FP8 - 3:30

Proceedings of the 27th Conference
on Decision and Control
Austin, Texas • December 1988

SENSOR SCHEDULING PROBLEMS

by

John S. Baras*
Electrical Engineering Department
and Systems Research Center
University of Maryland
College Park, MD 20742
USA

and

Alain Bensoussan
Université Paris Dauphine
and INRIA
75016 Paris
FRANCE

* Research supported in part by US Army contract DAAG-39-83-C-0028 and by NSF grants CDR-85-00108 and INT-8413793.

Abstract

We consider the nonlinear filtering problem of a vector difussion process, when several noisy vector observations with possibly different dimension of their range space are available. At each time any number of these observations (or sensors) can be utilized in the signal processing performed by the nonlinear filter. The problem considered is the optimal selection of a schedule of these sensors from the available set, so as to optimally estimate a function of the state at the final time. Optimality is measured by a combined performance measure that allocates penalties for errors in estimation, for switching between sensor schedules and for running a sensor. The solution is obtained in the form of a system of quasi-variational inequalities in the space of solutions of certain Zakai equations.

1. Introduction

The problem considered is as follows. A signal (or state) process $x(\cdot)$ is given, modelled by the diffusion

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dw(t) \\ x(0) &= \xi \end{aligned} \tag{1.1}$$

in $\mathbb{R}^n$. We further consider M noisy observations of $x(\cdot)$, described by

$$\begin{aligned} dy^i(t) &= h^i(x(t))dt + R_i^{1/2}dv^i(t), \\ y^i(0) &= 0 \end{aligned} \tag{1.2}$$

with values in $\mathbb{R}^{d_i}$. Here $w(\cdot)$, $v^i(\cdot)$ are independent, standard, Wiener processes in $\mathbb{R}^n$, $\mathbb{R}^{d_i}$ respectively, and $R_i = R_i^T > 0$ are $d_i \times d_i$ matrices. Let us consider a finite time horizon $[O,T]$. To formulate the problem of determining an *optimal utilization schedule* for the available sensors, so as to *simultaneously minimize* the cost of errors in estimating a function of $x(\cdot)$ and the costs of using as well as of switching between various sensors, we need to specify these costs. To this end, let $c_i(x)$ denote the cost per unit time when using sensor i, and the state of the system is x; $k_{io}(x)$, $k_{oi}(x)$ denote the cost for turning off, respectively on, the ith sensor when the state of the system is x. The objective of the performed signal processing is to compute, at time T, an estimate $\hat{\phi}(T)$ of a given function $\phi(x(T))$ of the state. Penalties for errors in estimation are assesed according to the cost function

$$E\{c_e(\phi(x(T)) - \hat{\phi}(T))\} := E\{|\phi(x(T)) - \hat{\phi}(T)|^2\} \tag{1.3}$$

We consider next, the set of all possible *sensor activation configurations*, denoted here by $\mathcal{N}$. An element $\nu \in \mathcal{N}$ is a *word* of length M from the alphabet $\{0,1\}$. If the ℓ^{th} position is occupied by an 1, the ℓth sensor is activated (used), if by a 0 the ℓ^{th} sensor is off. There are $N = 2^M$ elements in $\mathcal{N}$. A *schedule of sensors is then a piecewise constant function* $u(\cdot) : [O,T] \to \mathcal{N}$. We let $\tau_j \in [O,T]$ denote the instants of changing schedule; i. e. , the moments when at least one sensor is turned on or off. At such a switching moment, suppose the schedule before is characterized by $\nu \in \mathcal{N}$, and after by $\nu' \in \mathcal{N}$. Then the *switching cost associated with such a scheduling change* will be

$$\mathbf{k}_{\nu\nu'}(x) := \sum_{\{i\in\nu\}\{i\notin\nu'\}} k_{io}(x) + \sum_{\{j\notin\nu\}\{j\in\nu'\}} k_{oj}(x). \tag{1.4}$$

The *total running cost, associated* with schedule $\nu \in \mathcal{N}$ will be

$$\mathbf{c}_\nu(x) := \sum_{\{j\in\nu\}} c_j(x) \tag{1.5}$$

In (1.4), (1.5), *the symbol* $\{i \in \nu\}$ *denotes the set of all indices (from the set* $\{1, 2, \ldots, M\}$ *which are occupied by an 1 in* ν (i. e. the indices corresponding to the sensors which are on); similarly the symbol $\{i \notin \nu\}$ denotes the set of indices corresponding to sensors that are off.

Using the above notation the available observations, *under sensor schedule* $u(\cdot)$ are described by

$$dy(t, u(t)) := h(x(t), u(t))dt + r(u(t))dv(t), \tag{1.6}$$

where it is apparent that the available observations depend explicitly on the sensor schedule $u(\cdot)$. In (1.6), for $x \in \mathbb{R}^n$, $\nu \in \mathcal{N}$,

$$h(x,\nu) := \begin{bmatrix} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix}, \tag{1.7}$$

a block column vector, where in standard notation

$$\chi_{\{\nu\}}(i) := \left\{ \begin{array}{ll} 1, & \text{if the } i^{th} \text{ position in the word } \nu \text{ is} \\ & \quad \text{occupied by an 1} \\ 0, & \text{otherwise} \end{array} \right\} \tag{1.8}$$

Similarly for $\nu \in \mathcal{N}$

$$r(\nu) := \text{Block diagonal}\{R_i^{1/2}\chi_{\{\nu\}}(i)\}, \qquad (1.9)$$

where R_i are the symmetric, positive matrices defined above. Finally

$$v(t) := \begin{bmatrix} v^1(t) \\ \vdots \\ v^M(t) \end{bmatrix} \qquad (1.10)$$

is a higher dimensional standard Wiener process.

Following established terminology we see that a sensor scheduling strategy is defined by an increasing sequence of switching times $\tau_j \in [O,T]$ and the corresponding sequence $\nu_j \in \mathcal{N}$ of sensor activation configurations. We shall denote such a strategy by $u(\cdot)$, where

$$u(t) = \nu_j, \ \ t \in [\tau_j, \tau_{j+1}); \quad j = 1, 2, \dots \qquad (1.11)$$

As stated earlier we are interested in the *simultaneous* minimization of costs due to estimation errors as well as sensor scheduling. We shall therefore consider *joint estimation and sensor scheduling strategies.* Such a strategy consists of two parts: the sensor scheduling strategy u (see (1.11)) and the estimator $\hat{\phi}$. The set of admissible strategies U_{ad} is the customary set of strategies adapted to the sequence of σ-algebras

$$\mathcal{F}_t^{y(\cdot,u(\cdot))} := \sigma\{y(s,u(\cdot)),\ s \le t\}. \qquad (1.12)$$

That is, we consider *strict sense* admissible controls in the sense of [4]. For the problem under investigation this last statement must be interpreted very carefully.

Given such a strategy the corresponding cost is

$$\begin{aligned} J(u(\cdot),\hat{\phi}) \ &:= \ E\{|\phi(x(T)) - \hat{\phi}(T)|^2 \qquad (1.13) \\ &+ \int_0^T c(x(t),u(t))dt \\ &+ \sum_j k(x(t),u(\tau_{j-1}),u(\tau_j))\}. \end{aligned}$$

Here for $x \in \mathbb{R}^n$, $\nu, \nu' \in \mathcal{N}$

$$c(x,\nu) := \mathbf{c}_\nu(x), \qquad (1.14)$$

(c.f. Eq. (1.5)), and

$$k(x,\nu,\nu') = \mathbf{k}_{\nu,\nu'}(x), \qquad (1.15)$$

(c.f. Eq. (1.4)).

The optimal sensor scheduling in nonlinear filtering is thus formulated as the determination of a strategy achieving

$$\inf_{u(\cdot),\hat{\phi}} J(u(\cdot),\hat{\phi}) \qquad (1.16)$$

among all admissible strategies. To simplify the notation a little, let us order the elements of $\mathcal{N}$ according to the numbers they represent in binary form.

More details of the results can be found in [8].

2. The Stochastic Control Formulation

For the dynamical system described in 1, we consider now the cost functional (1.13) where the underlying probability measure is $P^{u(\cdot)}$. As indicated in the introduction, the general problem where the function ϕ will be in a nice class, e.g., bounded C^2, or polynomial, or C^∞ can be treated along identical lines. To simplify the notation we have chosen to formulate the problem for $\phi(x) = x$. The technical difficulties for this case are identical to the ones in the more general cases discussed above, particularly since this $\phi(\cdot)$ is unbounded on $\mathbb{R}^n$. For this choice the selection of the optimal estimator $\hat{\phi}(T)$ is the conditional mean

$$\hat{\phi}(T) = E^{u(\cdot)}\{x(T) \mid \mathcal{F}_T^{y(\cdot,u(\cdot))}\}, \qquad (2.1)$$

where $E^{u(\cdot)}$ denotes expectation with respect to $P^{u(\cdot)}$. Let $\mu(u,t)$ denote the conditional probability measure of $x(t)$, given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, on $\mathbb{R}^n$. It is convenient to express (2.17) as a vector valued functional of $\mu(u,t)$

$$\hat{\phi}(T) = \Phi(\mu(u,T)) = \int_{\mathbb{R}^n} x d\mu(u,T). \qquad (2.2)$$

We shall further assume that the running and switching cost functions $\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot)$, $i,\ j \in \{1,\dots,N\}$, introduced in (1.4) and (1.5) have the following regularity

$$\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot) \ \text{ are in } \ C_b(\mathbb{R}^n) \ \text{(i. e. bounded and continuous)} \qquad (2.3)$$

As a result of this simple transformation we can rewrite the cost as a function of the impulsive control $u(\cdot)$ only (i.e. the selection of $\hat{\phi}(\cdot)$ has been eliminated):

$$\begin{aligned} J(u(\cdot)) \ &= \ E^{u(\cdot)}\{\|x(T) - \Phi(\mu(u,T))\|^2 + \int_0^T c(x(t),u(t))dt \\ &\quad + \sum_{j=1}^{\infty} k(x(\tau_j),u(\tau_{j-1}),u(\tau_j))\chi_{\tau_j<T}\}, \qquad (2.4) \end{aligned}$$

where $\chi_{\tau_i<T}$ is the characteristic function of the Ω-set $\{\omega;\ \tau_i(\omega) < T\}$. We further assume that the switching costs are uniformly bounded below

$$k(x,i,j) \ge k_o, \ \ x \in \mathbb{R}^n,\ i,\ j \in \{1,\dots,N\} \qquad (2.5)$$

with k_0 a positive constant. Note that as a consequence if for some admissible $u(\cdot)$ with positive probability, the number of times $\tau_i < T$ is infinite, then the cost $J(u(\cdot))$ will be infinite. Therefore for T finite the optimal policy will exhibit a finite number of sensor switchings.

The optimal sensor selection problem can now be stated precisely as the optimization problem

$\mathcal{P}$: *Find an admissible impulsive control* $u^*(\cdot)$ *such that*

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in U_{ad}} J(u(\cdot)), \qquad (2.6)$$

where U_{ad} are all impulsive control strategies adapted to $\mathcal{F}^{y(\cdot,u(\cdot))}$, or equivalently satisfying. Problem $\mathcal{P}$ is a *non-standard* stochastic control problem of a partially observed diffusion.

In this section we transform the problem to a fully observed stochastic control problem, by introducing appropriate Zakai equations. As is customary in the theory of nonlinear filtering [1], [2], [3], [4], let us introduce the operator

$$p(u(\cdot),t)(\psi) = E\{\zeta(t)\psi(x(t)) \mid \mathcal{F}_t^{y(\cdot,u(\cdot))}\} \qquad (2.7)$$

for each impulsive control $u(\cdot)$. It is the *unnormalized conditional probability measure* of $x(t)$ given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, [1], [2].

With the help of these measures we can rewrite the various cost terms as follows:

$$E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2\} = E\{\Psi(p(u(\cdot),T))\} \quad (2.8)$$

where Ψ is the functional on finite measures on $\mathbb{R}^n$ defined by

$$\Psi(\mu) = \mu(\chi^2) - \frac{\|\mu(\chi)\|^2}{\mu(\mathbf{1})} \quad (2.9)$$

where $\chi^2(x) = \|x\|^2$, $x \in \mathbb{R}^n$, and μ is any finite measure on $\mathbb{R}^n$ such that the quantities $\mu(\chi^2)$ and $\mu(\chi)$ make sense.

Next define

$$C(u_i) := \mathbf{c}_{u_i}(\cdot),\ u_i \in \{1,2,\ldots,N\}. \quad (2.10)$$

and

$$K(u_i,u_j) = \mathbf{k}_{u_i,u_j}(\cdot), \quad u_i,\ u_j \in \{1,2,\ldots,N\}, \quad (2.11)$$

We can rewrite the cost (2.4) as follows

$$\begin{aligned} J(u(\cdot)) &= E\{\Psi(p(u(\cdot),T)) + \int_0^T (p(u(\cdot),t),C(u(t)))dt \\ &+ \sum_{i=1}^{\infty} \chi_{\tau_i<T}(p(u(\cdot),\tau_i),K(u_{i-1},u_i))\} \end{aligned} \quad (2.12)$$

Letting

$$\delta(x,\nu) = \begin{bmatrix} R_1^{-1}h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1}h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1}h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix} \quad (2.13)$$

the system dynamics are

$$\begin{aligned} dp(u(\cdot),t) &= L^*p(u(\cdot),t)dt + p(u(\cdot),t)\delta(\cdot,u(t))^T dy(t,u(\cdot)) \\ p(u(\cdot),0) &= p_0, \end{aligned} \quad (2.14)$$

where $y(t,u(t))$ is defined in (1.6). This makes precise the construction of a Zakai equation driven by "controlled" observations alluded to in the introduction.

3. The solution of the optimization problem

Let us consider the Banach space $H = L^2(\mathbb{R}^n;\mu) \cap L^1(\mathbb{R}^n;\mu)$ and the metric space H^+ of positive elements of H. Let

$$\begin{aligned} \mathcal{B} &:= \text{space of Borel measurable, bounded functions on } H^+ \\ \mathcal{C} &:= \text{space of uniformly continuous, bounded functions on } H^+. \end{aligned} \quad (3.1)$$

Let us now define semigroups $\Phi_j(t)$ on $\mathcal{B}$ or $\mathcal{C}$ as follows. Consider (2.14) with fixed schedule $u(t) = j$, and let p_j denote the corresponding density $p(\cdot,j)$. Then for $j \in \{1,2,\ldots,N\}$

$$dp_j = L^*p_j\,dt + p_j\tilde{h}^{j^T}dz(t),\ p_j(0) = \pi, \quad (3.2)$$

where

$$\tilde{h}^j := \tilde{h}(\cdot,j). \quad (3.3)$$

We set

$$\Phi_j(t)(F)(\pi) = E\{F(p_{j,\pi}(t))\},\ F \in \mathcal{B} \text{ or } \mathcal{C}, \quad (3.4)$$

where $p_{j,\pi}$ indicates the solution of (3.2) with initial value π. It is easy to see that Φ_j is a semigroup since $p_j(t)$ is a Markov process with values in H^+. It is also useful to introduce the subspaces $\mathcal{B}_1$ and $\mathcal{C}_1$ of functions such that

$$\|F\|_1 \sup_{\pi\in H^+} \frac{|F(\pi)|}{1+\|\pi\|_\mu} < \infty \quad (3.5)$$

where $\|\ \pi\ \|_\mu = \|\ \pi\ \|_{L^1(\mathbb{R}^n;\mu)}$. The spaces $\mathcal{B}_1$ and $\mathcal{C}_1$ are also Banach spaces. They are needed, because we shall encounter functionals with linear growth in the cost function (2.24). To simplify the statement and analysis of the quasi-variational inequalities that solve the optimization problem considered here, we give the details for the case N =2 only in the sequel. We shall insert remarks to indicate how the results should be modified for the general case. Let us introduce the notation

$$\begin{aligned} C_i &:= C(i,\cdot),\quad i=1,2, \\ K_1 &:= K(1,2) \\ K_2 &:= K(2,1). \end{aligned} \quad (3.6)$$

Since C_1, C_2, K_1, K_2 are bounded functions, one can utilize them to define elements of $\mathcal{C}_1$ via (for example)

$$C_1(\pi) = (C_1,\pi) \quad (3.7)$$

where a slight abuse of notation, in denoting the functional and the function by the same symbol, has been allowed. Similarly the functional on H^+

$$\Psi(\pi) = (\pi,\chi^2) - \frac{\|(\pi,\chi)\|^2}{(\pi,\mathbf{1})} \quad (3.8)$$

belongs to $\mathcal{C}_1$ since it is positive and

$$\Psi(\pi) \le (\pi,\chi^2) \le \|\pi\|_\mu. \quad (3.9)$$

Consider now the set of functionals $U_1(\pi,t)$, $U_2(\pi,t)$ such that

$$\begin{aligned} U_1,U_2 &\in C(0,T;\mathcal{C}_1) \\ U_1(\cdot,t)\ge 0, &\quad U_2(\cdot,t)\ge 0 \\ U_1(\pi,T) &= U_2(\pi,T) = \Psi(\pi) \\ U_1(\pi,t) &\le \Phi_1(s-t)U_1(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\ U_2(\pi,t) &\le \Phi_2(s-t)U_2(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\ \forall s &\ge t \\ U_1(\pi,t) &\le K_1(\pi) + U_2(\pi,t) \\ U_2(\pi,t) &\le K_2(\pi) + U_1(\pi,t). \end{aligned} \quad (3.10)$$

In the sequel we will occasionally use the notation $U_i(s)(\pi) = U_i(\pi,s)$, $\quad i=1,2$.

We shall refer to (3.10) as the system of quasi-variational inequalities (QVI). We first establish the following.

Theorem 3.1. *We assume that the conditions on the data f,g,h^i introduced in section 2.1 hold. Then the set of functionals U_1,U_2 satisfying (3.10) is non-empty and has a maximum element, in the sense that if $\tilde{U}_1,\tilde{U}_2$ denotes this maximum element and U_1,U_2 satisfies (3.10), then*

$\tilde{U}_1 \ge U_1, \tilde{U}_2 \ge U_2.$

The proof is carried out in several steps.

Remark: The extension of this result to the general case $N \neq 2$ is straightforward. The system (3.10) has N functionals $U_1, \ldots, U_N$. Everything in (3.10) is the same except for the last two inequalities which are replaced by

$$U_i(\pi, t) \leq \min_{\substack{j \neq i \\ j=1,\ldots,N}} (K_{ij}(\pi) + U_j(\pi, t)), \qquad i = 1, \ldots, N \tag{3.11}$$

Next we show that the maximum element U_1, U_2 of the QVI (3.10) provides the value function for the optimization problem. Furthermore we want to show how an admissible optimal sensor schedule is determined once the pair U_1, U_2 is known.

We establish that

$$U_i(\pi, 0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J(u(\cdot)), \qquad i = 1, 2 \tag{3.12}$$

where $\pi \in H^+$ satisfies $(\pi, \mathbb{1}) = 1$. An optimal schedule will be constructed as follows. Suppose, to fix ideas that $i = 1$. Then define

$$\tau_1^* = \inf_{t \leq T} \{U_1(p_1(t), t) = K_1(p_1(t)) + U_2(p_1(t), t)\} \tag{3.13}$$

where again $p_i(t)$ is the solution of (3.2). We write

$$p^*(t) = p_1(t), \qquad t \in [0, \tau_1^*]. \tag{3.14}$$

Next define

$$\tau_2^* = \inf_{\tau_1^* \leq t \leq T} \{U_2(p_2(t), t) = K_2(p_2(t)) + U_1(p_2(t), t)\} \tag{3.15}$$

In (3.15), it must be kept in mind that $p_2(t)$ represents the solution of (3.2) with j=2, starting at τ_1^* with value $p_1(\tau_1^*)$. We then define

$$p^*(t) = p_2(t), \qquad t \in [\tau_1^*, \tau_2^*] \tag{3.16}$$

Note that, unless $\tau_1^* = T$,

$$\tau_2^* > \tau_1^*, \tag{3.17}$$

otherwise

$$\begin{aligned} U_1(p_1(\tau_1^*), \tau_1^*) &= K_1(p_1(\tau_1^*)) + U_2(p_1(\tau_1^*), \tau_1^*) \\ U_2(p_1(\tau_1^*), \tau_1^*) &= K_2(p_1(\tau_1^*)) + U_1(p_1(\tau_1^*), \tau_1^*) \end{aligned} \tag{3.18}$$

which is impossible since

$$K_1(p_1(\tau_1^*)) > 0, K_2(p_1(\tau_1^*)) > 0 \qquad a.s. \tag{3.19}$$

Similarly one proceeds to construct a sequence of $\tau_1^* < \tau_2^* < \tau_3^* < \ldots$ and the process $p^*(\cdot)$. We can then establish the following.

Theorem 3.2. *With the same assumptions as in Theorem 3.1 the sequence of stopping times $\tau_1^*, \tau_2^*, \ldots$ defines an optimal admissible sensor schedule.*

References

[1] M. Hazewinkel and J.C. Willems, (Eds), *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, Proceedings of NATO Advanced Study Institute, Les Arcs, France, Reidel 1981.

[2] W.H. Fleming and L.G. Gorostiza, (Eds), *Advances in Filtering and Optimal Stochastic Control*, Proc. IFIP-WG 7/1 Working Conference, Cocoyoc, Mexico, 1982, Lect. Notes Control and Inf. Sciences, No. 42, Springer-Verlag, Berlin, 1982.

[3] M. Metivier and E. Pardoux, (Eds), *Stochastic Differential Systems: Filtering and Control*, Proc. IFIP-WG 7/1 Working Conference, Marseille-Luminy, France, 1984, Lect. Notes Control and Inf. Sciences, No. 69, Springer-Verlag, Berlin, 1985.

[4] W.H. Fleming and E. Pardoux, "Optimal Control for Partially Observed Diffusions", *SIAM J. Control and Optim.*, vol. 20, No. 2, March 1982, pp. 261-285.

[5] J.M. Bismut, "Partially Observed Diffusions and Their Control", *Ibid.*, vol. 20, No. 2, March 1982, pp. 302-309.

[6] A. Bensoussan and J.L. Lions, *Contrôle Impulsionnel et Inéquations Quasi-variationelles*, Dunod, Paris, 1982.

[7] E. Pardoux, "Stochastic Partial Differential Equations and Filtering of Diffusion Processes", *Stochastics*, vol 9, no. 3, (1983).

[8] John S. Baras and Alain Bensoussan, "Optimal Sensor Scheduling in Nonlinear Filtering of Diffusion Processes", accepted for publication SIAM J. Control and Optimization.