2 A}{\partial t^2},
\]

and \( A \) obeys the wave equation.

\( k = \omega/c \) and \( c = 1/\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0} \), so \( c = 2.998 \times 10^8 / 3 = 10^8 \text{m/s} \). \( k = 2\pi m^{-1} \).

(2) Use \( \text{curl} E = -\partial B/\partial t \). If the electric field only has a \( y \) component and the magnetic field only has an \( x \) component (which was given) then the \( x \) component of the curl must satisfy

\[
\text{curl} E_x = -\mu_r \mu_0 \frac{\partial H_x}{\partial t},
\]

which gives

\[
-\frac{\partial E_y}{\partial z} = -\mu_r \mu_0 \frac{\partial H_x}{\partial t}.
\]

For a wave of the form

\[
E_y = E_0 e^{-j(\omega t - kz)},
\]
this gives \(jkE_y = -j\omega \mu_r \mu_0 H_x\) and the required result follows

\[
\frac{E_y}{H_x} = -\sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}.
\]

For the \(E_\theta\) and \(H_\phi\) given, which are orthogonal field components, the impedance is \(Z = E_\theta / H_\phi\), which in this case where \(\mu_r = 1\) and \(\epsilon_r = 1\), gives \(Z = 376.7\) ohm.

To get the total power radiated over a sphere centered at the origin you must integrate the time averaged value of \(E_\theta H_\phi\) over the sphere, which gives

\[
P = \int_0^\pi \int_0^{2\pi} \frac{1}{2} \text{Re}(E_\theta H_\phi^*) \sin^2 \theta d\theta d\phi = \int_0^\pi \int_0^{2\pi} \frac{\omega k(I_0 h)^2}{32\pi^2 r^2} \sin^2 \theta dS,
\]

where \(dS = r^2 \sin \theta d\theta d\phi\) is the area element in spherical coordinates.
Writing \((\omega k(I_0 h)^2/32\pi^2) = A\), this gives

\[
P = A \int_0^\pi \int_0^{2\pi} \sin^3 \theta d\theta d\phi = 2\pi A \int_0^\pi \sin^3 \theta d\theta.
\]

Use \(\sin^2 \theta = (1 - \cos^2 \theta)\) to integrate, which gives \(P = 8\pi A/3\). The Poynting vector is \(\mathbf{S} = \mathbf{E} \times \mathbf{H}\), and its time average is \(\left|\mathbf{S}\right| = E^2/2Z = ZH^2/2\), where \(E\) and \(H\) are the field amplitudes. Therefore, \(E^2 = (2 \times 377 \times 10^{-5})\) V/m, and \(H^2 = (2 \times 10^{-5}/377)\) A/m.

Answers: \(E = 0.0868\) V/m; \(H = 0.00023\) A/m.

(3) The first part of this question is the standard derivation of the skin depth (see book). One way of doing this is by using

\[
k = \frac{\omega}{c} = \omega \sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0} = \omega \sqrt{\mu_r \mu_0 \epsilon_0} \sqrt{\epsilon' - j \epsilon''}.
\]

Using \(\epsilon'' = \sigma/\omega \epsilon_0\) this gives for \(\epsilon' \ll \epsilon''\)

\[
k = \sqrt{\mu_r \mu_0 \omega \sigma} \sqrt{-j} = \sqrt{\mu_r \mu_0 \omega \sigma (j - 1)}/\sqrt{2}.
\]

Recognize that \(1/\sqrt{\mu_r \mu_0 \omega \sigma}/2 = \delta\), where \(\delta\) is the skin depth the wave varies as

\[
E_x = E_0 e^{-jkz} = e^{-z/\delta} e^{-jz/\delta}
\]

The current in the skin also varies exponentially from the surface going into the conductor. The total current per unit width is

\[
J_s = \int_0^\infty jdz = \sigma E_0 \int_0^\infty e^{-z/\delta} dz,
\]

which gives \(J_s = \sigma \delta E_0\).
If \( \nu = 100 \text{GHz} = 10^{11} \text{Hz} \), and \( \sigma = 10^7 \text{S/m} \), then \( \delta = 5.03 \times 10^{-7} \text{m} \). The wavelength is \( \lambda = c/\nu = 3 \text{mm} \). At this depth the field is down by a factor \( e^{-5961} \).

(4)(a) The charging current of a capacitor is

\[
i_c = \frac{dQ}{dt} = C\frac{dV}{dt},
\]

and since the field in the capacitor is \( E = V/d \), where \( d \) is the plate spacing, the charging current density is

\[
j_c = \frac{Cd}{A} \frac{dE}{dt} = \frac{Cd}{A\varepsilon_r\varepsilon_0} \frac{dD}{dt} = \frac{dD}{dt}.
\]

(b) The reflection coefficient is

\[
\rho = \frac{Z_L - Z_0}{Z_L + Z_0}.
\]

The hint that \( \rho = 0 \) if \( Z_L = Z_0 \) was designed to help you remember this. Rearranging the equation gives

\[
Z_L = \frac{1 + \rho}{1 - \rho} Z_0.
\]

Substitution with \( \rho = 0.3e^{j\pi/4} \) and \( Z_0 = 75 \text{ohm} \) gives \( Z_L = 102.52 + j47.8 \text{ ohm} \).

(c) Motion of the conductor through the magnetic field produces a force \( \mathbf{F} = -e(\mathbf{v} \times \mathbf{B}) \) on the electrons in the conductor. To an observer moving with the conductor there is no apparent motion, and this observer will see the motion of the electrons as resulting from an apparent emf

\[
V = \int_1^2 (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}.
\]

(d) At the surface of a perfect conductor the tangential electric field must vanish: \( E_t = 0 \). The normal electric field obeys \( \varepsilon_r\varepsilon_0 E_n = \rho_s \), where \( \rho_s \) is any surface charge density \((\text{C/m}^2)\). The normal magnetic field is zero since there is no such thing as surface magnetic charge density. Any time-varying tangential magnetic field must be balanced by a surface current according to Amperés Law. \( H_t = \hat{n} \times \mathbf{J}_s \). There could be a dc magnetic field at the surface without any surface current.