ENEE 380 Spring 2003. Homework #2, 2/18/03

Due 2/25/03

(1) Cheng Problem (3.17)

(2) A charge of 1C is uniformly distributed in the xy plane between $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Calculate and plot the electric field in the z-direction as z varies from 0 to 1000m.

(3) In Question (2) How far up the z-axis do you need to be before the field can be calculated within 1% accuracy by treating all the charge as if it were at the point (0,0,0).

(4) Cheng Problem (3.20)

(5) One type of quadrupole is an arrangement of 4 charges of magnitudes $+q$, $-q$, $+q$, $-q$ arranged at the corners of a square. The spacing of the charges is negligible compared to the distance to an observation point where the field is measured. Derive expressions for the potential distribution from a quadrupole and thereby the various field components. Plot the equipotential surfaces in the plane of the quadrupole, and in a plane perpendicular to the plane of the quadrupole passing through the center of the square where the charges are located. Plot the radial electric field variation in the plane of the quadrupole.

(6) Repeat (5) for a linear quadrupole, where the charges are $+q$, $-2q$, $+q$ arranged in a straight line.
(1)\[ dV_p(x) = \frac{p_0dx}{4\pi\varepsilon_0 \sqrt{(x-x')^2 + b^2}} \]

\[ V_p(x) = \frac{p_0}{4\pi\varepsilon_0} \left[ \frac{L}{\sqrt{(x-x')^2 + b^2}} \right] \]

\[ = \frac{p_0}{4\pi\varepsilon_0} \left[ \sinh^{-1}\left(\frac{L-x}{b}\right) + \sinh^{-1}\left(\frac{x}{b}\right) \right] \]

(4)\[ P_3-20\] Positive charge \( Ne \) uniformly distributed over a sphere of radius \( R_0 \): \( P = \frac{Ne}{\frac{4}{3}\pi R_0^3} = \frac{3Ne}{4\pi R_0^3} \).

Inside the sphere (applying Gauss's law): \( \vec{E} = \vec{E}_R = \frac{Ne}{4\pi\varepsilon_0 R_0^3} \).

a) Force experienced by an electron \( -e \): \( \vec{F} = -\vec{E}_R \frac{Ne^2}{4\pi\varepsilon_0 R_0^3} \).

b) Equation of motion for an electron with mass \( m \):
\[ m \frac{d^2r}{dt^2} = -\frac{Ne^2r}{4\pi\varepsilon_0 R_0^3}, \quad \text{or} \quad \frac{d^2r}{dt^2} + \left( \frac{Ne^2}{4\pi\varepsilon_0 m R_0^3} \right) r = 0, \]

or \( \frac{d^2r}{dt^2} + \omega_e^2 r = 0 \), where \( \omega_e = \sqrt{\frac{Ne^2}{4\pi\varepsilon_0 m R_0^3}} \).

Hence the electrons would undergo a simple harmonic motion with an angular frequency \( \omega_e \).

(4) The oscillating electrons would lose power through radiation and lead to an unstable atomic model.
The field in the z-direction is

\[
E_z = \int_{-1}^{1} \int_{-1}^{1} \frac{z}{4 \left( x^2 + y^2 + z^2 \right)^{\frac{3}{2}}} \, dx \, dy
\]

The field only points in the z-direction because of the symmetry of the problem, but during the integration the z-component of each contribution to the total field must be found by using the cosine of the vertical angle between the line to the charge element at \(dx\,dy\) and the z-axis.

\[
z : = 0.001, 1 \ldots 1000
\]

\[
E(z) := \int_{-1}^{1} \int_{-1}^{1} \frac{z}{4 \left( x^2 + y^2 + z^2 \right)^{\frac{3}{2}}} \, dx \, dy
\]

Assuming that the sheet of charge can be treated as a point charge

\[
E_1(z) := \frac{1}{z^2}
\]
Test different values of $z$

For $z > 10$ the result is within 1% of the correct value if the charge is assumed to be a point charge of value 1C.
In plane potential surfaces of a square quadrupole

\[ \theta := 0, 0.01 \ldots 2\pi \]
\[ V = \frac{3 \cdot qdd \cdot \sin(\theta) \cdot \cos(\theta)}{4 \cdot \pi \cdot \varepsilon_0 \cdot r^3} \]

\[ \varepsilon_0 := 1 \]

Change to a scaled system of units to avoid big numbers
\[ qdd := 1 \]
\[ i := 1, 2, \ldots, 10 \]
\[ V_i := i \quad \text{and} \quad V_{\neg i} := -i \]

Choose both positive and negative numbers to see full equipotential pattern

\[ r_{1}(\theta) := \left( \frac{3 \cdot qdd \cdot \sin(\theta) \cdot \cos(\theta)}{4 \cdot \pi \cdot \varepsilon_0 \cdot V_1} \right)^{\frac{1}{3}} \]
\[ r_{10}(\theta) := \left( \frac{3 \cdot qdd \cdot \sin(\theta) \cdot \cos(\theta)}{4 \cdot \pi \cdot \varepsilon_0 \cdot V_{10}} \right)^{\frac{1}{3}} \]

Some examples of equipotentials in the plane
\[
    E_r = \frac{9 \cdot q \cdot d \cdot \sin(\theta) \cdot \cos(\theta)}{4 \cdot \pi \cdot \epsilon_0 \cdot r^4}
\]

Radial Electric Field Strength

In terms of \( x \) and \( y \) coordinates

\[
    i := 1, 2, \ldots, 100
\]

\[
    j := 1, 2, \ldots, 100
\]

\[
    x_i := -1 + (i - 1) \cdot \frac{2}{99} \quad y_j := -1 + (j - 1) \cdot \frac{2}{99}
\]

\[
    E_{r_{i,j}} := \frac{9 \cdot q \cdot d \cdot x_i \cdot y_j}{4 \cdot \pi \cdot \epsilon_0 \left[ (x_i)^2 + (y_j)^2 \right]^2}
\]

\[
    E_{R_{i,j}} := \ln(E_{r_{i,j}}) \quad \text{Take log to make plots look better}
\]

Log of Radial electric strength in the plane of the quadrupole
Numerical Approach

\[ x_1 := 10^{-4} \]
\[ y_1 := 10^{-4} \]
\[ x_2 := 10^{-4} \]
\[ y_2 := -10^{-4} \]
\[ x_3 := -10^{-4} \]
\[ y_3 := -10^{-4} \]
\[ x_4 := -10^{-4} \]
\[ y_4 := 10^{-4} \]
\[ q_1 := 10^8 \]
\[ q_2 := -10^8 \]
\[ q_3 := 10^8 \]
\[ q_4 := -10^8 \]
\[ x := 0.1, 0.11 \ldots 1 \]
\[ y := 0.1, 0.11 \ldots 1 \]

\[ V_1(x, y, z) := \frac{q_1}{4 \cdot \pi \cdot \varepsilon_0 \left[ (x-x_1)^2 + (y-y_1)^2 + (z)^2 \right]^\frac{1}{2}} \]

\[ V_2(x, y, z) := \frac{q_2}{4 \cdot \pi \cdot \varepsilon_0 \left[ (x-x_2)^2 + (y-y_2)^2 + (z)^2 \right]^\frac{1}{2}} \]

\[ V_3(x, y, z) := \frac{q_3}{4 \cdot \pi \cdot \varepsilon_0 \left[ (x-x_3)^2 + (y-y_3)^2 + (z)^2 \right]^\frac{1}{2}} \]

\[ V_4(x, y, z) := \frac{q_4}{4 \cdot \pi \cdot \varepsilon_0 \left[ (x-x_4)^2 + (y-y_4)^2 + (z)^2 \right]^\frac{1}{2}} \]

\[ V(x, y, z) := V_1(x, y, z) + V_2(x, y, z) + V_3(x, y, z) + V_4(x, y, z) \]

\[ x_i := 0 + (i) \cdot \frac{1}{99} \]
\[ y_j := 0 + (j) \cdot \frac{1}{99} \]
\[ \phi_{i,j} := \ln(V(x_i, y_j, 0)) \]
Equipotentials in $xy$ plane
Log of Potential Function in first quadrant of xy plane
\[ x_i := 0 + (i) \cdot \frac{1}{99} \]
\[ z_j := 0 + (j) \cdot \frac{1}{99} \]
\[ \phi_{i,j} := \left( V \left( x_i \cdot 10^{-6}, z_j \right) \right) \]

\[ \phi_{i,j} := \ln(\phi_{i,j}) \]

Equipotentials in the xz plane
(6) Linear Quadrupole

\[ \theta := 0, 0.01 \ldots 2\pi \]

\[ R := 0.01, 0.02 \ldots 1 \]

\[ V = \frac{q}{4\pi \varepsilon_0} \left( \frac{3}{R^3} d^2 \sin^2(\theta) - \frac{d^2}{R^3} \right) \]

Change to different scaled unit so that

\[ V(R, \theta) := \frac{3 \cdot \sin^2(\theta) - 1}{R^3} \]

polar plot of potential function

\[ i := 1, 2 \ldots 10 \quad v_i := -4.999 + i \]

\[ j := 1, 2 \ldots 100 \]

\[ R_{i,j} := \left[ \frac{3 \cdot (\sin(\theta_j))^2 - 1}{v_i} \right]^3 \]
Examples of some equipotentials

Radial electric field

\[ ER(R, \theta) = \frac{-3}{R^4} \left( (3 \cdot \sin(\theta))^2 - 1 \right) \]
Alternative Numerical approach: set up 4 charges and draw equipotentials

\[ i := 1, 2, \ldots, 100 \]

\[ j := 1, 2, \ldots, 100 \]

\[ y_1 := 10^{-6} \]

\[ y_2 := 0 \]

\[ y_3 := -10^{-6} \]

\[ x_i := \frac{i}{100} - \frac{1}{1.999}, y_j := \frac{j}{100} - \frac{1}{1.999} \]

Location of charges

different measurement locations

\[ R_{1, i, j} := \sqrt{(x_i)^2 + (y_j - y_1)^2} \]

\[ x_0 := x_1 \]

\[ y_0 := y_1 \]

\[ R_{2, i, j} := \sqrt{(x_i)^2 + (y_j - y_2)^2} \]

\[ R_{3, i, j} := \sqrt{(x_i)^2 + (y_j - y_3)^2} \]

\[ V_{i, j} := \left( \frac{1}{R_{1, i, j}} - \frac{2}{R_{2, i, j}} + \frac{1}{R_{3, i, j}} \right) \]

normalized units

\[ \psi_{i, j} := \log(V_{i, j}) \]
Equipotentials in xy plane