Solutions

ENEE 457: Computer Systems Security
PRF Class Exercise 10/5/22

Let F be a length-preserving pseudorandom function. For the following constructions of a keyed function $F': \{0,1\}^n \times \{0,1\}^{n-1} \to \{0,1\}^{2n}$, state whether F' is a pseudorandom function. If yes, prove it; if not, show an attack.

1. a) How many functions are there from $\{0,1\}^n \to \{0,1\}^n$?

   Truth table has 2^n number of rows. For each row there are 2^n number of choices. So the total number is (2^n)^{2^n} = 2^{n*2^n}.

   b) How many *permutations* are there from $\{0,1\}^n \to \{0,1\}^n$?

   Truth table has 2^n rows. For row i there are (2^n - i + 1) choices.
   So the total number of choices is 2^n * (2^n-1) * (2^n-2)... =
   (2^n)!

   c) What is the expected number of bits needed to describe a random function f?
   log_2(2^{n*2^n}) = n*2^n.

   d) What is the expected number of bits needed to describe a random permutation f?
   log_2 ((2^n)!). By Stirling's approximation, log(x!) \approx log(x^x) so this is also log((2^n)^{2^n}) = log(2^{n*2^n}) = n*2^n.

   e) Let F be a length-preserving pseudorandom function, $F: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$. Assuming the description of F is public, how many bits are needed to represent a function F_k?
   n bits.

2. Consider a keyed function $F: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$.

   a) If F has the property that for all k, x, y: $F_k(x \oplus y) = F_k(x) \oplus F_k(y)$, can F be a pseudorandom function? Justify your answer.
   No. Because given x, y \neq 0 and F_k(x) and F_k(y), we can predict the value of F_k(x \oplus y) = F_k(x) \oplus F_k(y). Whereas for a (pseudo) random function, knowing the value of the function on 2 points should give no information about its value at a third distinct point.

   b) If F has the property that for k, ℓ, x: $F_{k \oplus \ell}(x) = F_k(x) \oplus F_\ell(x)$, can F be a pseudorandom function? Assume the above relation holds for any k and x and some particular value of ℓ. Justify your answer.

   Yes, this is possible. In the security game the attacker *only* gets access to F with a particular secret key k. Therefore, the attacker would not be able to obtain the values of F_k(x) in a security game with secret key k \oplus \ell. (It would only be able to obtain the values F_{k \oplus \ell}(x) and F_{k'}(x) for known k'.)