Introduction to Cryptology

Lecture 9
Announcements

• HW4 up on course webpage, due Tuesday, 3/1
Agenda

• Last time:
  – CPA Security (Sec. 3.4)
  – Pseudorandom Functions (PRF) (Sec. 3.5)
  – Constructing CPA-secure encryption from PRF (Sec. 3.5)

• This time:
  – Proof of security for the construction (3.5)
  – Class exercise on PRFs
CPA-Security

The CPA Indistinguishability Experiment $PrivK^{cpa}_{A,\Pi}(n)$:

1. A key $k$ is generated by running $Gen(1^n)$.

2. The adversary $A$ is given input $1^n$ and oracle access to $Enc_k(\cdot)$, and outputs a pair of messages $m_0, m_1$ of the same length.

3. A random bit $b \leftarrow \{0,1\}$ is chosen, and then a challenge ciphertext $c \leftarrow Enc_k(m_b)$ is computed and given to $A$.

4. The adversary $A$ continues to have oracle access to $Enc_k(\cdot)$, and outputs a bit $b'$.

5. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.
**CPA-Security**

Definition: A private-key encryption scheme $\Pi = (Gen, Enc, Dec)$ has indistinguishable encryptions under a chosen-plaintext attack if for all ppt adversaries $A$ there exists a negligible function $negl$ such that

$$\Pr \left[ PrivK^{cpa}_{A,\Pi}(n) = 1 \right] \leq \frac{1}{2} + negl(n),$$

where the probability is taken over the random coins used by $A$, as well as the random coins used in the experiment.
CPA-security for multiple encryptions

Theorem: Any private-key encryption scheme that has indistinguishable encryptions under a chosen-plaintext attack also has indistinguishable multiple encryptions under a chosen-plaintext attack.
Theorem: If $\Pi = (Gen, Enc, Dec)$ is an encryption scheme in which $Enc$ is a deterministic function of the key and the message, then $\Pi$ cannot be CPA-secure.

Why not?
Constructing CPA-Secure Encryption Scheme
Pseudorandom Function

Definition: A keyed function $F: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$ is a two-input function, where the first input is called the key and denoted $k$. 
Pseudorandom Function

Definition: Let $F: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be an efficient, length-preserving, keyed function. We say that $F$ is a pseudorandom function if for all ppt distinguishers $D$, there exists a negligible function $\text{negl}$ such that:

$$\left| \text{Pr}[D^{F_k}(\cdot)(1^n) = 1] - \text{Pr}[D^f(\cdot)(1^n) = 1] \right| \leq \text{negl}(n).$$

where $k \leftarrow \{0,1\}^n$ is chosen uniformly at random and $f$ is chosen uniformly at random from the set of all functions mapping $n$-bit strings to $n$-bit strings.
Construction of CPA-Secure Encryption from PRF
Formal Description of Construction

Let $F$ be a pseudorandom function. Define a private-key encryption scheme for messages of length $n$ as follows:

• *Gen*: on input $1^n$, choose $k \leftarrow \{0,1\}^n$ uniformly at random and output it as the key.

• *Enc*: on input a key $k \in \{0,1\}^n$ and a message $m \in \{0,1\}^n$, choose $r \leftarrow \{0,1\}$ uniformly at random and output the ciphertext

$$c := \langle r, F_k(r) \oplus m \rangle.$$

• *Dec*: on input a key $k \in \{0,1\}^n$ and a ciphertext $c = \langle r, s \rangle$, output the plaintext message

$$m := F_k(r) \oplus s.$$
Security Analysis

Theorem: If $F$ is a pseudorandom function, then the Construction above is a CPA-secure private-key encryption scheme for messages of length $n$. 
Recall: CPA-Security

The CPA Indistinguishability Experiment $\text{PrivK}^{\text{cpa}}_{A,\Pi}(n)$:

1. A key $k$ is generated by running $\text{Gen}(1^n)$.
2. The adversary $A$ is given input $1^n$ and oracle access to $\text{Enc}_k(\cdot)$, and outputs a pair of messages $m_0, m_1$ of the same length.
3. A random bit $b \leftarrow \{0,1\}$ is chosen, and then a challenge ciphertext $c \leftarrow \text{Enc}_k(m_b)$ is computed and given to $A$.
4. The adversary $A$ continues to have oracle access to $\text{Enc}_k(\cdot)$, and outputs a bit $b'$.
5. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.
Recall: CPA-Security

Definition: A private-key encryption scheme \( \Pi = (Gen, Enc, Dec) \) has indistinguishable encryptions under a chosen-plaintext attack if for all ppt adversaries \( A \) there exists a negligible function \( negl \) such that

\[
\Pr \left[ PrivK^{cpa}_{A,\Pi}(n) = 1 \right] \leq \frac{1}{2} + negl(n),
\]

where the probability is taken over the random coins used by \( A \), as well as the random coins used in the experiment.
Security Analysis

Let $A$ be a ppt adversary trying to break the security of the construction. We construct a distinguisher $D$ that uses $A$ as a subroutine to break the security of the PRF.

Distinguisher $D$:

$D$ gets oracle access to oracle $O$, which is either $F_{k}$, where $F$ is pseudorandom or $f$ which is truly random.

1. Instantiate $A^{En_{k}(\cdot)}(1^{n})$.
2. When $A$ queries its oracle, with message $m$, choose $r$ at random, query $O(r)$ to obtain $z$ and output $c := \langle r, z \oplus m \rangle$.
3. Eventually, $A$ outputs $m_0, m_1 \in \{0,1\}^n$.
4. Choose a uniform bit $b \in \{0,1\}$. Choose $r$ at random, query $O(r)$ to obtain $z$ and output $c := \langle r, z \oplus m \rangle$.
5. Give $c$ to $A$ and obtain output $b'$. Output 1 if $b' = b$, and output 0 otherwise.
Security Analysis

Consider the probability $D$ outputs 1 in the case that $O$ is truly random function $f$ vs. $O$ is a pseudorandom function $F_k$.

- When $O$ is pseudorandom, $D$ outputs 1 with probability $\Pr[\text{PrivK}_{A,\Pi}^{cpa}(n) = 1] = \frac{1}{2} + \rho(n)$, where $\rho$ is non-negligible.
- When $O$ is random, $D$ outputs 1 with probability at most $\frac{1}{2} + \frac{q(n)}{2^n}$, where $q(n)$ is the number of oracle queries made by $A$. Why?
Security Analysis

$D$’s distinguishing probability is:

$$\left| \frac{1}{2} + \frac{q(n)}{2^n} - \left( \frac{1}{2} + \rho(n) \right) \right| = \rho(n) - \frac{q(n)}{2^n}.$$ 

Since, $\frac{q(n)}{2^n}$ is negligible and $\rho(n)$ is non-negligible, $\rho(n) - \frac{q(n)}{2^n}$ is non-negligible.

This is a contradiction to the security of the PRF.