# Cryptography

Lecture 25

# Announcements

- HW9 due on 5/9

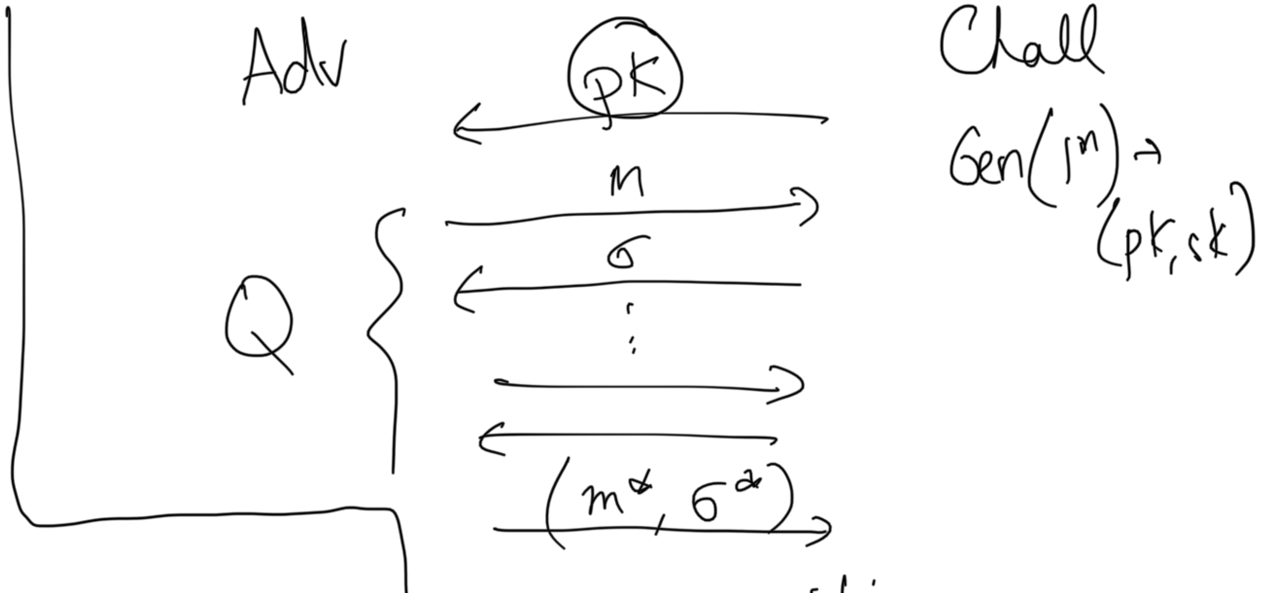
Digital Signatures as the public key analogue of MACs

(Gen, Sign, Vrfy)

$\text{Gen}(1^n) \to (pk, sk)$

$\text{Sign}_{sk}(m) \to \sigma$

$\text{Vrfy}_{pk}(m, \sigma) = 0/1$



Adv wins if:

1. $m^* \notin Q$
2. $\text{Vrfy}_{pk}(m^*, \sigma^*) = 1$

Correctness: $\forall pk, sk$ output by Gen, $\forall m$: $\text{Vrfy}_{pk}(m, \sigma \leftarrow \text{Sign}_{sk}(m)) = 1$

# Agenda

- Last time:
  - Diffie-Hellman Key Exchange
  - Public Key Encryption
  - ElGamal, Textbook RSA Encryption (11.5)

- This time:
  - Padded RSA Encryption
  - Digital Signatures Definitions (12.2-12.3)
  - RSA Signatures (12.4)

# Padded RSA

Bleichenbacher's Attack.

↓

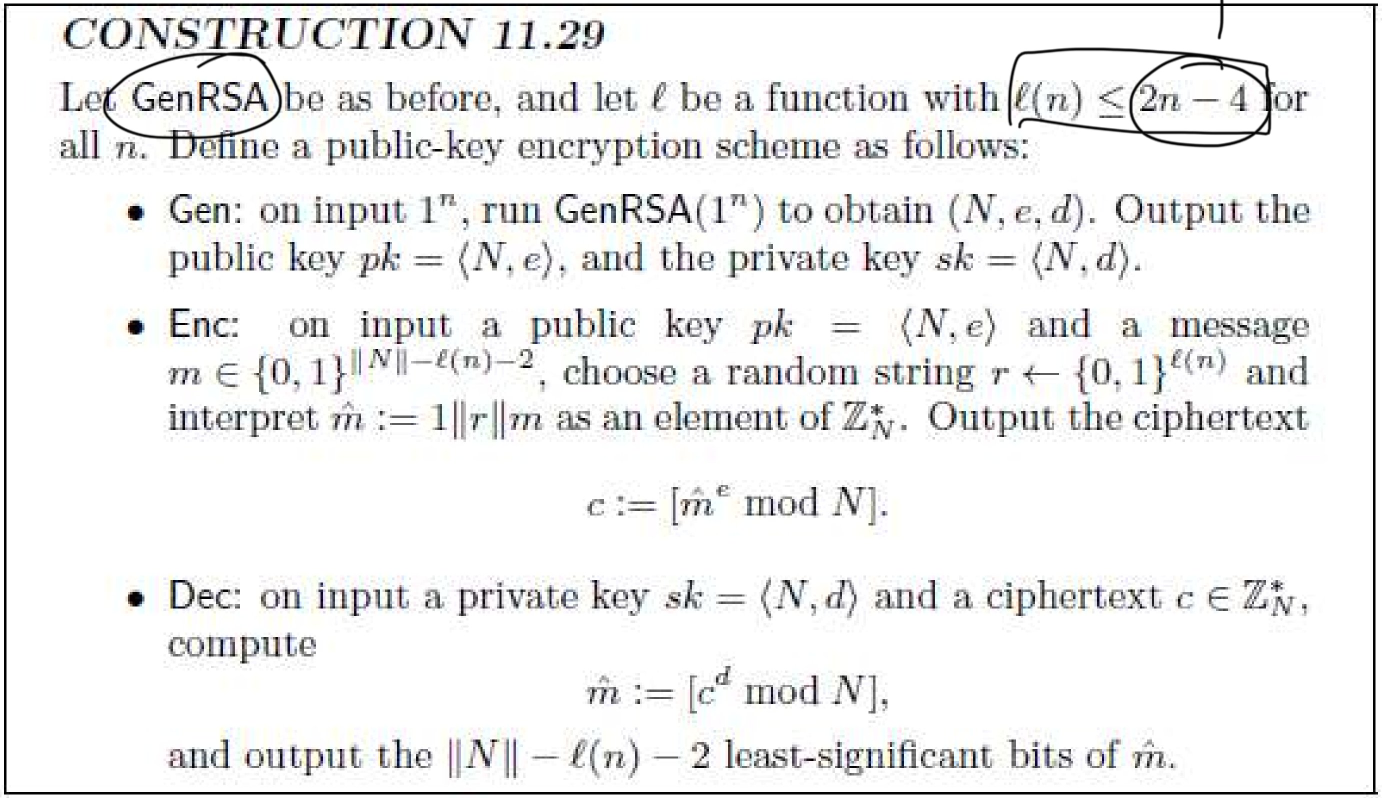
OAEP

$\boxed{N} = p \cdot q$

$\cup \; \cup$

n is the # of bits in prime.

$2^{2n-4}$

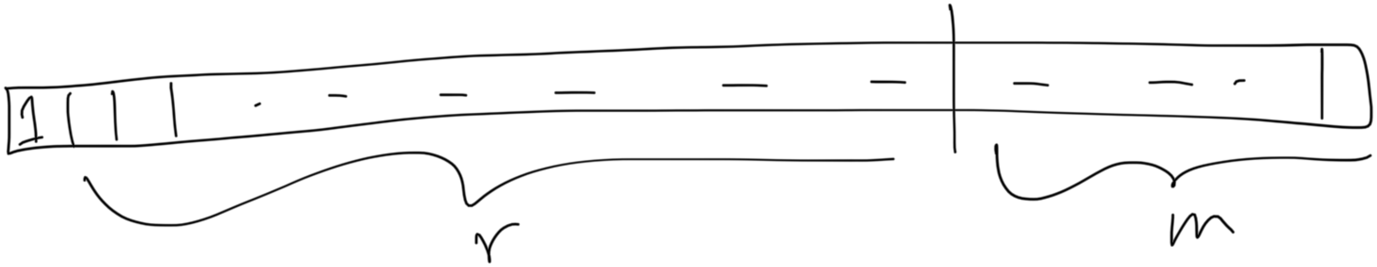


*CONSTRUCTION 11.29*

Let GenRSA be as before, and let $\ell$ be a function with $\ell(n) \leq 2n - 4$ for all $n$. Define a public-key encryption scheme as follows:

- Gen: on input $1^n$, run $\mathsf{GenRSA}(1^n)$ to obtain $(N, e, d)$. Output the public key $pk = \langle N, e \rangle$, and the private key $sk = \langle N, d \rangle$.
- Enc: on input a public key $pk = \langle N, e \rangle$ and a message $m \in \{0,1\}^{\|N\| - \ell(n) - 2}$, choose a random string $r \leftarrow \{0,1\}^{\ell(n)}$ and interpret $\hat{m} := 1 \| r \| m$ as an element of $\mathbb{Z}_N^*$. Output the ciphertext
$$c := [\hat{m}^e \bmod N].$$
- Dec: on input a private key $sk = \langle N, d \rangle$ and a ciphertext $c \in \mathbb{Z}_N^*$, compute
$$\hat{m} := [c^d \bmod N],$$
and output the $\|N\| - \ell(n) - 2$ least-significant bits of $\hat{m}$.

The padded RSA encryption scheme.



$2^{4000}$

# Digital Signatures Definition

A digital signature scheme consists of three ppt algorithms $(Gen, Sign, Vrfy)$ such that:

1. The key-generation algorithm $Gen$ takes as input a security parameter $1^n$ and outputs a pair of keys $(pk, sk)$. We assume that $pk, sk$ each have length at least $n$, and that $n$ can be determined from $pk$ or $sk$.
2. The signing algorithm $Sign$ takes as input a private key $sk$ and a message $m$ from some message space (that may depend on $pk$). It outputs a signature $\sigma$, and we write this as $\sigma \leftarrow Sign_{sk}(m)$.
3. The deterministic verification algorithm $Vrfy$ takes as input a public key $pk$, a message $m$, and a signature $\sigma$. It outputs a bit $b$, with $b = 1$ meaning valid and $b = 0$ meaning invalid. We write this as $b \coloneqq Vrfy_{pk}(m, \sigma)$.

Correctness: It is required that except with negligible probability over $(pk, sk)$ output by $Gen(1^n)$, it holds that $Vrfy_{pk}\big(m, Sign_{sk}(m)\big) = 1$ for every message $m$.

# Digital Signatures Definition: Security

Experiment $SigForge_{A,\Pi}(n)$:

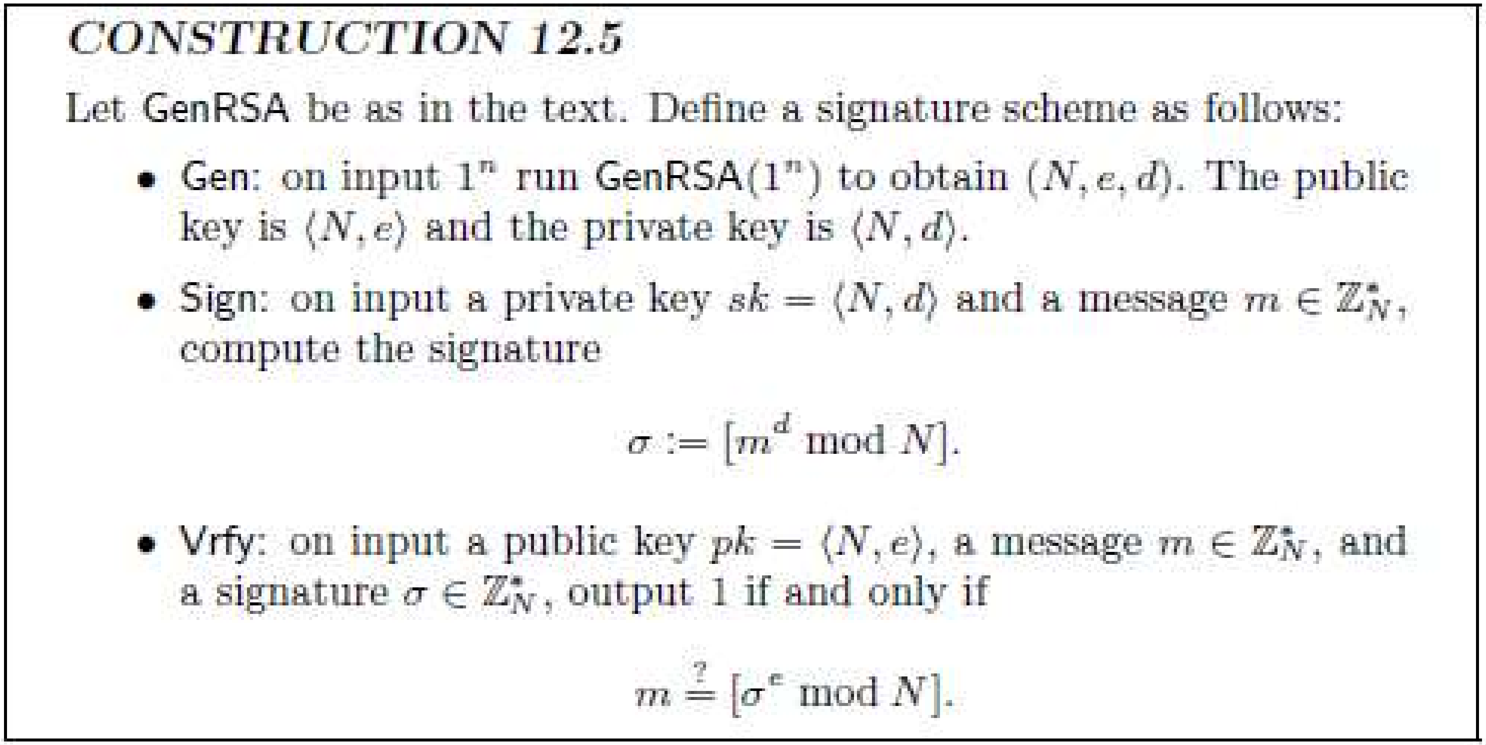
1. $Gen(1^n)$ is run to obtain keys $(pk, sk)$.
2. Adversary $A$ is given $pk$ and access to an oracle $Sign_{sk}(\cdot)$. The adversary then outputs $(m, \sigma)$. Let $Q$ denote the set of all queries that $A$ asked to its oracle.
3. $A$ succeeds if and only if
   1. $Vrfy_{pk}(m, \sigma) = 1$
   2. $m \notin Q$.

In this case the output of the experiment is defined to be 1.

Definition: A signature scheme $\Pi = (Gen, Sign, Vrfy)$ is existentially unforgeable under an adaptive chosen-message attack, if for all ppt adversaries $A$, there is a negligible function $neg$ such that:

$$\Pr\left[SigForge_{A,Pi}(n) = 1\right] \leq neg(n) .$$

# RSA Signatures



**CONSTRUCTION 12.5**

Let GenRSA be as in the text. Define a signature scheme as follows:

- Gen: on input $1^n$ run GenRSA$(1^n)$ to obtain $(N, e, d)$. The public key is $\langle N, e\rangle$ and the private key is $\langle N, d\rangle$.
- Sign: on input a private key $sk = \langle N, d\rangle$ and a message $m \in \mathbb{Z}_N^*$, compute the signature
$$\sigma := [m^d \bmod N].$$
- Vrfy: on input a public key $pk = \langle N, e\rangle$, a message $m \in \mathbb{Z}_N^*$, and a signature $\sigma \in \mathbb{Z}_N^*$, output 1 if and only if
$$m \overset{?}{=} [\sigma^e \bmod N].$$

The plain RSA signature scheme.

No message attack: pick $\sigma$ first $\quad (m, \sigma)$

set $m = \sigma^e \bmod N$

2-message attack: $\sigma_1 = m_1^d \bmod N \quad \sigma_2 = m_2^d \bmod N \quad (m_1 \cdot m_2, \sigma_1 \cdot \sigma_2)$

$\sigma_1 \cdot \sigma_2 = m_1^d \cdot m_2^d \bmod N = (m_1 \cdot m_2)^d \bmod N$

# Attacks

No message attack:

Choose $s \in Z_N^*$, compute $s^e$.

Ouput $(m = s^e, \sigma = s)$ as the forgery.

# Attacks

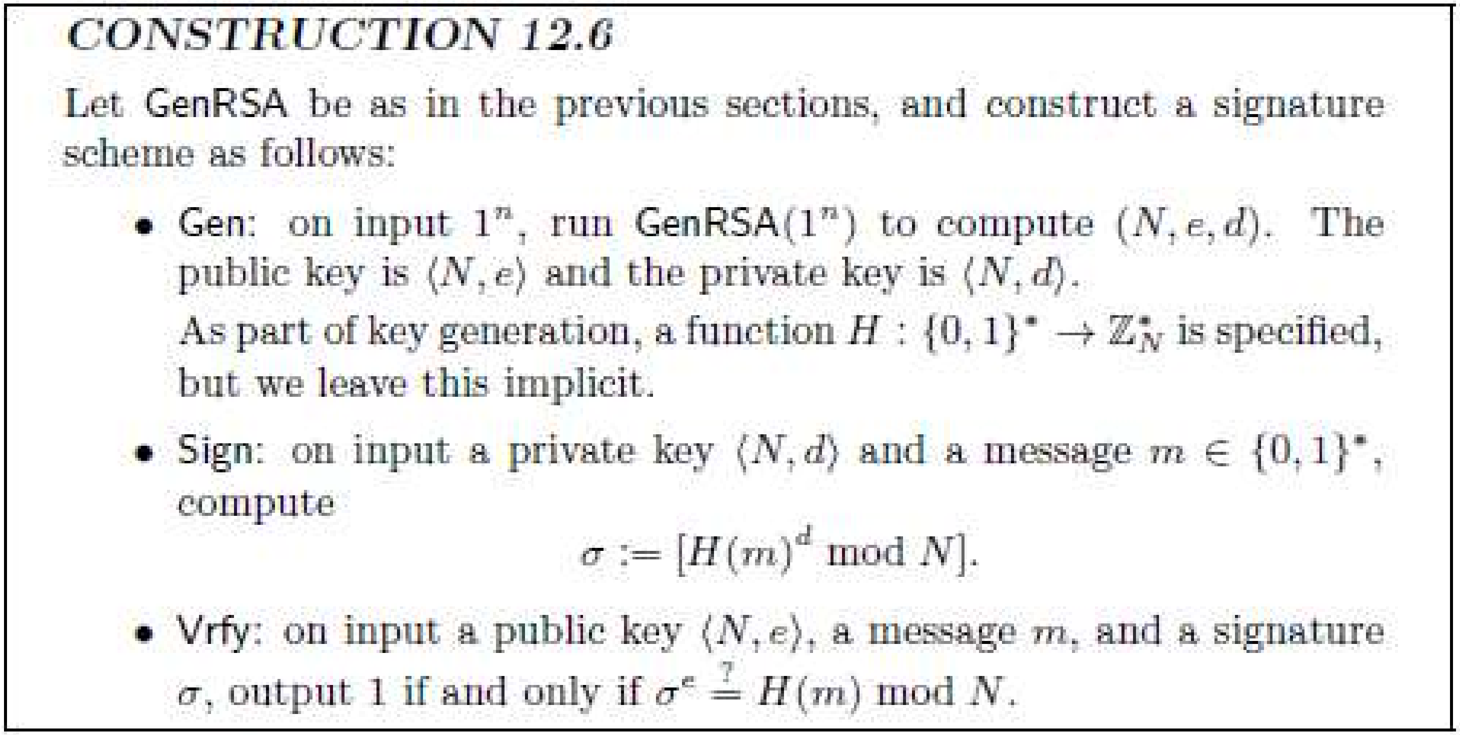
Forging a signature on an arbitrary message:

To forge a signature on message $m$, choose arbitrary $m_1, m_2 \neq 1$ such that $m = m_1 \cdot m_2$.

Query oracle for $(m_1, \sigma_1), (m_2, \sigma_2)$.

Output $(m, \sigma)$, where $\sigma = \sigma_1 \cdot \sigma_2$.

# RSA-FDH



***CONSTRUCTION 12.6***

Let GenRSA be as in the previous sections, and construct a signature scheme as follows:

- Gen: on input $1^n$, run GenRSA$(1^n)$ to compute $(N, e, d)$. The public key is $\langle N, e\rangle$ and the private key is $\langle N, d\rangle$.
  As part of key generation, a function $H : \{0,1\}^* \to \mathbb{Z}_N^*$ is specified, but we leave this implicit.
- Sign: on input a private key $\langle N, d\rangle$ and a message $m \in \{0,1\}^*$, compute
  $$\sigma := [H(m)^d \bmod N].$$
- Vrfy: on input a public key $\langle N, e\rangle$, a message $m$, and a signature $\sigma$, output 1 if and only if $\sigma^e \overset{?}{=} H(m) \bmod N$.

The RSA-FDH signature scheme.

$\sigma^e \bmod N \overset{?}{=} H(m)$ can't invert H to get m.

$\sigma_1^e \cdot \sigma_2^e = H(m_1) \cdot H(m_2) \neq H(m_1 \cdot m_2)$

previous attacks don't work

# Random Oracles

- Assume certain hash functions behave exactly like a random oracle.
- The “oracle” is a box that takes a binary string as input and returns a binary string as output.
- The internal workings of the box are unknown.
- All parties (honest parties and adversary) have access to the box.
- The box is consistent.
- Oracle implements a random function by choosing values of $H(x)$ “on the fly.”

# Principles of RO Model

1. If $x$ has not been queried to $H$, then the value of $H(x)$ is uniform.
2. If $A$ queries $x$ to $H$, the reduction can see this query and learn $x$.
3. The reduction can set the value of $H(x)$ to a value of its choice, as long as this value is correctly distributed, i.e., uniform.

# Security of RSA-FDH

Theorem: If the RSA problem is hard relative to $GenRSA$ and $H$ is modeled as a random oracle, then the construction above is secure.

# PKCS #1 v2.1

- Uses an instantiation of RSA-FDH for signing.
- SHA-1 should not be used “off-the-shelf” as an instantiation of $H$ because output length is too small and so practical short-message attacks apply.
- In PKCS #1 v2.1, $H$ is constructed via repeated application of an underlying cryptographic hash function.