Cryptography

Lecture 21

Announcements

- HW8 due 5/3

Agenda

- Last time:
  - Elliptic Curve Groups
  - Key Exchange Definitions (10.3)
- This time:
  - More on Key Exchange Definitions
  - Diffie-Hellman Key Exchange (10.3)
  - El Gamal Encryption (11.4)
  - RSA Encryption (11.5)

Key Agreement

The key-exchange experiment ${KE^{eav}}_{A,\Pi}(n)$:

1. Two parties holding 1^n execute protocol Π. This results in a transcript $trans$ containing all the messages sent by the parties, and a key k output by each of the parties.
2. A uniform bit $b \in \{0,1\}$ is chosen. If $b = 0$ set $\hat{k} \coloneqq k$, and if $b = 1$ then choose $\hat{k} \in \{0,1\}^n$ uniformly at random.
3. A is given $trans$ and $\hat{k}$, and outputs a bit b'.
4. The output of the experiment is defined to be 1 if $b' = b$ and 0 otherwise.

Definition: A key-exchange protocol Π is secure in the presence of an eavesdropper if for all ppt adversaries A there is a negligible function neg such that

$$\Pr\left[{KE^{eav}}_{A,\Pi}(n) = 1\right] \leq \frac{1}{2} + neg(n).$$

Key Agreement.

Key Establishment

Key Exchange

$\Rightarrow$ Missing authentication

Protocol Π

S ⇄ R

(K) (K')

m —— c ——→ $m = Dec_K(c)$

$c \leftarrow Enc_K(m)$

Eve

Adv

Chall

generate a transcript

$(trans, K)$

$b \xleftarrow{R} \{0,1\}$

←—— $(trans, K)$ —— If $b=0$

←—— $(trans, \tilde{K})$ —— If $b=1$

return b' —— b' ——→ ↳ unrelated to transcript, uniform random

Discussion of Definition

- Why is this the “right” definition?
- Why does the adversary get to see $\hat{k}$?

Diffie-Hellman Key Exchange

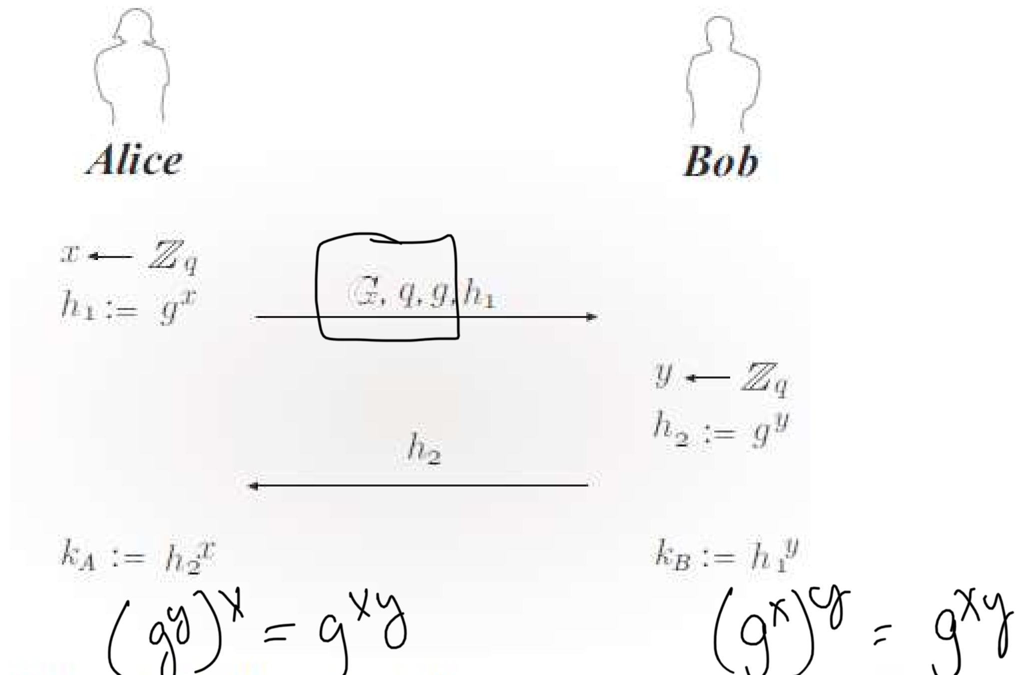


FIGURE 10.2: The Diffie-Hellman key-exchange protocol.

Recall DDH problem

We say that the DDH problem is hard relative to $\boldsymbol{G}$ if for all ppt algorithms A, there exists a negligible function neg such that

$$|\Pr[A(G, q, g, g^x, g^y, g^z) = 1] - \Pr[A(G, q, g, g^x, g^y, g^{xy}) = 1]| \leq neg(n).$$

Security Analysis

Theorem: If the DDH problem is hard relative to $\boldsymbol{G}$, then the Diffie-Hellman key-exchange protocol Π is secure in the presence of an eavesdropper.

Assuming the DHKE protocol is a secure Key Exchange

$\Longleftrightarrow$

DDH problem.

$(\text{trans}, K) = (g^x, g^y, g^{xy})$

$(\text{trans}, \tilde{K}) = (g^x, g^y, g^z)$

Distinguishing these is exactly the DDH problem

S R

$\xleftarrow{pk}$ $Gen(1^n) \to (sk, pk)$

$\boxed{m} = K$

$c \leftarrow Enc_{pk}(m)$ $\xrightarrow{c}$ $m = Dec_{sk}(c)$

$\boxed{K}$

Eve

Correctness

$\forall m, \forall (sk, pk) \leftarrow Gen(1^n):$

$Dec_{sk}(c \leftarrow Enc_{pk}(m)) = m$

CPA-Security for PKE

Adv Chall

$Gen(1^n) \to (pk, sk)$

replaced key having pk

~~missing oracle queries to Encryption~~

$\xleftarrow{pk}$

$\xrightarrow{m_0, m_1}$ $b \xleftarrow{R} \{0,1\}$

$c^* \leftarrow Enc_{pk}(m_b)$

$\xleftarrow{c^*}$

$\xrightarrow{b'}$

Adv wins if $b' = b$.

Public Key Encryption

Definition: A public key encryption scheme is a triple of ppt algorithms (Gen, Enc, Dec) such that:

1. The key generation algorithm Gen takes as input the security parameter 1^n and outputs a pair of keys (pk, sk). We refer to the first of these as the public key and the second as the private key. We assume for convenience that pk and sk each has length at least n, and that n can be determined from pk, sk.
2. The encryption algorithm Enc takes as input a public key pk and a message m from some message space. It outputs a ciphertext c, and we write this as $c \leftarrow Enc_{pk}(m)$.
3. The deterministic decryption algorithm Dec takes as input a private key sk and a ciphertext c, and outputs a message m or a special symbol $\perp$ denoting failure. We write this as $m \coloneqq Dec_{sk}(c)$.

Correctness: It is required that, except possibly with negligible probability over (pk, sk) output by $Gen(1^n)$, we have $Dec_{sk}\left(Enc_{pk}(m)\right) = m$ for any legal message m.

CPA-Security

The CPA experiment ${PubK^{cpa}}_{A,\Pi}(n)$:

1. $Gen(1^n)$ is run to obtain keys (pk, sk).
2. Adversary A is given pk, and outputs a pair of equal-length messages m_0, m_1 in the message space.
3. A uniform bit $b \in \{0,1\}$ is chosen, and then a challenge ciphertext $c \leftarrow Enc_{pk}(m_b)$ is computed and given to A.
4. A outputs a bit b'. The output of the experiment is 1 if $b' = b$, and 0 otherwise.

Definition: A public-key encryption scheme $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ is CPA-secure if for all ppt adversaries A there is a negligible function neg such that

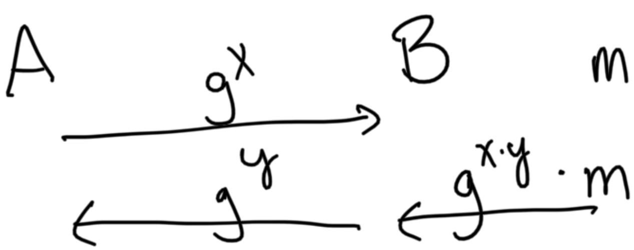
$$\Pr\left[{PubK^{cpa}}_{A,\Pi}(n) = 1\right] \leq \frac{1}{2} + neg(n)\,.$$

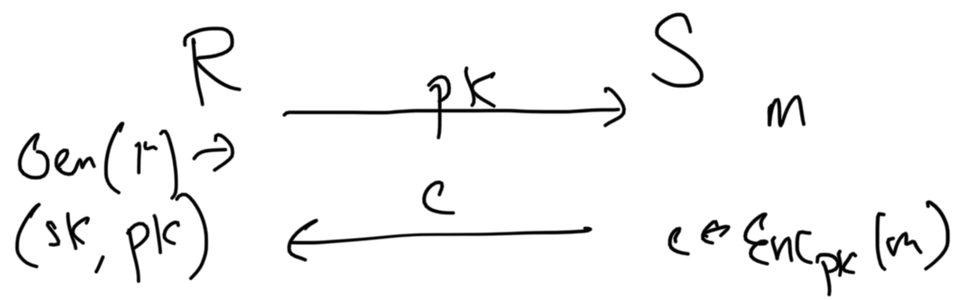
Discussion

- Discuss how in the public key setting security in the presence of an eavesdropper and CPA security are equivalent (since anyone can encrypt using the public key).
- Discuss how CPA-secure encryption cannot be deterministic!!
  - Why not?

El Gamal Encryption

--Show how we can derive El Gamal PKE from Diffie-Hellman Key Exchange





$g^{xy} \approx g^z$

secret key: x

public key: $h = g^x$

ciphertext: $(g^y, h^y \cdot m)$, where $c_1 = g^y$ and $c_2 = h^y \cdot m$

Decryption: $= (c_1^x)^{-1} \cdot c_2$ $[c_2 / c_1^x]$

Important Property

One time pad property for arbitrary groups

Lemma: Let G be a finite group, and let $m \in G$ be arbirary. Then choosing uniform $k \in G$ and setting $k' := k \cdot m$ gives the same distribution for k' as choosing uniform $k' \in G$. Put differently, for any $\hat{g} \in G$ we have

$$\Pr[k \cdot m = \hat{g}] = 1/|G|\,.$$

El Gamal Encryption Scheme

CONSTRUCTION 11.16

Let $\mathcal{G}$ be as in the text. Define a public-key encryption scheme as follows:

- Gen: on input 1^n run $\mathcal{G}(1^n)$ to obtain $(\mathbb{G}, q, g)$. Then choose a uniform $x \leftarrow \mathbb{Z}_q$ and compute $h := g^x$. The public key is $\langle \mathbb{G}, q, g, h \rangle$ and the private key is $\langle \mathbb{G}, q, g, x \rangle$. The message space is $\mathbb{G}$.
- Enc: on input a public key $pk = \langle \mathbb{G}, q, g, h \rangle$ and a message $m \in \mathbb{G}$, choose a uniform $y \leftarrow \mathbb{Z}_q$ and output the ciphertext

$$\langle g^y,\ h^y \cdot m \rangle.$$

- Dec: on input a private key $sk = \langle \mathbb{G}, q, g, x \rangle$ and a ciphertext $\langle c_1, c_2 \rangle$, output

$$\hat{m} := c_2 / c_1^x.$$

The El Gamal encryption scheme.

Security Analysis

Theorem: If the DDH problem is hard relative to $\mathcal{G}$, then the El Gamal encryption scheme is CPA-secure.

Textbook/Plain ✓

RSA Encryption

$Gen(1^n)$: $N = p \cdot q$, $\phi(N) = (p-1)(q-1)$, e, d s.t. $e \cdot d \equiv 1 \bmod \phi(N)$

$pk = (N, e)$ $sk = (N, d)$

$Enc_{pk}(m)$: $c = m^e \bmod N$

$Dec_{sk}(c)$: $m = c^d \bmod N$

Textbook/Plain **RSA Sig**

$\mathbb{Z}_N^*$ has order $\phi(N)$ $\forall x \in \mathbb{Z}_N^*$ $x^{\phi(N)} \underset{\bmod N}{=} 1$

$Vrfy_{pk}(m, \sigma)$: $m \stackrel{?}{=} \sigma^e \bmod N$

$Sign_{sk}(m)$: $\sigma = m^d \bmod N$

Textbook RSA Encryption

CONSTRUCTION 11.25

Let GenRSA be as in the text. Define a public-key encryption scheme as follows:

- Gen: on input 1^n run GenRSA(1^n) to obtain N, e, and d. The public key is $\langle N, e \rangle$ and the private key is $\langle N, d \rangle$.
- Enc: on input a public key $pk = \langle N, e \rangle$ and a message $m \in \mathbb{Z}_N^*$, compute the ciphertext
$$c := [m^e \bmod N].$$
- Dec: on input a private key $sk = \langle N, d \rangle$ and a ciphertext $c \in \mathbb{Z}_N^*$, compute the message
$$m := [c^d \bmod N].$$

The plain RSA encryption scheme.

Is Plain-RSA Secure?

- It is deterministic so cannot be secure!

Additional Attacks

We will look at additional attacks in one of the upcoming class exercises.

Abridged version: Assume $e = 3$

$$c = m^3 \bmod \underbrace{N}_{\approx 2^{4096}} \qquad 3 \cdot d \equiv 1 \bmod \phi(N)$$

$$m < 2^{\frac{4096}{3}} \quad \therefore \quad c = m^3 \qquad m = c^{1/3}$$