CPA-Security

The CPA Indistinguishability Experiment $PrivK_{A,\Pi}^{cpa}(n)$:

1. A key $k$ is generated by running $Gen(1^n)$.
2. The adversary $A$ is given input $1^n$ and oracle access to $Enc_k(\cdot)$, and outputs a pair of messages $m_0, m_1$ of the same length.
3. A random bit $b \leftarrow \{0,1\}$ is chosen, and then a challenge ciphertext $c \leftarrow Enc_k(m_b)$ is computed and given to $A$.
4. The adversary $A$ continues to have oracle access to $Enc_k(\cdot)$, and outputs a bit $b'$.
5. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.
Definition: A private-key encryption scheme $\Pi = (Gen, Enc, Dec)$ has indistinguishable encryptions under a chosen-plaintext attack if for all ppt adversaries $A$ there exists a negligible function $negl$ such that

$$\Pr \left[ PrivK^{cpa}_{A,\Pi}(n) = 1 \right] \leq \frac{1}{2} + negl(n),$$

where the probability is taken over the random coins used by $A$, as well as the random coins used in the experiment.
CPA-security for multiple encryptions

Theorem: Any private-key encryption scheme that has indistinguishable encryptions under a chosen-plaintext attack also has indistinguishable multiple encryptions under a chosen-plaintext attack.
CPA-secure Encryption Must Be Probabilistic

Theorem: If \( \Pi = (Gen, Enc, Dec) \) is an encryption scheme in which \( Enc \) is a deterministic function of the key and the message, then \( \Pi \) cannot be CPA-secure.

Why not?
Constructing CPA-Secure Encryption Scheme
Pseudorandom Function

Definition: A keyed function $F: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ is a two-input function, where the first input is called the key and denoted $k$. 
Pseudorandom Function

Definition: Let $F: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$ be an efficient, length-preserving, keyed function. We say that $F$ is a pseudorandom function if for all ppt distinguishers $D$, there exists a negligible function $negl$ such that:

$$|\Pr[D^{F_k(\cdot)}(1^n) = 1] - \Pr[D^{f(\cdot)}(1^n) = 1]| \leq negl(n).$$

where $k \leftarrow \{0,1\}^n$ is chosen uniformly at random and $f$ is chosen uniformly at random from the set of all functions mapping $n$-bit strings to $n$-bit strings.
Construction of CPA-Secure Encryption from PRF
Formal Description of Construction

Let $F$ be a pseudorandom function. Define a private-key encryption scheme for messages of length $n$ as follows:

- **Gen**: on input $1^n$, choose $k \leftarrow \{0,1\}^n$ uniformly at random and output it as the key.
- **Enc**: on input a key $k \in \{0,1\}^n$ and a message $m \in \{0,1\}^n$, choose $r \leftarrow \{0,1\}$ uniformly at random and output the ciphertext
  \[ c := \langle r, F_k(r) \oplus m \rangle. \]
- **Dec**: on input a key $k \in \{0,1\}^n$ and a ciphertext $c = \langle r, s \rangle$, output the plaintext message
  \[ m := F_k(r) \oplus s. \]
Theorem: If $F$ is a pseudorandom function, then the Construction above is a CPA-secure private-key encryption scheme for messages of length $n$. 
Recall: CPA-Security

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1. A key $k$ is generated by running $Gen(1^n)$.
2. The adversary $A$ is given input $1^n$ and oracle access to $Enc_k(\cdot)$, and outputs a pair of messages $m_0, m_1$ of the same length.
3. A random bit $b \leftarrow \{0,1\}$ is chosen, and then a challenge ciphertext $c \leftarrow Enc_k(m_b)$ is computed and given to $A$.
4. The adversary $A$ continues to have oracle access to $Enc_k(\cdot)$, and outputs a bit $b'$.
5. The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.
Recall: CPA-Security

Definition: A private-key encryption scheme $\Pi = (Gen, Enc, Dec)$ has indistinguishable encryptions under a chosen-plaintext attack if for all ppt adversaries $A$ there exists a negligible function $negl$ such that

$$\Pr \left[ PrivK_{A,\Pi}^{cpa}(n) = 1 \right] \leq \frac{1}{2} + negl(n),$$

where the probability is taken over the random coins used by $A$, as well as the random coins used in the experiment.
Security Analysis

Let $A$ be a ppt adversary trying to break the security of the construction. We construct a distinguisher $D$ that uses $A$ as a subroutine to break the security of the PRF.

Distinguisher $D$:

$D$ gets oracle access to oracle $O$, which is either $F_k$, where $F$ is pseudorandom or $f$ which is truly random.

1. Instantiate $A^{Enc_k(\cdot)}(1^n)$.
2. When $A$ queries its oracle, with message $m$, choose $r$ at random, query $O(r)$ to obtain $z$ and output $c := \langle r, z \oplus m \rangle$.
3. Eventually, $A$ outputs $m_0, m_1 \in \{0,1\}^n$.
4. Choose a uniform bit $b \in \{0,1\}$. Choose $r$ at random, query $O(r)$ to obtain $z$ and output $c := \langle r, z \oplus m \rangle$.
5. Give $c$ to $A$ and obtain output $b'$. Output 1 if $b' = b$, and output 0 otherwise.
Security Analysis

Consider the probability $D$ outputs 1 in the case that $O$ is truly random function $f$ vs. $O$ is a pseudorandom function $F_k$.

- When $O$ is pseudorandom, $D$ outputs 1 with probability $\Pr[\text{PrivK}_{A,\Pi}^{\text{cpa}}(n) = 1] = \frac{1}{2} + \rho(n)$, where $\rho$ is non-negligible.
- When $O$ is random, $D$ outputs 1 with probability at most $\frac{1}{2} + \frac{q(n)}{2^n}$, where $q(n)$ is the number of oracle queries made by $A$. Why?
Security Analysis

$D$’s distinguishing probability is:

$$\left| \frac{1}{2} + \frac{q(n)}{2^n} - \left( \frac{1}{2} + \rho(n) \right) \right| = \rho(n) - \frac{q(n)}{2^n}.$$

Since, $\frac{q(n)}{2^n}$ is negligible and $\rho(n)$ is non-negligible, $\rho(n) - \frac{q(n)}{2^n}$ is non-negligible.

This is a contradiction to the security of the PRF.
Block Ciphers/Pseudorandom Permutations

Definition: Pseudorandom Permutation is exactly the same as a Pseudorandom Function, except for every key $k$, $F_k$ must be a permutation and it must be indistinguishable from a random permutation.
Definition: Let $F: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$ be an efficient, length-preserving, keyed permutation. We say that $F$ is a strong pseudorandom permutation if for all ppt distinguishers $D$, there exists a negligible function $negl$ such that:

$$\Pr[D^{F_k(.)}, F^{-1}_k(.) (1^n) = 1] - \Pr[D^{f(.)}, f^{-1}(.) (1^n) = 1] \leq negl(n).$$

where $k \leftarrow \{0,1\}^n$ is chosen uniformly at random and $f$ is chosen uniformly at random from the set of all permutations mapping $n$-bit strings to $n$-bit strings.
Modes of Operation—Stream Cipher

If sender and receiver are willing to maintain state, can encrypt multiple messages.
Modes of Operation—Block Cipher

**FIGURE 3.5:** Electronic Code Book (ECB) mode.

**FIGURE 3.6:** An illustration of the dangers of using ECB mode. The middle figure is an encryption of the image on the left using ECB mode; the figure on the right is an encryption of the same image using a secure mode.

**FIGURE 3.7:** Cipher Block Chaining (CBC) mode.
Modes of Operation—Block Cipher

**FIGURE 3.9:** Output Feedback (OFB) mode.

**FIGURE 3.10:** Counter (CTR) mode.