An Introduction to Lattice-Based Cryptography

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Traditional Crypto Assumptions

- Factoring: Given $N = pq$, find p, q
  - RSA Given $N = pq, e, x^e \ mod \ N$, find x.

- Discrete Log: Given $g^x \ mod \ p$, find x.
  - Diffie-Hellman Assumptions (g^x, g^y, g^{xy}), (g^x, g^y, g^z)

Are They Secure?

- Algorithmic Advances:
  - Factoring: Best algorithm time $2^{\tilde{O}(n^{\frac{1}{3}})}$ to factor n-bit number.
  - Discrete log: Best algorithm $2^{\tilde{O}(n^{\frac{1}{3}})}$ for groups Z_p^*, where p is n bits.
    - [Adrian et al. 2015] With preprocessing could possibly be feasible for nation-states and $n = 1024$.
    - Quasipolynomial time algorithms for small characteristic fields. Not known to apply in practice.

- Quantum Computers:
  - Shor's algorithm solves both factoring and discrete log in quantum polynomial time ($\tilde{O}(n^2)$).

Are They Secure?

"For those partners and vendors that have not yet made the transition to Suite B algorithms (ECC), we recommend not making a significant expenditure to do so at this point but instead to **prepare for the upcoming quantum resistant algorithm transition**.... Unfortunately, the growth of elliptic curve use has bumped up against the fact of continued progress in the research on quantum computing, necessitating a re-evaluation of our cryptographic strategy. "—NSA Statement, August 2015

NIST Kicks Off Effort to Defend Encrypted Data from Quantum Computer Threat

April 28, 2016

Google Dabbles in Post-Quantum Cryptography

By Richard Adhikari
Jul 12, 2016 2:06 PM PT

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Post-Quantum Approach

- New set of assumptions based on finding short vectors in lattices.
- Believed to be hard for quantum computers.
- Evidence of hardness "worst case to average case reduction".
- Versatile: Can essentially construct all cryptosystems out of these assumptions.

My Research

- New efficient cryptosystems from post-quantum and FHE assumptions [1], [7]
- Concrete hardness of post-quantum cryptosystems (with or without side information) [2], [3], [4], [5], [6], [8], [9]
- Concrete hardness of FHE (with or without side information) [10]

[1] Constant-Round Group Key-Exchange from the Ring-LWE Assumption. D. Apon, D. Dachman-Soled, H. Gong, J. Katz. PQCrypto 2019.

[2] LWE with Side Information: Attacks and Concrete Security Estimation. D. Dachman-Soled, L. Ducas, H. Gong, M. Rossi, CRYPTO 2020.

[3 Security of NewHope under Partial Key Exposure. D. Dachman-Soled, H. Gong, M. Kulkarni, A. Shahverdi. Research in Mathematics and Public Policy, 2020

[4] (In)Security of Ring-LWE Under Partial Key Exposure. D. Dachman-Soled, H. Gong, M. Kulkarni, A. Shahverdi. Journal of Mathematical Cryptology, 2020.

[5] Towards a Ring Analogue of the Leftover Hash Lemma. D. Dachman-Soled, H. Gong, M. Kulkarni, A. Shahverdi. Journal of Mathematical Cryptology, 2020.

[6] BKW Meets Fourier: New Algorithms for LPN with Sparse Parities. D. Dachman-Soled, H. Gong, H. Kippen, A. Shahverdi. TCC 2021

[7] Compressed Oblivious Encoding for Homomorphically Encrypted Search. S. G. Choi, D. Dachman-Soled, D. Gordon, L. Liu, A. Yerukhimovich. CCS 2021

[8] When Frodo Flips: End-to-End Key Recovery on FrodoKEM via Rowhammer. M. Fahr Jr., H. Kippen, A. Kwong, T. Dang, J. Lichtinger, D. Dachman-Soled, D. Genkin, A. Nelson, R. Perlner, A. Yerukhimovich, D. Apon. CCS 2022, RWC 2023

[9] Refined Security Estimation for LWE with Hints via a Geometric Approach. D Dachman-Soled, H Gong, T Hanson, H Kippen, Cryptology ePrint Archive

[10] Concrete Security Loss in IND-CPA^D Attacks on Approximate FHE (In preparation).

Math Prelim

Matrix Multiplication

$$\begin{matrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{matrix} \times \begin{matrix} v_{1,j} \\ v_{2,j} \\ v_{3,j} \end{matrix} = \sum_{i=1}^{3} v_{i,j} \cdot \begin{matrix} m_{1,i} \\ m_{2,i} \\ m_{3,i} \end{matrix}$$

$$\begin{matrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{matrix} \times \begin{matrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{matrix}:$$

For $j \in \{1,2,3\}$, j-th column of the output is computed as :

$$\sum_{i=1}^{3} v_{i,j} \cdot \begin{matrix} m_{1,i} \\ m_{2,i} \\ m_{3,i} \end{matrix}$$

Lattices

An n-dimensional lattice L is an additive discrete subgroup of R^n. A basis $\boldsymbol{B} \in R^{n \times n}$ defines a lattice $\mathrm{L}(\boldsymbol{B})$ in the following way:

$L(\boldsymbol{B}) = \{\boldsymbol{v} \in R^n \ s.t.\ \boldsymbol{v} = \boldsymbol{Bz}$ for some $\boldsymbol{z} \in Z^n\}$.

“integer linear combinations of the basis vectors”

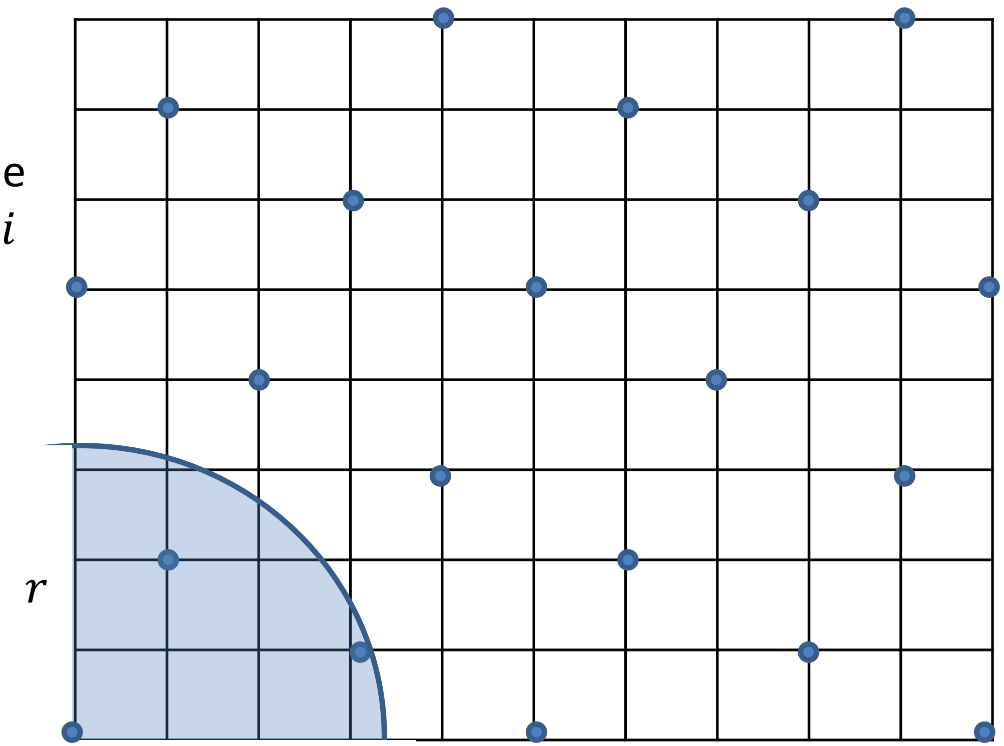
***i*-th successive minima $\boldsymbol{\lambda_i(L(B))}$**: The smallest radius r such that there are i linearly independent vectors $\{v_1, \dots, v_i\}$ of length at most r.

Shortest vector: (1,2)

$$\lambda_1 = \sqrt{5}$$

Shortest basis: $\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix}$

$$\lambda_2 = \sqrt{10}$$



Lattices

An n-dimensional lattice L is an additive discrete subgroup of R^n. A basis $\boldsymbol{B} \in R^{n \times n}$ defines a lattice $\mathrm{L}(\boldsymbol{B})$ in the following way:

$$L(\boldsymbol{B}) = \{\boldsymbol{v} \in R^n \ s.t. \ \boldsymbol{v} = \boldsymbol{Bz} \text{ for some } \boldsymbol{z} \in Z^n\}.$$

"integer linear combinations of the basis vectors"

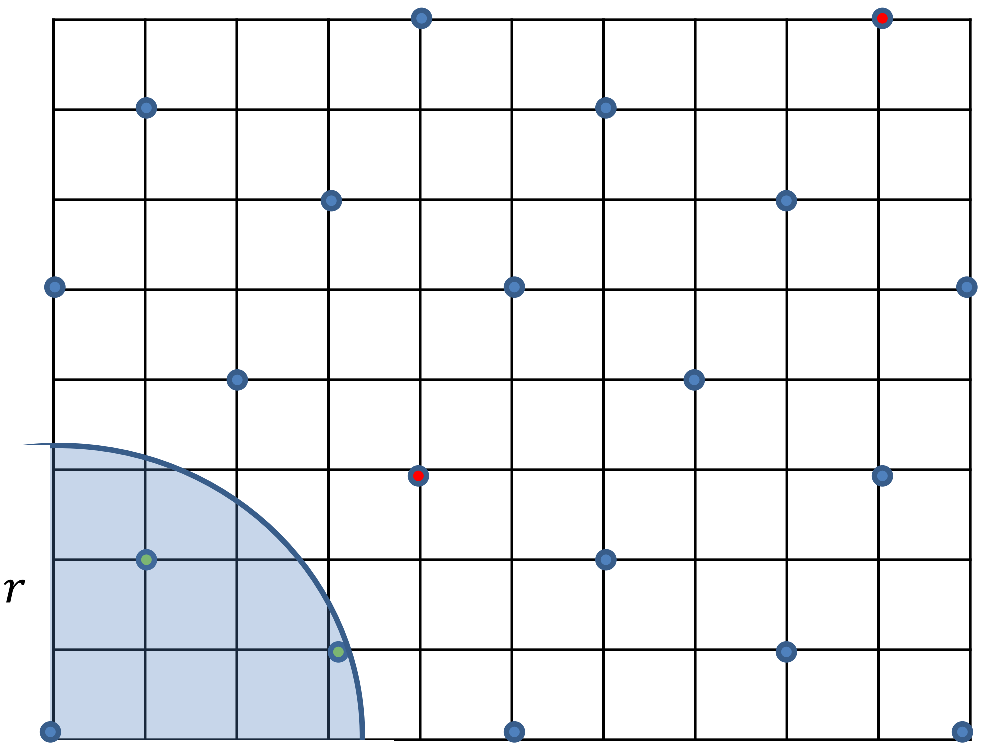
Basis is not unique!

For the lattice to the right,

$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix}$ form a basis

$\begin{matrix} 4 & 9 \\ 3 & 8 \end{matrix}$ also form a basis

Given two bases B, B', they define the same lattice iff $B' = BU$, where U is a r unimodular matrix (determinant ± 1).



Hard Lattice Problems

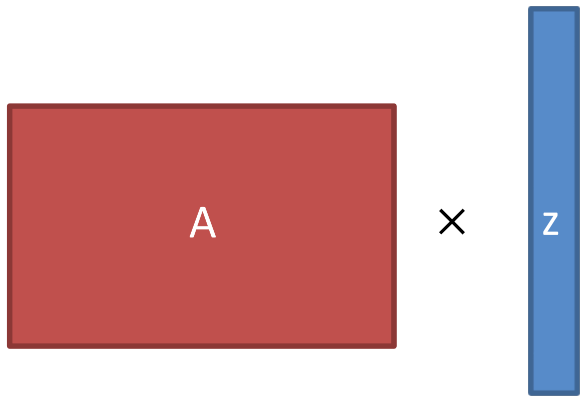
- Are all parameterized by "approximation factor" $\gamma > 1$.
- Shortest Vector Problem (**SVP**): Given a basis B, find a non-zero vector $\boldsymbol{v} \in L(\boldsymbol{B})$ whose length is at most $\gamma \cdot \lambda_1(L(\boldsymbol{B}))$.
- Shortest Independent Vector Problem (**SIVP**): Given a basis B, find a linearly independent set $\{v_1, \dots, v_n\}$ such that all vectors have length at most $\gamma \cdot \lambda_n(L(\boldsymbol{B}))$.
- Gap Shortest vector problem (**GapSVP**): Given a basis B, and a radius r > 0
  - Return YES if $\lambda_1\big(L(B)\big) \leq r$
  - Return NO if $\lambda_1\big(L(B)\big) > \gamma \cdot r$.

Believed hard even for a quantum computer!

Cryptographic Hard Problems

The SIS Problem

Dimension m



$$= \vec{0} \; mod \; p$$

Public $n \times m$ matrix A, with entries chosen at random over Z_p

Dimension n

$n \ll m$

Problem: Given A, find z $\in \{0,1\}^m$
(or sufficiently "short" z)

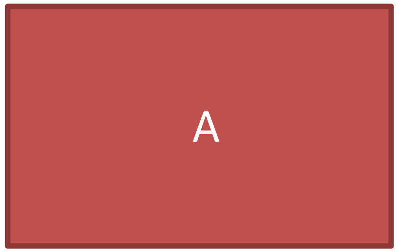
Relation to Lattices

- Worst-Case to Average-Case Reduction: Breaking the cryptosystem on average is as hard as breaking the hardest instance of the underlying lattice problem.
- SIS:
  - Worst-Case to Average-Case Reduction from SIVP.

CRHF from Lattices

CRHF from Lattices

Public Matrix:

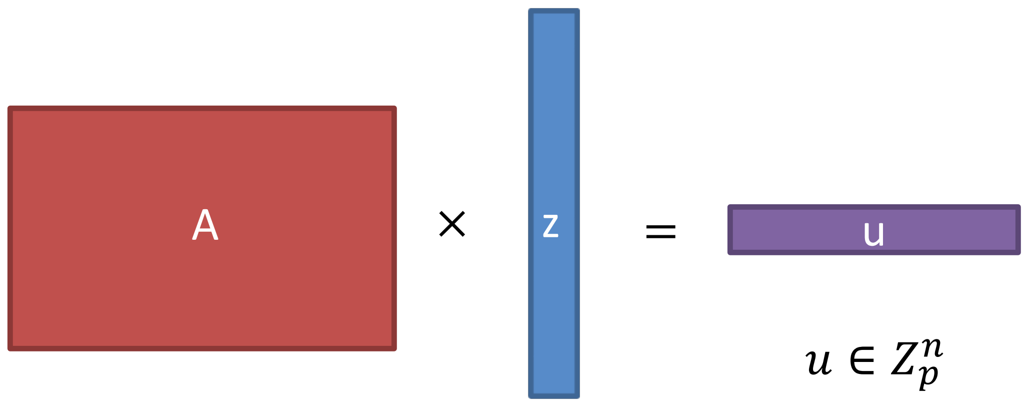


Input:

$z \in \{0,1\}^m$

Public $n \times m$ matrix A, with entries chosen at random over Z_p

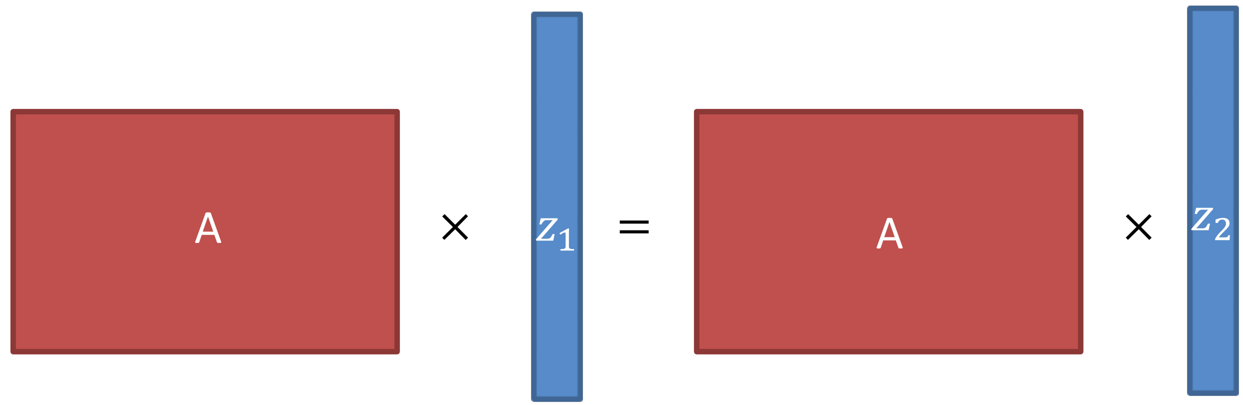
To evaluate the hash on z output:



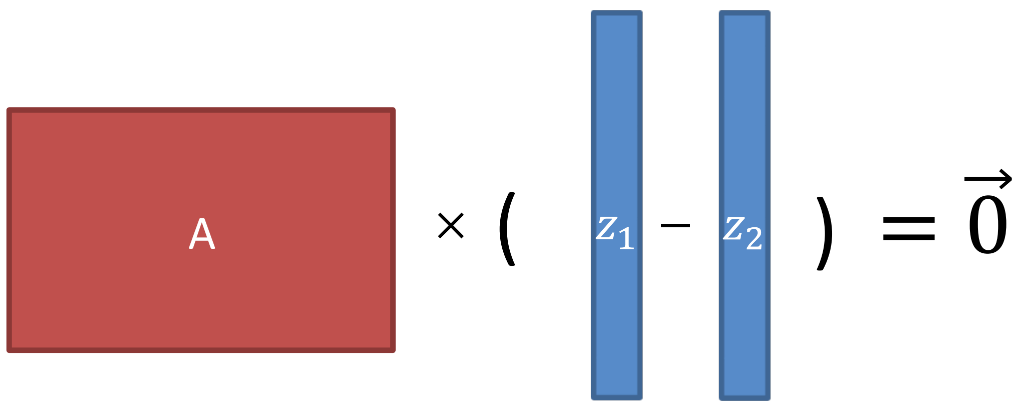
$u \in Z_p^n$

CRHF from Lattices

Given a collision $z_1, z_2 \in \{0,1\}^m$:



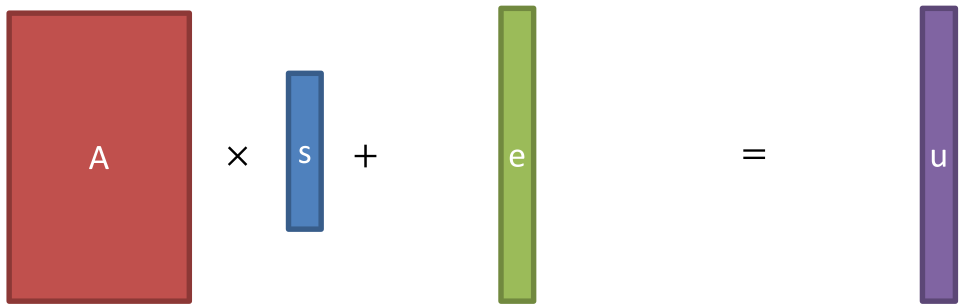
Obtain $(z_1 - z_2) \in \{-1,0,1\}^m$:



The LWE Problem (Search)

Secret n-dimension vector s with entries chosen at random

Operations are mod p.



Public $m \times n$ matrix A, with entries chosen at random over Z_p

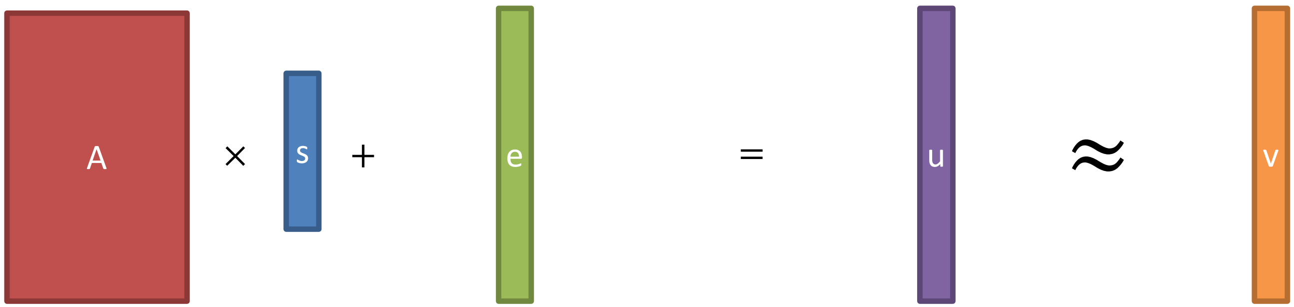
m-dimension error vector e, with entries sampled from χ.

Problem: Given, A, u = As+e, find s.

The LWE Problem (Decision)

Secret n-dimension vector s with entries chosen at random

Operations are mod p.



Public $m \times n$ matrix A, with entries chosen at random over Z_p

m-dimension error vector e, with entries sampled from χ.

Problem: Distinguish (A , u) from (A, v)

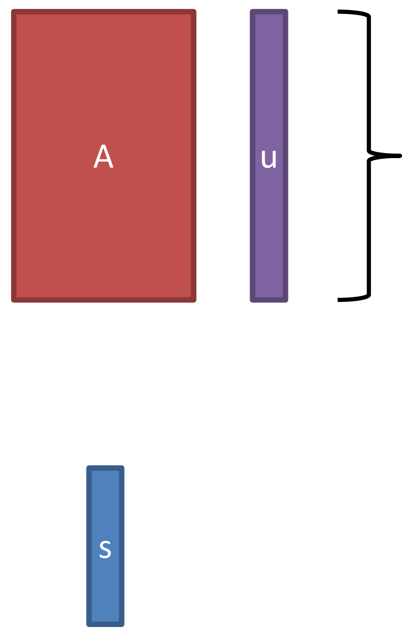
Relation to Lattices

- Worst-Case to Average-Case Reduction: Breaking the cryptosystem on average is as hard as breaking the hardest instance of the underlying lattice problem.
- LWE:
  - Worst-Case to Average-Case Quantum Reduction from SIVP.
  - Worst-Case to Average-Case Classical Reductions from GapSVP.

Lattice-Based Encryption

Regev's Cryptosystem [Regev '04]

Public Key:

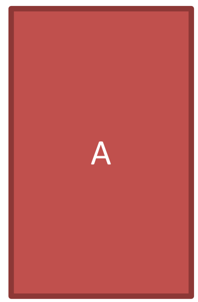


$u = As + e$

Secret Key:

Regev's Cryptosystem—Encryption of $m \in \{0,1\}$

(1) 

$\times$ 

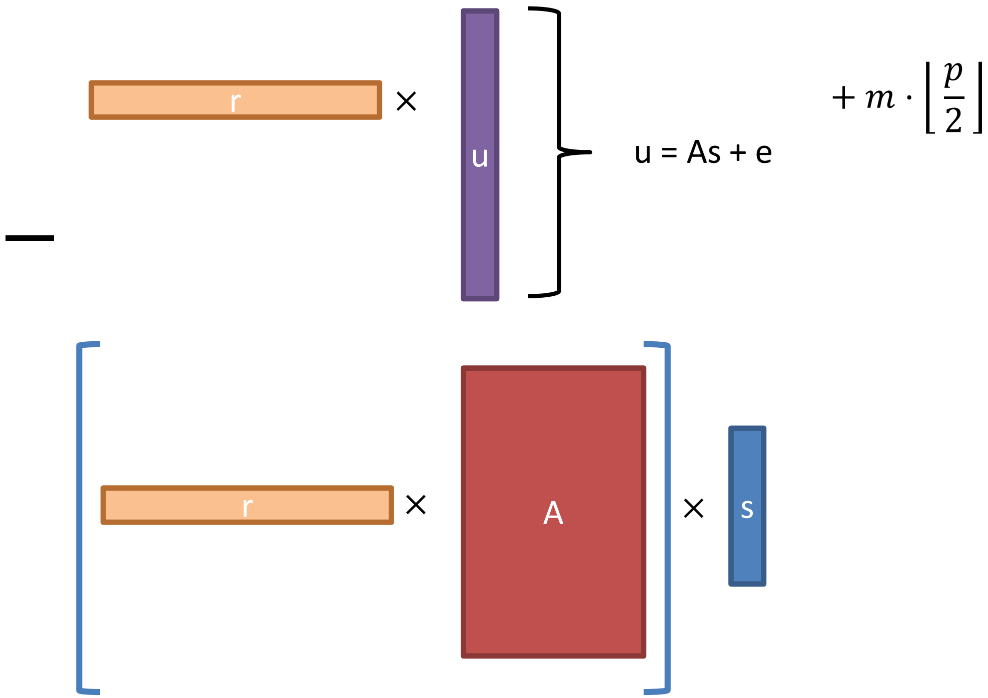
r $\in \{0,1\}^m$ chosen at random.

(2) 

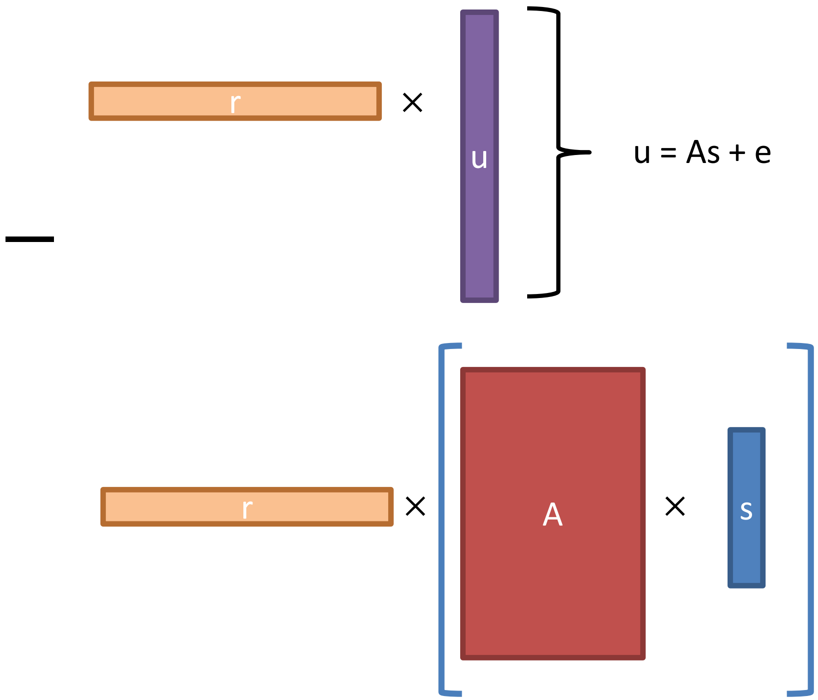
$\times$ 

$+ m \cdot \left\lfloor \frac{p}{2} \right\rfloor$

Regev's Cryptosystem—Decryption

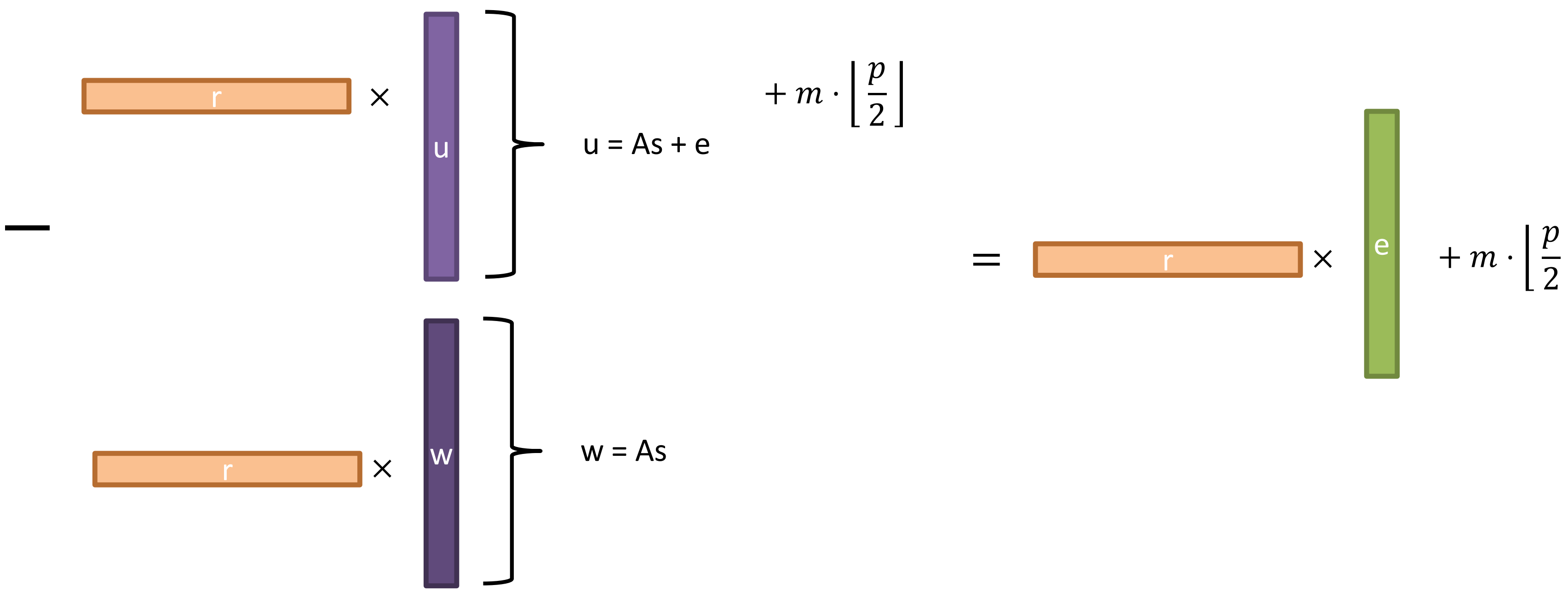


Regev's Cryptosystem—Decryption

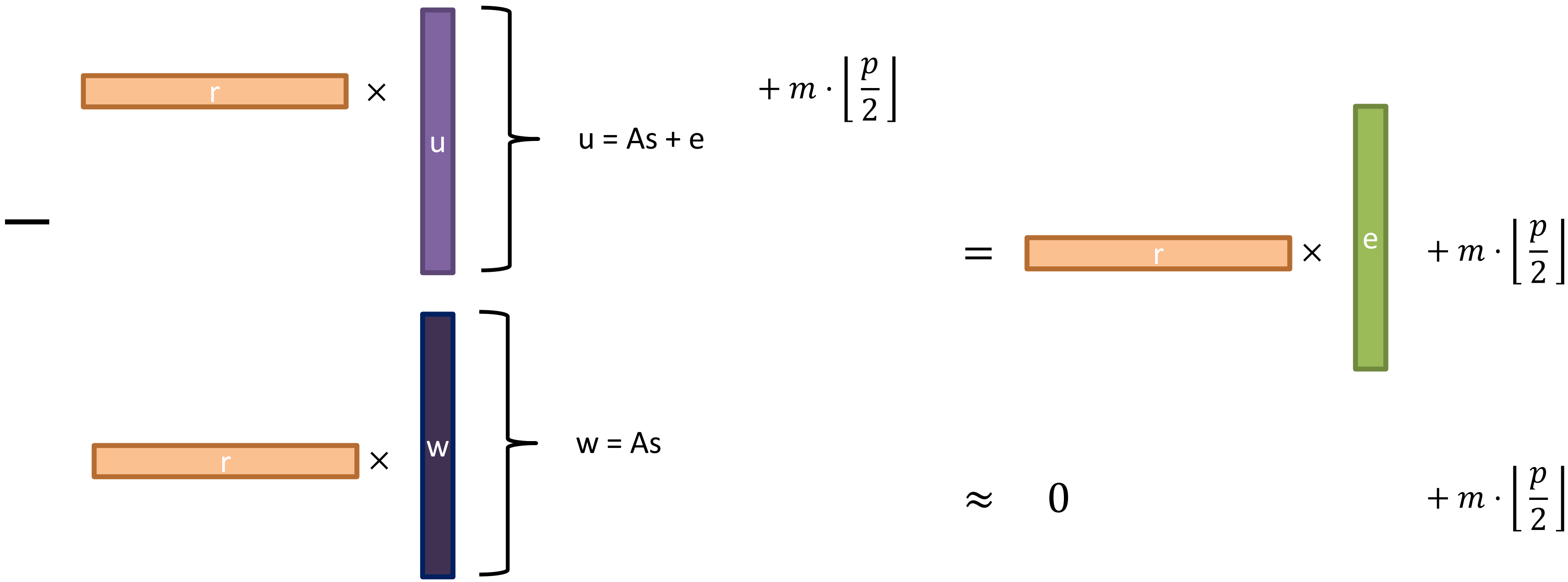


$$+ m \cdot \left\lfloor \frac{p}{2} \right\rfloor$$

Regev’s Cryptosystem—Decryption



Regev’s Cryptosystem—Decryption



Properties of LWE

- Equivalance of Search/Decision LWE
- Equivalence of LWE with random secret/secret drawn from error distribution

Efficiency

- Efficiency is a main concern in lattice-based cryptosystems.
- In both SIS and LWE-based cryptosystems, the public key consists of a random matrix of size $\mathrm{m} \times \mathrm{n}$ $(m \geq n \log p)$, requiring space $O(n^2 \log^2 p)$.
  - RSA and discrete-log based cryptosystems: public key size is linear in the security parameter.
- To reduce the public key size, consider lattices with structure.
- This is the Ring-LWE setting.

Ring-LWE Setting

- Highly efficient key exchange protocols are possible in the Ring-LWE setting.
  - Similar to Diffie-Hellman Key Exchange
- It is likely that at least one such scheme will be standardized by NIST.
- Details in the slides, but will skip in the lecture.

Summary

- Lattice-based cryptography is a promising approach for efficient, post-quantum cryptography.
- All the basic public key primitives can be constructed from these assumptions:
  - Public key encryption, Key Exchange, Digital Signatures
- For more information on research projects, please contact me at: danadach@umd.edu

Thank you!

The Ring Setting

- Quotient ring $\mathrm{Z}_q[x]/\Phi_m(x)$, where Φ_m is the m-th cyclotomic polynomial of degree $\varphi(m)$
  – e.g.,$\Phi_{2n} = x^n + 1, n = 2, q = 13.$
  – $x^2 = -1\ mod\ (x^2 + 1)$
  – $12x^3 + 15x^2 + 9x + 25 \rightarrow 12x^3 + 2x^2 + 9x + 12 \rightarrow x - 2 + 9x + 12 \rightarrow (10{,}10).$
- Lattice is defined as an ideal $I \subseteq Z[x]/\Phi_m(x)$.
- Ring-LWE and ring-SIS problems are defined by substituting the matrix A with polynomials from the quotient ring and substituting polynomial multiplication for matrix-vector multiplication.
- The public key is now a polynomial in $\mathrm{Z}_q[x]/\Phi_m(x)$, and so can be described using $O(n \log q)$ bits.

NTT Transform

Consider Φ_m, where m is a power of 2. Then degree is equal to n, power of 2, $m = 2n$. $\Phi_{2n} = x^n + 1$

- Consider prime q s.t. $q = 1 \; mod \; 2n$.
- Then we have n $2n$-th primitive roots modulo q
  - Why? Z_q^* is cyclic with order $q - 1$. $2n \mid (q - 1)$.
  - Let g be a generator of Z_q^*. g is a $(q - 1)$-th primitive root.
  - $g^{a \cdot 2n} = g^{q-1}$, since $2n \mid (q - 1)$. g^a is a $2n$-th primitive root. Also $(g^a)^i$, where i is relatively prime to $2n$.
  - Note that $(g^a)^n = -1 \; mod \; q$. Modulo $x^n + 1$ means $x^n = -1$.
  - Let $\gamma_1, \dots, \gamma_n$ be the n number of $2n$-th primitive roots
- For a polynomial $p(x) \in Z_q[x]/x^n$+1
- For every $\gamma_i, p(\gamma_i) \; mod \; p$ is equal to taking $p(x)$ modulo $x^n + 1$ and modulo q and then evaluating the reduced polynomial at γ_i.

NTT Transform

- For a polynomial $p(x) \in Z_q[x]/x^n$+1
- Evaluate $p(x)$ on all n number of $2n$-th primitive roots. Obtain a vector $p(\gamma_1) \ldots p(\gamma_n)$.
- Can now do both addition and multiplication coordinate-wise.

Key Exchange from Ring-LWE

Simple Key Exchange

P_1 $\qquad\qquad\qquad\qquad\qquad\qquad$ P_2

s_1 $\xrightarrow{(a,u_1 \;=\; a \cdot\, s_1 \;+\; e_1)}$ s_2

$\xleftarrow{(a,u_2 \;=\; a \cdot\, s_2 \;+\; e_2)}$

RECONCILIATION

$u_2 \cdot\, s_1 \approx a \cdot\, s_2 \cdot\, s_1$

$u_1 \cdot\, s_2 \approx a \cdot\, s_1 \cdot\, s_2$