Cryptography

Lecture 2

Announcements

- HW1 due on Wed, 2/8 at beginning of class
- Discrete Math Readings/Quizzes on Canvas due on Friday, 2/10 at 11:59pm

Agenda

- Last time:
  - Historical ciphers and their cryptanalysis (K/L 1.3)
- This time:
  - More cryptanalysis (K/L 1.3)
  - Discussion on defining security
  - Basic terminology
  - Formal definition of symmetric key encryption (K/L 2.1)
  - Information-theoretic security (K/L 2.1)

Shift Cipher

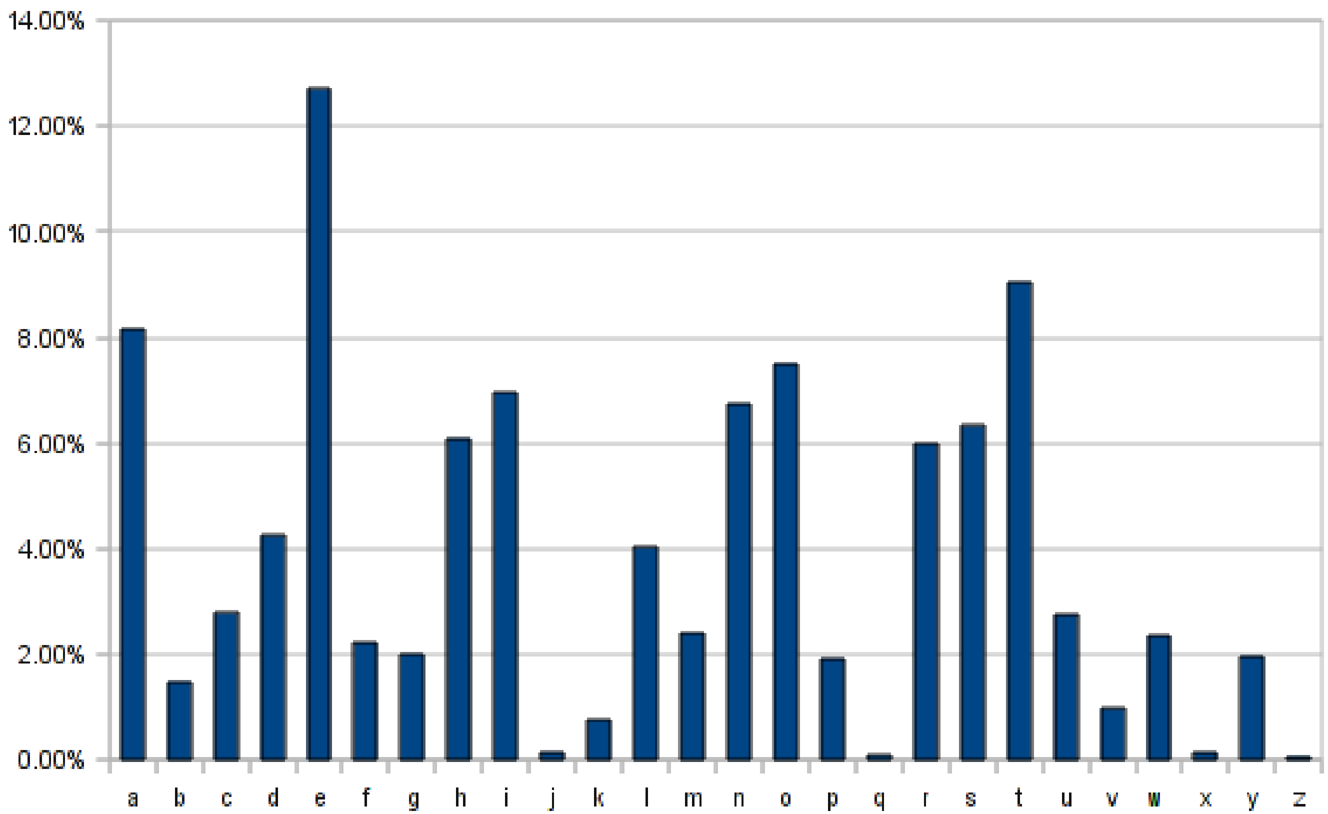
- For $0 \leq i \leq 25$, the ith plaintext character is shifted by some value $0 \leq k \leq 25\ (mod\ 26)$.
  - E.g. k = 3

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

goodmorning → **JRRGPRUQLQJ**

Frequency Analysis

If plaintext is known to be grammatically correct English, can use frequency analysis to break monoalphabetic substitution ciphers:



An Improved Attack on Shift/Caesar Cipher using Frequency Analysis

- Associate letters of English alphabet with numbers $0\ldots25$
- Let p_i denote the probability of the i-th letter in English text.
- Using the frequency table:

$$\sum_{i=0}^{25} {p_i}^2 \approx 0.065$$

- Let q_i denote the probability of the i-th letter in this ciphertext: # of occurrences/length of ciphertext
- Compute $I_j = \sum_{i=0}^{25} p_i \cdot q_{i+j}$ for each possible shift value j
- Output the value k for which I_k is closest to 0.065.

Vigenere Cipher (1500 A.D.)

- Poly-alphabetic shift cipher: Maps the same plaintext character to different ciphertext characters.
- Vigenere Cipher applies multiple shift ciphers in sequence.
- Example:

Plaintext:	t	e	l	l	h	i	m	a	b	o	u	t	m	e
Key:	c	a	f	e	c	a	f	e	c	a	f	e	c	a
Ciphertext:	V	E	Q	P	J	I	R	E	D	O	Z	X	O	E

Breaking the Vigenere cipher

- Assume length of key t is known.
- Ciphertext $C = c_1, c_2, c_3, \ldots$
- Consider sequences
  - $c_1, c_{1+t}, c_{1+2t}, \cdots$
  - $c_2, c_{2+t}, c_{2+2t}, \cdots$
  - . .
- For each one, run the analysis from before to determine the shift k_j for each sequence j.

Index of Coincidence Method

- How to determine the key length?
- Consider the sequence: $c_1, c_{1+t}\ c_{1+2t}, \ldots$ where t is the true key length
- We expect $\sum_{i=0}^{25} q_i{}^2 \approx \sum_{i=0}^{25} p_i{}^2 \approx 0.065$
- To determine the key length, try different values of τ and compute $S_\tau = \sum_{i=0}^{25} q_i{}^2$ for subsequence $c_1, c_{1+\tau}, c_{1+2\tau}, \cdots$
- When $\tau = t$, we expect S_τ to be ≈ 0.065
- When $\tau \neq t$, we expect that all characters will occur with roughly the same probability so we expect S_τ to be $\approx \frac{1}{26} \approx 0.038$.

What have we learned?

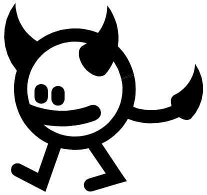
- Sufficient key space principle:
  - A secure encryption scheme must have a key space that cannot be searched exhaustively in a reasonable amount of time.
- Designing secure ciphers is a hard task!!
  - All historical ciphers can be completely broken.
- First problem: What does it mean for an encryption scheme to be secure?

Recall our setting

Sender

k

$c \leftarrow Enc_k(m)$



Receiver

k

c

$m = Dec_k(c)$

Coming up with the right definition

After seeing various encryption schemes that are clearly not secure, can we formalize what it means to for a private key encryption scheme to be secure?

Coming up with the right definition

First Attempt:

"An encryption scheme is secure if no adversary can find the secret key when given a ciphertext"

Problem: The aim of encryption is to protect the message, not the secret key.

Ex: Consider an encryption scheme that ignores the secret key and outputs the message.

Coming up with the right definition

Second Attempt:

"An encryption scheme is secure if no adversary can find the plaintext that corresponds to the ciphertext"

Problem: An encryption scheme that reveals 90% of the plaintext would still be considered secure as long as it is hard to find the remaining 10%.

Coming up with the right definition

Third Attempt:

“An encryption scheme is secure if no adversary learns meaningful information about the plaintext after seeing the ciphertext”

How do you formalize learns meaningful information?

Coming Up With The Right Definition

How do you formalize learns meaningful information?

Two ways:

- An information-theoretic approach of Shannon (next couple of lectures)
- A computational approach (the approach of modern cryptography)

New Topic:
Information-Theoretic Security

Probability Background

Terminology

- Discrete Random Variable: A discrete random variable is a variable that can take on a value from a finite set of possible different values each with an associated probability.
- Example: Bag with red, blue, yellow marbles. Random variable X describes the outcome of a random draw from the bag. The value of X can be either red, blue or yellow, each with some probability.

More Terminology

- **A discrete probability distribution** assigns a probability to each possible outcomes of a discrete random variable.
  - Ex: Bag with red, blue, yellow marbles.
- An **experiment** or **trial** (see below) is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes, known as the sample space.
  - Ex: Drawing a marble at random from the bag.
- An **event** is a set of outcomes of an experiment (a subset of the sample space) to which a probability is assigned
  - Ex: A red marble is drawn.
  - Ex: A red or yellow marble is drawn.

Formally Defining a Symmetric Key Encryption Scheme

Syntax

- An encryption scheme is defined by three algorithms
  - Gen, Enc, Dec
- Specification of message space $\boldsymbol{M}$ with $|\boldsymbol{M}| > 1$.
- Key-generation algorithm Gen:
  - Probabilistic algorithm
  - Outputs a key k according to some distribution.
  - Keyspace $\boldsymbol{K}$ is the set of all possible keys
- Encryption algorithm Enc:
  - Takes as input key $k \in \boldsymbol{K}$, message $m \in \boldsymbol{M}$
  - Encryption algorithm may be probabilistic
  - Outputs ciphertext $c \leftarrow Enc_k(m)$
  - Ciphertext space $\boldsymbol{C}$ is the set of all possible ciphertexts
- Decryption algorithm Dec:
  - Takes as input key $k \in \boldsymbol{K}$, ciphertext $c \in \boldsymbol{C}$
  - Decryption is deterministic
  - Outputs message $m \coloneqq Dec\_k(c)$

Distributions over K, M, C

- Distribution over $\boldsymbol{K}$ is defined by running Gen and taking the output.
  - For $k \in \boldsymbol{K}, \Pr[K = k]$ denotes the prob that the key output by Gen is equal to k.
- For $m \in \boldsymbol{M}, \Pr[M = m]$ denotes the prob. That the message is equal to m.
  - Models a priori knowledge of adversary about the message.
  - E.g. Message is English text.
- Distributions over $\boldsymbol{K}$ and $\boldsymbol{M}$ are independent.
- For $c \in \boldsymbol{C}, \Pr[C = c]$ denotes the probability that the ciphertext is c.
  - Given Enc, distribution over $\boldsymbol{C}$ is fully determined by the distributions over $\boldsymbol{K}$ and $\boldsymbol{M}$.