Lattice-Based Crypto

Lecture 27

Traditional Crypto Assumptions

- Factoring: Given $N = pq$, find p, q
  - RSA Given $N = pq, e, x^e \mod N$, find x.

- Discrete Log: Given $g^x \mod p$, find x.
  - Diffie-Hellman Assumptions (g^x, g^y, g^{xy}), (g^x, g^y, g^z)

Are They Secure?

- Algorithmic Advances:
  - Factoring: Best algorithm time $2^{\tilde{O}(n^{\frac{1}{3}})}$ to factor n-bit number.
  - Discrete log: Best algorithm $2^{\tilde{O}(n^{\frac{1}{3}})}$ for groups Z_p^*, where p is n bits.
    - [Adrian et al. 2015] With preprocessing could possibly be feasible for nation-states and $n = 1024$.
    - Quasipolynomial time algorithms for small characteristic fields. Not known to apply in practice.

- Quantum Computers:
  - Shor's algorithm solves both factoring and discrete log in quantum polynomial time ($\tilde{O}(n^2)$).

Are They Secure?

"For those partners and vendors that have not yet made the transition to Suite B algorithms (ECC), we recommend not making a significant expenditure to do so at this point but instead to **prepare for the upcoming quantum resistant algorithm transition**.... Unfortunately, the growth of elliptic curve use has bumped up against the fact of continued progress in the research on quantum computing, necessitating a re-evaluation of our cryptographic strategy. "—NSA Statement, August 2015

NIST Kicks Off Effort to Defend Encrypted Data from Quantum Computer Threat

April 28, 2016

Google Dabbles in Post-Quantum Cryptography

By Richard Adhikari
Jul 12, 2016 2:06 PM PT

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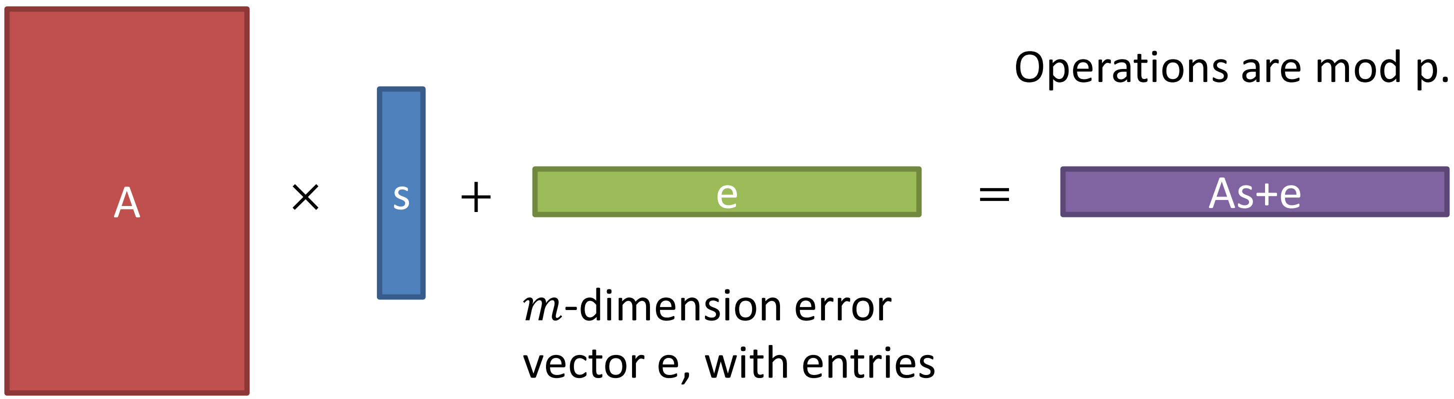
Post-Quantum Approach

- New set of assumptions based on finding short vectors in lattices.
- Believed to be hard for quantum computers.
- Evidence of hardness "worst case to average case reduction".
- Versatile: Can essentially construct all cryptosystems out of these assumptions.

The LWE Problem (Search)

Secret n-dimension vector s with entries chosen at random

Operations are mod p.



Public $m \times n$ matrix A, with entries chosen at random over Z_p

m-dimension error vector e, with entries sampled from χ. **Distribution χ depends on dimension of A and the modulus.**

Problem: Given, A, As+e, find s.

The LWE Problem Decision

Secret n-dimension vector s with entries chosen at random

Operations are mod p.

$$A \times s + e = As{+}e$$

Public $m \times n$ matrix A, with entries chosen at random over Z_p

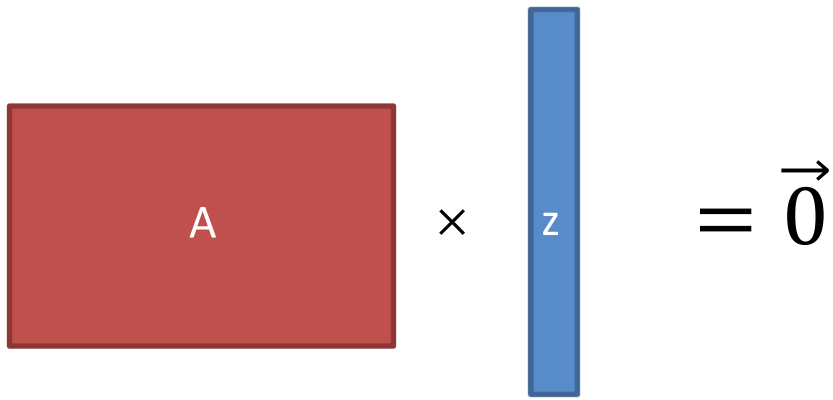
m-dimension error vector e, with entries sampled from χ

$$\approx$$

$$u$$

m uniform random elements from Z_p

The SIS Problem



Public $n \times m$ matrix A, with entries chosen at random over Z_p

Problem: Given A, find z $\in \{0,1\}^m$

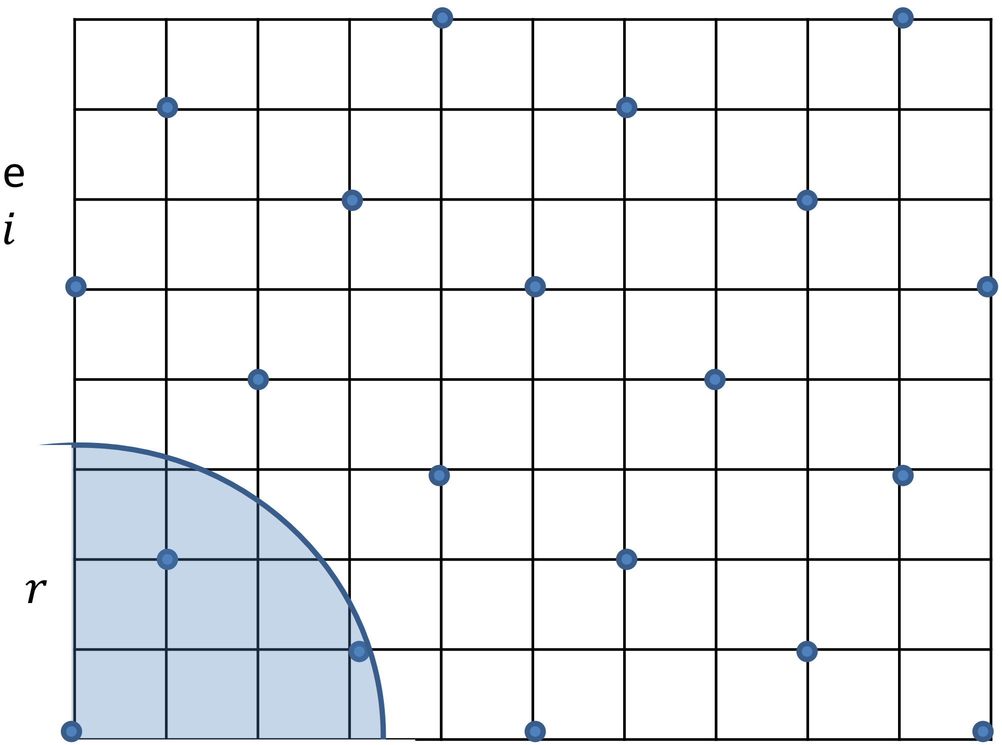
Lattices

An n-dimensional lattice L is an additive discrete subgroup of R^n. A basis $\boldsymbol{B} \in R^{n \times n}$ defines a lattice $\mathrm{L}(\boldsymbol{B})$ in the following way:

$L(\boldsymbol{B}) = \{\boldsymbol{v} \in R^n\ s.t.\ \boldsymbol{v} = \boldsymbol{Bz}$ for some $\boldsymbol{z} \in Z^n\}$.

"integer linear combinations of the basis vectors"

$\boldsymbol{i}$-th successive minima $\boldsymbol{\lambda_i(L(B))}$: The smallest radius r such that there are i linearly independent vectors $\{v_1, \dots, v_i\}$ of length at most r.



Hard Lattice Problems

- Are all parameterized by "approximation factor" $\gamma > 1$.
- Shortest Vector Problem (**SVP**): Given a basis B, find a non-zero vector $\boldsymbol{v} \in L(\boldsymbol{B})$ whose length is at most $\gamma \cdot \lambda_1(L(\boldsymbol{B}))$.
- Shortest Independent Vector Problem (**SIVP**): Given a basis B, find a linearly independent set $\{v_1, \ldots, v_n\}$ such that all vectors have length at most $\gamma \cdot \lambda_n(L(\boldsymbol{B}))$.
- Gap Shortest vector problem (**GapSVP**): Given a basis B, and a radius r > 0
  - Return YES if $\lambda_1\big(L(B)\big) \leq r$
  - Return NO if $\lambda_1\big(L(B)\big) > \gamma \cdot r$.

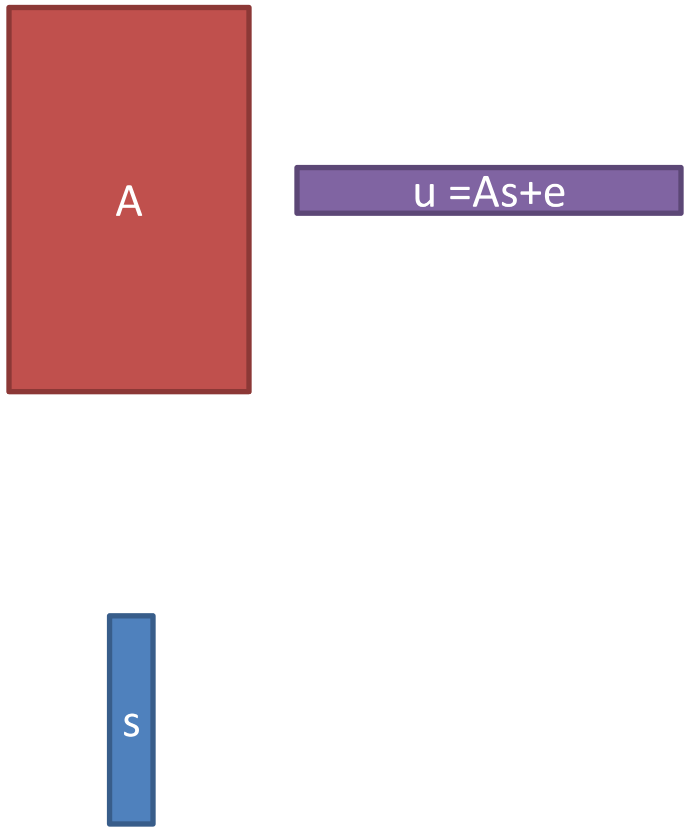
Relation to LWE, SIS

- Worst-Case to Average-Case Reduction: Breaking the cryptosystem on average is as hard as breaking the hardest instance of the underlying lattice problem.
- SIS:
  - Worst-Case to Average-Case Reduction from SIVP.
- LWE:
  - Worst-Case to Average-Case Quantum Reduction from SIVP.
  - Worst-Case to Average-Case Classical Reductions from GapSVP.

Lattice-Based Encryption

Regev’s Cryptosystem

Public Key:

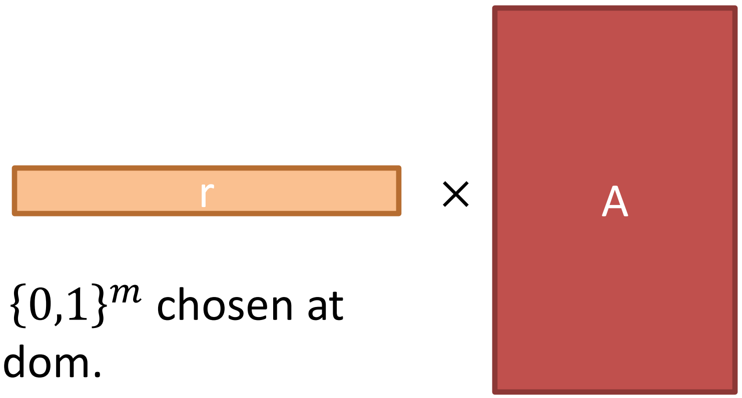


Secret Key:

Regev's Cryptosystem

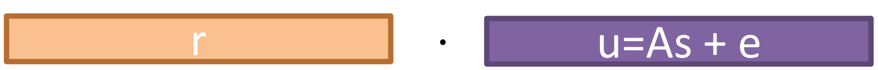
Encryption of $m \in \{0,1\}$

(1)



r $\in \{0,1\}^m$ chosen at random.

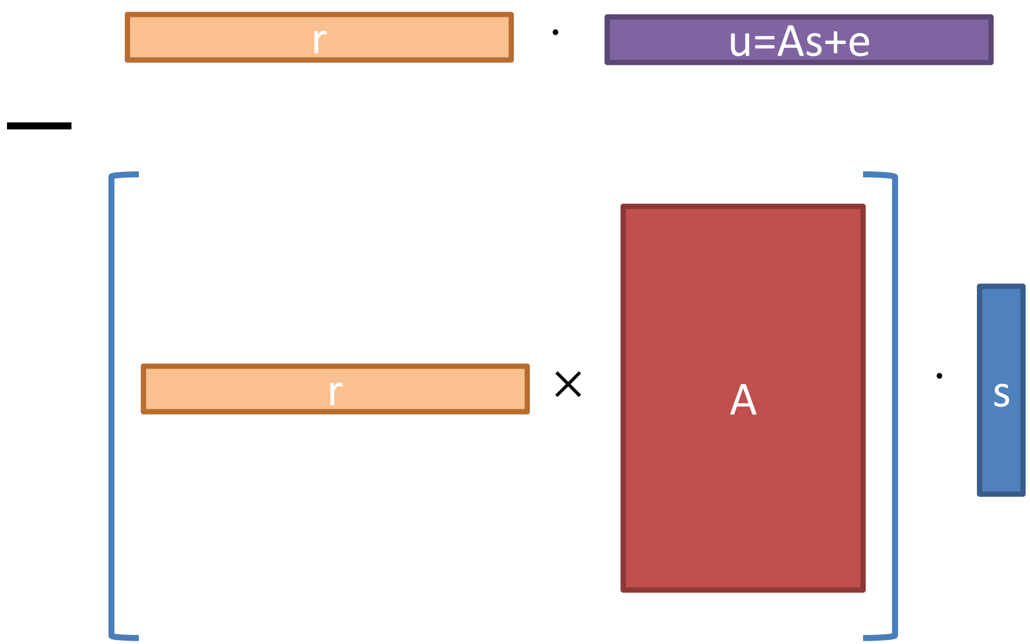
(2)



$+ m \cdot \left\lfloor \frac{p}{2} \right\rfloor$

Regev’s Cryptosystem

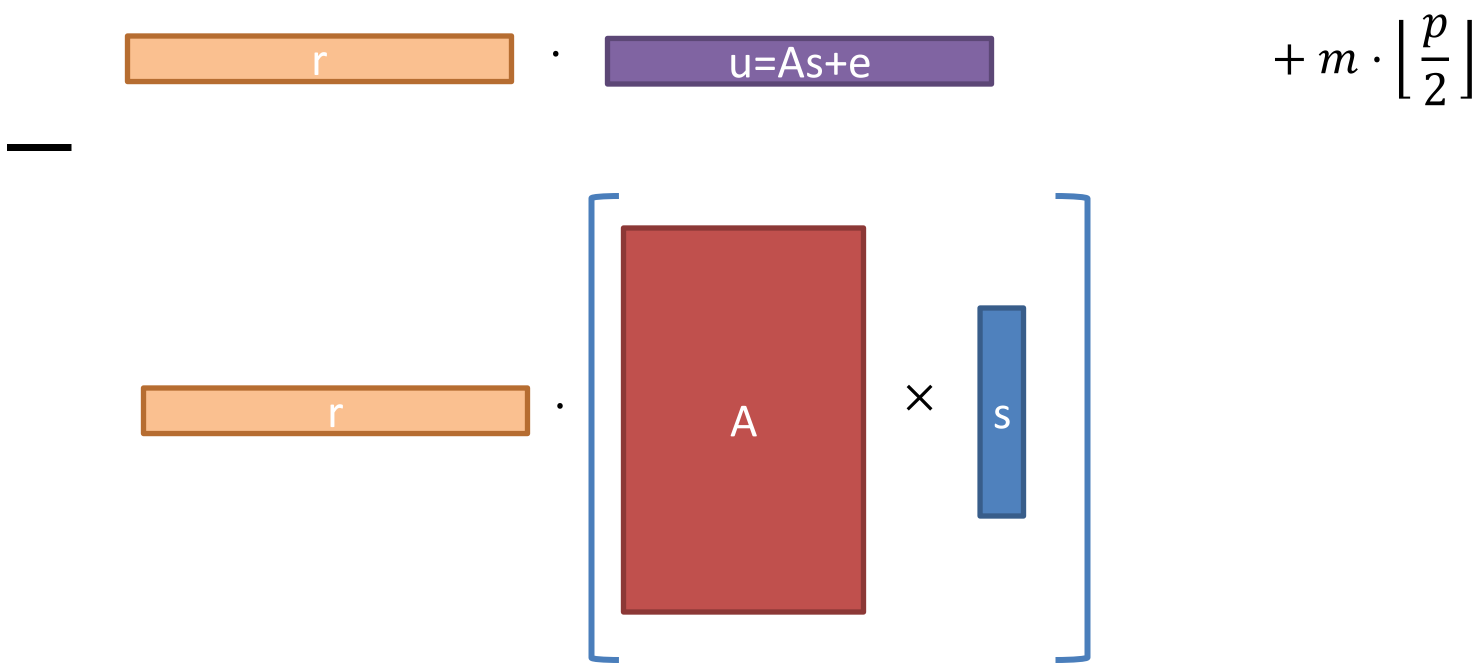
Decryption



$$+ m \cdot \left\lfloor \frac{p}{2} \right\rceil$$

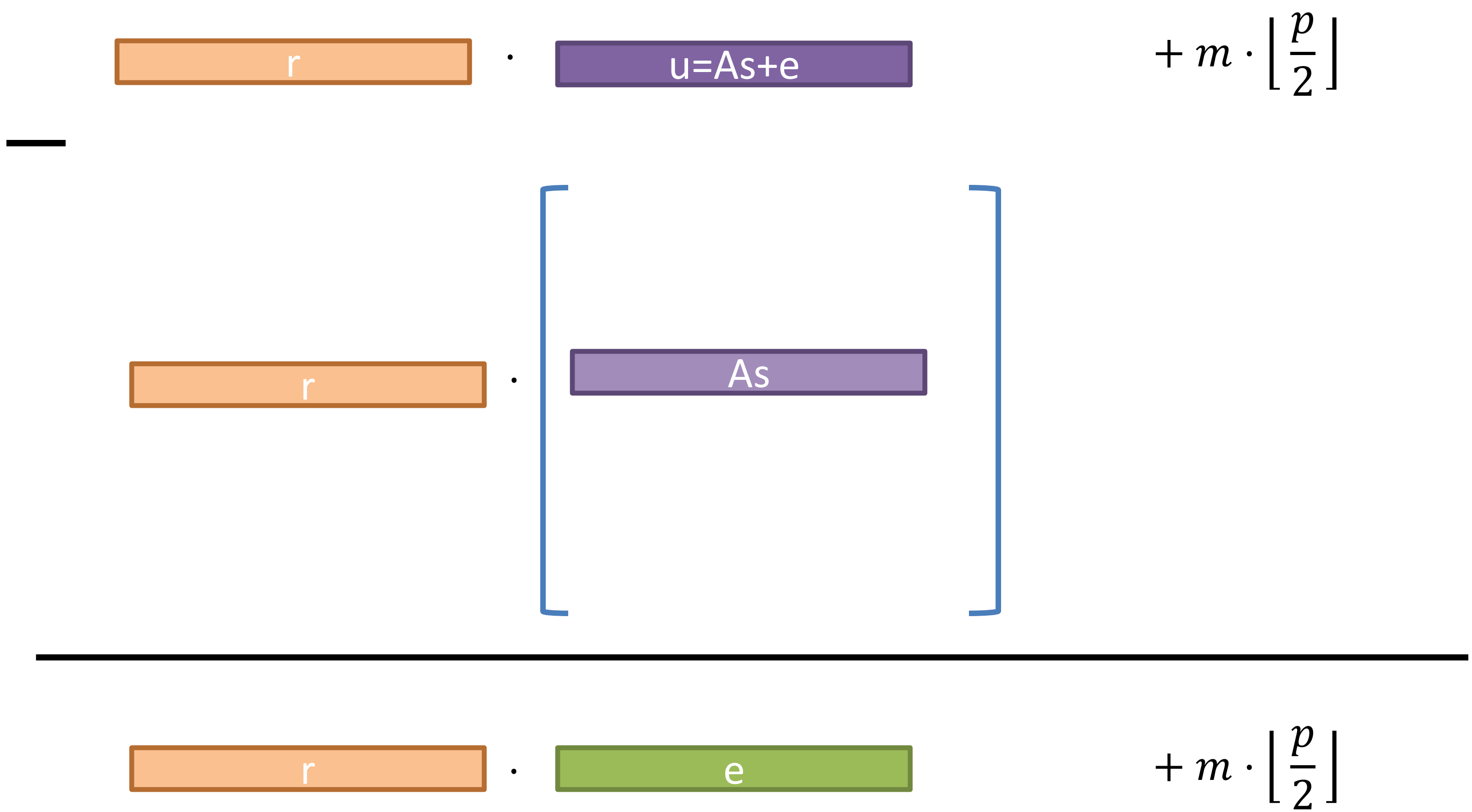
Regev’s Cryptosystem

Decryption



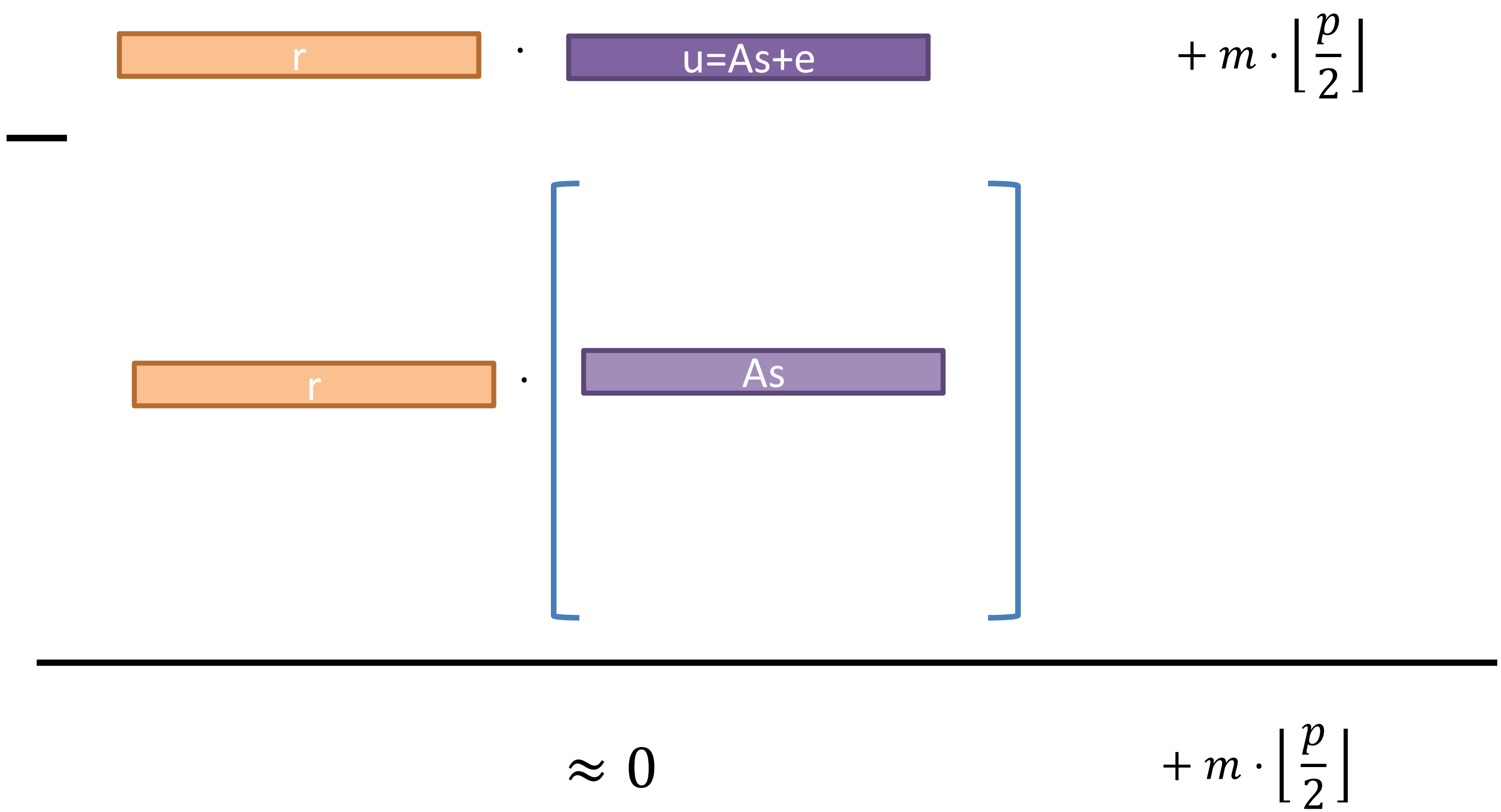
Regev's Cryptosystem

Decryption



Regev's Cryptosystem

Decryption



Thank You!