Introduction to Cryptology

Lecture 6

Announcements

- HW2 due Thursday, 2/15
- Readings/Quizzes on Canvas due today (11:59pm)
- Pick up graded HW1 after class

Agenda

- Last time:
  - One time pad (OTP) (K/L 2.2)
  - Limitations of perfect secrecy (K/L 2.3)
- This time:
  - Shannon’s Theorem and examples (K/L 2.4)
  - The Computational Approach (K/L 3.1)
  - Defining computationally secure SKE (K/L 3.2)

Shannon's Theorem

Let (Gen, Enc, Dec) be an encryption scheme with message space $\boldsymbol{M}$, for which $|\boldsymbol{M}| = |\boldsymbol{K}| = |\boldsymbol{C}|$. The scheme is perfectly secret if and only if:

1. Every key $k \in \boldsymbol{K}$ is chosen with equal probability $1/|\boldsymbol{K}|$ by algorithm Gen.
2. For every $m \in \boldsymbol{M}$ and every $c \in \boldsymbol{C}$, there exists a unique key $k \in \boldsymbol{K}$ such that $Enc_k(m)$ outputs c.

**Theorem only applies when $|\boldsymbol{M}| = |\boldsymbol{K}| = |\boldsymbol{C}|$.

Some Examples

- Is the following scheme perfectly secret?
- Message space $\boldsymbol{M} = \{0,1,\ldots,n-1\}$. Key space $\boldsymbol{K} = \{0,1,\ldots,n-1\}$.
- $\mathrm{Gen}()$ chooses a key k at random from $\boldsymbol{K}$.
- $\mathrm{Enc}_k(m)$ returns $m + k$.
- $Dec_k(c)$ returns $c - k$.

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- $\mathrm{Gen}()$ chooses a key k at random from $\boldsymbol{K}$.
- $\mathrm{Enc}_k(m)$ returns $m + k \; mod \; n$.
- $Dec_k(c)$ returns $c - k \; mod \; n$.

The Computational Approach

Two main relaxations:

1. Security is only guaranteed against efficient adversaries that run for some feasible amount of time.
2. Adversaries can potentially succeed with some very small probability.

Security Parameter

- Integer valued security parameter denoted by n that parameterizes both the cryptographic schemes as well as all involved parties.
- When honest parties initialize a scheme, they choose some value n for the security parameter.
- Can think of security parameter as corresponding to the length of the key.
- Security parameter is assumed to be known to any adversary attacking the scheme.
- View run time of the adversary and its success probability as functions of the security parameter.

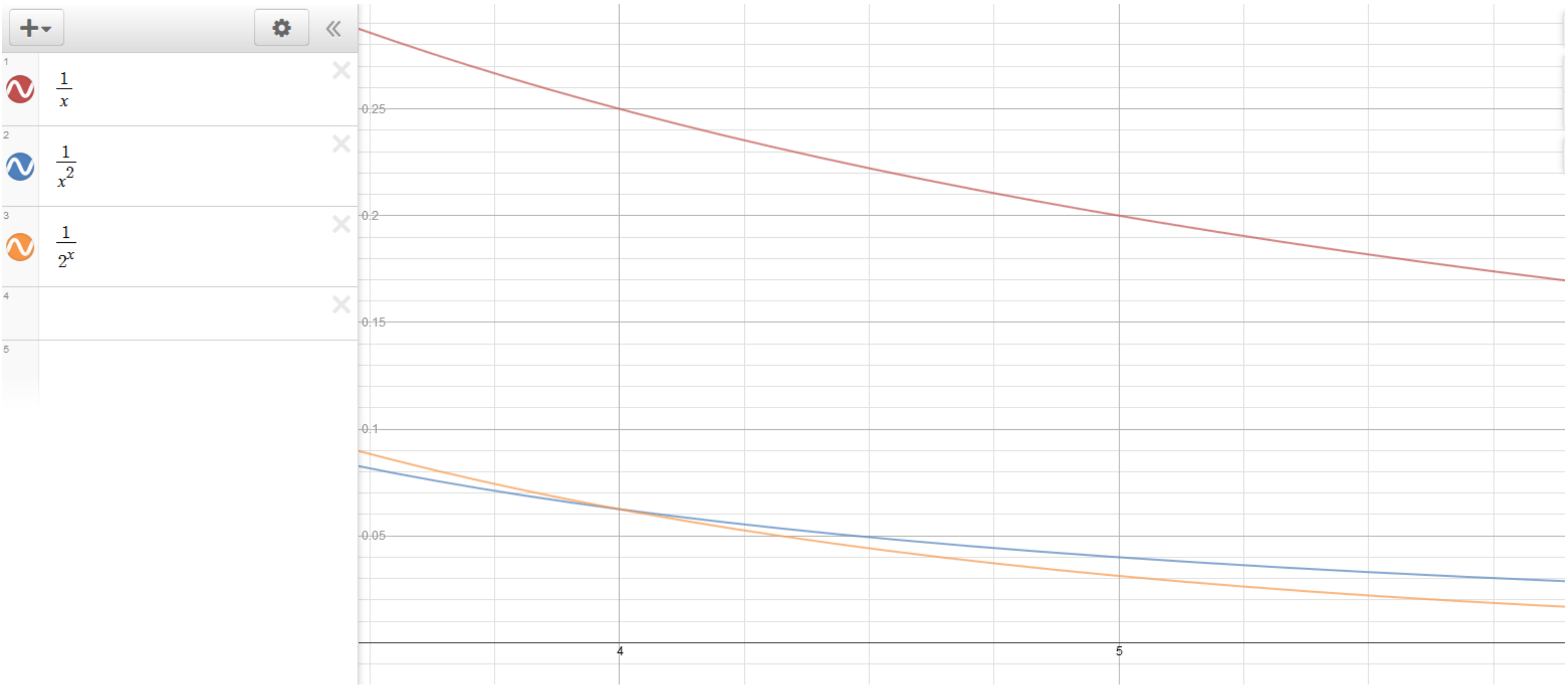
Polynomial Time

- Efficient adversaries = Polynomial time adversaries
  - There is some polynomial p such that the adversary runs for time at most $p(n)$ when the security parameter is n.
  - Honest parties also run in polynomial time.
  - The adversary may be much more powerful than the honest parties.

Negligible

- Small probability of success = negligible probability
  - A function f is negligible if for every polynomial p and all sufficiently large values of n it holds that $f(n) < \frac{1}{p(n)}$.
  - Intuition, $f(n) < n^{-c}$ for every constant c, as n goes to infinity.

Negligible



Practical Implications of Computational Security

- For key size n, any adversary running in time $2^{n/2}$ breaks the scheme with probability $1/2^{n/2}$.
- Meanwhile, Gen, Enc, Dec each take time n^2.
- If $n = 128$ then:
  - Gen, Enc, Dec take time 16,384
  - Adversarial run time is $2^{64} \approx 10^{18}$
- If n = 256 then:
  - Gen, Enc, Dec quadruples--takes time 65,536
  - Adversary run time is multiplied by 2^{64}. Becomes $2^{128} \approx 10^{38}$

Defining Computationally Secure Encryption

A private-key encryption scheme is a tuple of probabilistic polynomial-time algorithms (Gen, Enc, Dec) such that:

1. The key-generation algorithm Gen takes as input security parameter 1^n and outputs a key k denoted $k \leftarrow Gen(1^n)$. We assume WLOG that $|k| \geq n$.
2. The encryption algorithm Enc takes as input a key k and a message $m \in \{0,1\}^*$, and outputs a ciphertext c denoted $c \leftarrow Enc_k(m)$.
3. The decryption algorithm Dec takes as input a key k and ciphertext c and outputs a message m denoted by $m := Dec_k(c)$.

Correctness: For every n, every key $k \leftarrow Gen(1^n)$, and every $m \in \{0,1\}^*$, it holds that $Dec_k\big(Enc_k(m)\big) = m$.

Indistinguishability in the presence of an eavesdropper

Consider a private-key encryption scheme $\Pi = (Gen, Enc, Dec)$, any adversary A, and any value n for the security parameter.

The eavesdropping indistinguishability experiment $PrivK^{eav}{}_{A,\Pi}(n)$:

1. The adversary A is given input 1^n, and outputs a pair of messages m_0, m_1 of the same length.
2. A key k is generated by running $Gen(1^n)$, and a random bit $b \leftarrow \{0,1\}$ is chosen. A challenge ciphertext $c \leftarrow Enc_k(m_b)$ is computed and given to A.
3. Adversary A outputs a bit b'.
4. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise. If $PrivK^{eav}{}_{A,\Pi}(n) = 1$, we say that A succeeded.

Indistinguishability in the presence of an eavesdropper

Definition: A private key encryption scheme $\Pi = (Gen, Enc, Dec)$ has indistinguishable encryptions in the presence of an eavesdropper if for all probabilistic polynomial-time adversaries A there exists a negligible function $negl$ such that

$$\Pr\left[PrivK^{eav}{}_{A,\Pi}(n) = 1\right] \leq \frac{1}{2} + negl(n),$$

Where the prob. Is taken over the random coins used by A, as well as the random coins used in the experiment.