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UNIQUENESS AND STABILITY OF DISCRETE ZERO-CROSSINGS AND MAXIMA WAVELET REPRESENTATIONS

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A general signal description, called an inherently bounded Adaptive Quasi Linear Representation (AQLR), motivated by two important examples, namely the wavelet maxima representation, and the wavelet zero-crossings representation, is introduced. It is shown, that the dyadic wavelet maxima (zero-crossings) representation is, in general, nonunique. Using the idea of the inherently bounded AQLR, a global BIBO stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied.

1 Introduction[1]

S. Mallat in [4] and, together with Zhong, in [5], introduced zero-crossings and extrema of the wavelet transform as a multiscale edge representation. Two important advantages of these methods are low algorithmic complexity and flexibility in choosing the basic filter. Moreover, [4] and [5] propose reconstruction procedures and show accurate numerical reconstruction results from zero-crossings and maxima representations. In [4, 5], as in many other works in this area, the basic algorithms were developed using continuous variables. The continuous approach gives an excellent background to motivate and justify the use of either local extrema or zero-crossings as important signal features. Unfortunately, in the continuous framework, analytic tools to investigate the information content of the representation are not yet available. The knowledge about properties of the representations is mainly based on empirical reconstruction results. From the theoretical point of view, there are still important open problems, e.g. stability, uniqueness, and structure of a reconstruction set (a family of signals having the same representation).

Our objective is to analyze these theoretical questions using a model of an actual implementation. The main assumption is that the data is discrete and finite. The discrete multiscale maxima and zero-crossings representations are defined in a general set-up of a linear filter bank, however, the main goal is to consider a particular case where the filter bank describes the wavelet transform. Since reconstruction sets of both maxima and zero-crossings representations have a similar structure, a general form called the Adaptive Quasi Linear Representation (AQLR) is introduced. Moreover, many generalizations of the basic maxima and zero-crossings representations fit into the framework of the AQLR.

We first present conditions for uniqueness, then apply these conditions to the wavelet transform-based representation, and then obtain a conclusive result. It turns out, that neither the wavelet maxima representation nor the wavelet zero-crossings representation is, in general, unique.

The next subject is stability of the representation. This issue is of great importance because there are many known examples of unstable zero-crossings representations. In order to improve the stability properties, Mallat has included additional sums in the standard zero-crossings representation and, together with Zhong, they have introduced the wavelet maxima representation, as a stable alternative to the zero-crossings representation. Using the idea of the inherently bounded AQLR, global BIBO (bounded input, bounded output) stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied.

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2 Multiscale Maxima Representation

Consider $\mathcal{L}$, a linear space of real, finite sequences:

$$\mathcal{L} \triangleq \left\{f : f = \{f(n)\}_{n=0}^{N-1} \;\; f(n) \in \Re\right\}.$$

Let X and Y denote operators on $\mathcal{L}$ which provide the sets of local maxima and minima, respectively, of a sequence $f \in \mathcal{L}$. The formal definitions are :

$$Xf = \{k : f(k+1) \leq f(k) \text{ and } f(k-1) \leq f(k)\}$$

$$Yf = \{k : f(k+1) \geq f(k) \text{ and } f(k-1) \geq f(k)\}$$

In this work, in order to avoid boundary problems, an N-periodic extension of finite sequences is assumed.

Let $W_1, W_2, \ldots, W_J, S_J$ be linear operators on $\mathcal{L}$. The multiscale local extrema representation, $R_m f$ is defined as:

$$R_m f \triangleq \left\{\left\{XW_jf, YW_jf, \{W_jf(k)\}_{k\in EW_jf}\right\}_{j=1}^{J}, S_Jf\right\}.$$

where $EW_jf = XW_jf \cup YW_jf$.

Following [5], $R_m f$, will be called the multiscale maxima representation as well. In the particular case, when $W_1, W_2, \ldots, W_J, S_J$ correspond to a wavelet transform, $R_m f$ will be called the wavelet maxima representation.

The determination of the extrema point sets is highly nonlinear. However, for the given extrema sets, XW_jf and YW_jf, the remaining data, called the sampling information, are obtained by a linear operation of sampling an image of a linear operator at fixed points.

Let T_{mf} denote the linear operator associated with the sampling information. Then, $R_m f$ is written in an alternative way as:

$$R_m f = \left\{\{XW_jf, YW_jf\}_{j=1}^{J}, T_{mf}f\right\}. \qquad (1)$$

For a given representation Rf, a reconstruction set $\Gamma(Rf)$ is defined as a set of all sequences satisfying this representation, i.e.

$$\Gamma(Rf) \triangleq \{\gamma \in \mathcal{L} : R\gamma = Rf\}. \qquad (2)$$

It is clear that in order to satisfy a given maxima representation, a sequence $h \in \mathcal{L}$, in addition to obeying the sampling information $T_{mf}h = T_{mf}f$, needs to meet the requirement that W_jh has local extrema at the points of XW_jf and YW_jf. Loosely speaking, we have to assure that W_jh is increasing after a minimum and before a maximum and it is decreasing otherwise. Straightforward analysis yields:

Theorem 1 *$R_m f$ is a given multiscale maxima representation. $h \in \Gamma(R_m f)$ if and only if*

$$T_{mf}h = T_{mf}f \qquad (3)$$

$$t(k) \cdot (W_jh(k+1) - W_jh(k)) > 0 \qquad (4)$$

The last inequality should be satisfied for $j = 1, 2, \ldots, J$ and for almost all k (if two consecutive k's belong to EW_jf then the first is omitted here). $t(k)$ is called the type of k and can be either $+1$ or -1.

The maxima representations can be cast into the form $Rf = \{Vf, Tf\}$, where Vf is a set of points and T is a linear operator which may depend on Vf. However, the key feature of the maxima representation is the fact that the set Vf yields additional constraints, in the form of linear inequalities, which do not appear directly in Rf. Stimulated by this observation, we define the following general family of signal representations.

Definition 1 *$Rf = \{Vf, Tf\}$ is called an Adaptive Quasi Linear Representation (AQLR) if there exists a linear operator A and a sequence a such that:*

$$x \in \Gamma(Rf) \Leftrightarrow Tx = Tf \;\; \text{and} \;\; Ax > a. \qquad (5)$$

A, a may depend on Vf, but they must be independent of Tf.

The reasoning behind the name "Adaptive Quasi Linear Representation" (AQLR) is as follows. This representation is adaptive since T, A, a depend on the sequence f (via the set Vf). It is quasi linear because it is based on a set of linear equalities and inequalities.

Clearly, the following is true.

Proposition 1 *Any multiscale maxima representation is an AQLR.*

The next definition is a generalization of an essential boundness property of the wavelet maxima representation.

Definition 2 *An AQLR is called inherently bounded if there exists a real $K > 0$ such that*

$$x \in \Gamma(Rf) \Rightarrow \|x\| \leq K\|Tf\|.$$

The coefficient K can depend on the parameters of the representation e.g. $N, J, W_1, \ldots, W_J, S_J$ but it must be independent of Vf and Tf.

Proposition 2 *The wavelet maxima representation is an inherently bounded AQLR.*

It turns out that the wavelet maxima representation, as defined here, provides bounds for $\|W_jh\|$, $\|S_Jh\|$. These bounds, due to Parseval's equality, yield a bound for the original sequence h. For details, the reader is referred to [3].

3 The Multiscale Zero-Crossings

In defining the multiscale zero-crossings representation, we essentially follow [4], but minor changes are necessary due to our basic assumption that only a discrete signal version is available. Let Z be an operator which provides the set of zero-crossings of a given sequence $f \in \mathcal{L}$, i.e.

$$Zf \triangleq \{k : f(k-1) \cdot f(k) \leq 0\}. \qquad (6)$$

Mallat in [4] has stabilized the zero-crossings representation by including the values of the wavelet transform integral calculated between consecutive zero-crossing points. Therefore, the multiscale zero-crossings representation, $R_z f$, is defined as:

$$R_z f \triangleq \left\{ \{ZW_j f, U_j^{zf} f\}_{j=1}^{J}, S_J f \right\}. \qquad (7)$$

where

$$U_j^{zf} f(k) = \sum_{l=k}^{n(k)-1} W_j f(l).$$

k and $n(k)$ are two consecutive zero-crossings of $W_j f$.

As in the maxima representation case, for fixed sets $ZW_j f$, the remaining data $U_j^{zf} f$ and $S_J f$ are obtained by a linear operator, denoted by T_{zf}. The zero-crossings representation can also be written as:

$$R_z f = \left\{ \{ZW_j f\}_{j=1}^{J}, T_{zf} f \right\}. \qquad (8)$$

We have the following characterization of the reconstruction set.

Theorem 2 *Let $R_z f$ be a given multiscale zero-crossings representation. $h \in \Gamma(R_z f)$ if and only if*

$$T_{zf} h = T_{zf} f \qquad (9)$$
$$t(k) \cdot W_j h(k) > 0. \qquad (10)$$

$t(k)$ is the type of k and can be either $+1$ or -1. The last inequality should be satisfied for almost all k (if $W_j f(k) = 0$ then k is omitted).

As an immediate consequence of Theorem 2 we have:

Proposition 3 *Any multiscale zero-crossings representation is an AQLR.*

Moreover:

Theorem 3 *The wavelet zero-crossings representation is an inherently bounded AQLR.*

4 Nonuniqueness

A representation $Rf = \{Vf, Tf\}$ is said to be unique, if the reconstruction set $\Gamma(Rf)$ consists of exactly one element. We have the following uniqueness characterization for AQLR's.

Lemma 1 *Let $Rf = \{Vf, Tf\}$ be an AQLR. Then Rf is unique if and only if the kernel of the operator T is trivial, i.e. $\mathcal{N}T = \{0\}$.*

The proof is clear from topological considerations. Nevertheless, an elementary but constructive proof is given in [3].

This claim has some significant consequences. Using the above lemma, an algorithm which tests for uniqueness can be developed. Perhaps the most important consequence of Lemma 1 is the fact that uniqueness of the representation Rf is equivalent to uniqueness of the underlying irregular sampling Tf. In other words, in the unique case, all the information about the signal is already contained in Tf. Additional constraints $Af > a$ are redundant. On the other hand, from the signal compression, understanding and interpretation point of view, it seems to be desirable that a little information would be specified explicitly by Tf and as much as possible information about a signal should be described implicitly by $Af > a$. Therefore, in our opinion, the most important and interesting features of AQLR's appear in the nonunique case.

Using the previous lemma, we are able to show that:

Theorem 4 *A discrete dyadic wavelet maxima (zero-crossings) representation based on a discrete low pass filter $H(\omega)$ is given. If $H(\pi) = 0$, $J \geq 3$, and N is a multiple of 2^J then there exists a sequence f which has a nonunique maxima (zero-crossings) representation.*

As a universal example of nonuniqueness the following sequence is proposed.

$$f(n) = \cos(2\pi \frac{n}{2^J}) \quad n = 0, 1, \ldots, N-1. \qquad (11)$$

Observe that the same sequence is proposed for all dyadic wavelet transforms and for both the maxima representation and the zero-crossings representation.

The example of the nonunique maxima representation is described in [1, 2]. Now, let us consider the zero-crossings representation based on the wavelet transform defined in [4]. Let $J = 5$ and $N = 256$. Consider two sequences:

$$f(k) = \cos\left(\frac{2\pi k}{32}\right)$$

$$f_a(k) = \cos\left(\frac{2\pi k}{32}\right) + 0.1 \cdot \sin\left(\frac{2\pi k}{16}\right).$$

It can be shown that they have the same zero-crossings representation. Figure 1 describes these sequences, while Figure 2 gives their first level wavelet transforms (other levels wavelet transforms are very similar).

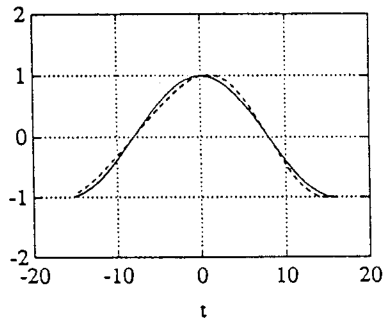

Figure 1: The sequences f and f_a

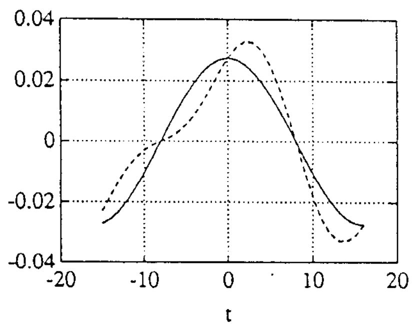

Figure 2: The sequences $W_1 f$ and $W_1 f_a$

5 Stability

To address the stability issue, the standard approach is to introduce the notion of perturbation of the representation, and of the reconstruction set. In addition, distance measures between distinct representations and between different reconstruction sets should be defined. In general, this is not an easy task. Observe that Vf, Tf may have different sizes for different representations. Fortunately, for inherently bounded representations, the following characterization of BIBO (bounded input, bounded output) stability is easily verified.

Proposition 4 *Let $Rf_i = \{Vf_i, T_i f_i\}$ $(i = 1,2)$ be inherently bounded AQLR's. Then for all $K_I > 0$ there exists K_O such that (i=1,2):*

$$\|T_i f_i\| \leq K_I \Rightarrow \|x_1 - x_2\| \leq K_O \quad \forall x_i \in \Gamma(Rf_i)$$

In many applications, the reasons for perturbations in a representation are arithmetic or quantization errors in a reconstruction algorithm. This kind of perturbations may change the continuous values of Tf but it preserves the discrete values of Vf. Therefore the perturbed representation, $(Rf)_p$, can be written as:

$$(Rf)_p = \{Vf, Tf + \Delta(Tf)\}. \qquad (12)$$

Let Γ_p be the corresponding reconstruction set. In general, the distance, d, between two reconstruction sets, Γ and Γ_p, is defined as:

$$d(\Gamma, \Gamma_p) \triangleq \sup\{\|\gamma - \gamma_p\| : \gamma \in \Gamma, \gamma_p \in \Gamma_p\}.$$

Observe, that for an inherently bounded AQLR, $d(\Gamma, \Gamma_p)$ is always finite. The measure of the perturbation in the reconstruction set is the difference between $d(\Gamma, \Gamma_p)$ and the size of Γ which is defined as follows:

$$s(\Gamma) \triangleq d(\Gamma, \Gamma) = \sup\{\|\gamma_1 - \gamma_2\| : \gamma_1, \gamma_2 \in \Gamma\}. \qquad (13)$$

$s(\Gamma)$ and $d(\Gamma, \Gamma_p)$ describe the largest possible Euclidian norm of a reconstruction error, from the original representation and from a perturbed one, respectively.

One remark is in order. In general, for an arbitrary $\Delta(Tf)$, the associated reconstruction set may be empty and then $d(\Gamma, \Gamma_p)$ would not be defined. In the sequel, it is assumed that this problem is treated by a reconstruction algorithm and hence $\Delta(Tf)$ yields a nonempty Γ_p. In this case, the following Lipschitz condition is satisfied.

Theorem 5 *For all inherently bounded AQLR, there exists $K > 0$ such that:*

$$d(\Gamma, \Gamma_p) \leq K \cdot \|\Delta(Tf)\| + s(\Gamma). \qquad (14)$$

Conclusions

The described theoretical results about uniqueness and stability are new. In our opinion, the most significant contribution of this work is to create a framework to define and analyze a wide family of representations. Important examples are generalizations of a basic maxima representation obtained by using only a subset of local extreme points. Their properties are the subject of the undergoing research.

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