1. Before HMAC was invented, it was quite common to define a MAC by \( \text{Mac}_k(m) = H^*(k||m) \) where \( H \) is a collision-resistant hash function. Show that this is not a secure MAC when \( H \) is constructed via the Merkle-Damgard transform.

2. For each of the following modifications to the Merkle-Damgard transform, determine whether the result is collision resistant. If yes, provide a proof; if not, demonstrate an attack.

   **Hint.** For both parts below, if your answer is that it is insecure, you should design a specific compression function \( h \) which is collision-resistant, but for which the Merkle-Damgard transform is not collision resistant.

   (a) Modify the construction so that the input length is not included at all (i.e., output \( z_B \) and not \( z_{B+1} = h^*(z_B||L) \)). (Assume the resulting hash is only defined for inputs whose length is an integer multiple of the block length.)

   (b) Modify the construction so that instead of outputting \( z = h^*(z_B||L) \), the algorithm outputs \( z_B||L \).

3. Generalize the Merkle-Damgard construction for any compression function that compresses by at least one bit. You should refer to a general input length \( \ell' \) and general output length \( \ell \) (with \( \ell' > \ell \)).

4. Let \((\text{Gen}; H)\) be a collision-resistant hash function and let \( F \) be a PRF. For each of the following, state whether \( \hat{H} \) is necessarily collision resistant. Justify your answer.

   (a) \( \hat{H}^* := H^*(x_1)||H^*(x_2) \).

   (b) \( \hat{H}^* := H^*(x_1 \oplus x_2) \).