Stochastic Propagation Delay Through a CMOS Inverter as a Consequence of Stochastic Power Supply Voltage

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June 15, 2016

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OUTLINE

• Brief Statement of Work Performed
• Context / Overall Modeling Approach
• Deterministic Inverter Propagation Delay Model
• Probabilistic Inverter Propagation Delay Model
• Circuit Malfunction Probability
• Example
• Summary and Future Work
WORK PERFORMED

We have derived analytical expressions for the probability distribution of the propagation delay time of logic signals through a CMOS inverter when the inverter power supply voltage $V_{dd}$ is probabilistic (because of, e.g., system noise or a random external EM disturbance).
CONTEXT

• MOTIVATION
  • Experimental observation that exposure of some digital electronic systems to pulsed EM irradiation results in upset of the system’s functionality
  • This upset appears to be subject to probabilistic behavior.

• FUNDAMENTAL QUESTION
  • Can we model this observed behavior via a hierarchy of models for the basic digital electronics building blocks:
    • Transistors (e.g., MOSFETS)
    • Logic gates (collections of transistors, e.g., a CMOS Inverter gate)
    • SSI chips (very small collections of logic gates)
    • Small circuit boards (small collections of SSI chips)
    • ...

• OBSERVATIONS REGARDING ROLE OF RANDOMNESS
  • At some level (TBD) of electronics integration, probabilistic models MUST be introduced in addition to deterministic ones to account for and predict the observed probabilistic upset behavior
  • Some possible sources of observed randomness
    • Insufficiently precise repeatability of the EM environment
    • “True” probabilistic behavior of the digital components and collections thereof at some level of electronic granularity
    • Randomness of the time synchronization between the instant of arrival of the EM field pulse at the digital component and the time-dependent evolution of the electronic processes occurring in that component
OVERALL MODELING APPROACH

**AT EACH LEVEL OF INTEGRATION**

- Import/Formulate deterministic models of component behavior when operating in expected/designed-for EM environments (e.g., for a CMOS Inverter, its designed-for propagation delay time)
- Extend the deterministic models of component behavior beyond the designed-for EM-environment regimes to “regimes of interest” attainable by externally impinging EM irradiation
- Examine the extended deterministic models for possible avenues of ingress of probabilistic behavior; incorporate said behavior into the deterministic models via newly-formulated probabilistic model extensions.

**EXAMPLE: CMOS INVERTER**

- Extended deterministic models for a CMOS Inverter exist (see later slide) to compute, e.g.,
  - Perturbed Inverter throughput signal propagation time delay resulting from perturbed CMOS input (gate) voltage $V_{dd}$
  - Some perturbed output signal time delays may result in bit errors and others not
  - Notionally, we may then make the following picture:

- The probabilistic aspect is introduced by asking: “What is the probability that the perturbed voltage at the Inverter gate has amplitude $\geq V^*$?”
  - We treat the perturbed input signal as probabilistic because of, for example:
    - The EM complexity of the environment surrounding the Inverter making an exact deterministic computation of the Inverter-local field impractical (e.g., inside a chaotic cavity) or minimally believable
    - The randomness of the time synchronization between the instant of arrival of the EM field pulse at the Inverter gate and the time-dependent evolution of the electronic process occurring in the Inverter
DETERMINISTIC MODEL: BACKGROUND

• WHY PROPAGATION DELAY IS IMPORTANT
  • Timing constraints (margins) must be met by propagating logic signals traversing the various logic gates for IC’s to function as intended.
  • Decreasing power supply voltages $V_{dd}$ imply shrinking noise margins for $V_{dd}$ so even small fluctuations in $V_{dd}$ can lead to failures in circuit function.
  • These fluctuations may arise as a result of system noise or from random external electromagnetic disturbances; often involve electromagnetic pulses having time widths of tens of nanoseconds and rise times of several nanoseconds (EFT’s).
  • Some values of $V_{dd}$ may produce values of inverter propagation delay time that cause the inverter to malfunction with respect to the circuit in which it resides (e.g., by having a propagation delay time value exceeding the delay time margins for that circuit).

• HISTORY
  • In [1 - 3] a model was presented to predict changes in propagation delay times through logic gates as a result of changes in $V_{dd}$ with goal of predicting when logic processing errors will occur as a result of EFT-induced variations in $V_{dd}$.
  • Both the $V_{dd}$ amplitude and the resulting propagation delay $r_\Delta$ were treated deterministically, with the latter given as a function of the former.

• WHAT’S NEW
  • Since the value of $V_{dd}$ at any instant during an EFT may in fact be available only probabilistically we have extended the previous deterministic treatment to a probabilistic one in which both $V_{dd}$ and $r_\Delta$ are treated as random variables in some probability space,
  • Currently we have restrict ourselves to a single CMOS inverter.

DETERMINISTIC MODEL

- PROPAGATION DELAY TIME $\tau_\Delta$ AS A FUNCTION OF (GATE) VOLTAGE $V_{dd}$
  \[ \tau_\Delta = A \left[ V_{dd}^2 / \left( |V_{dd}| - |V_{th}| \right)^{\alpha+1} \right] |V_{dd}| > |V_{th,p}| \]
  
  - $(V_{th}, \alpha)$ is the threshold-voltage/velocity-saturation-index pair of the nMOSFET or of the pMOSFET of the CMOS inverter:
  \[ |V_{th}| > 0 \quad 1 \leq \alpha \leq 2 \]
  
  - $A > 0$ is a constant which does not vary as $V_{dd}$ varies--depends only upon
    a fixed $V_{dd, reference}$ a fixed $V_{th}$ the fixed load capacitance driven by the inverter
    WLOG we take $A = 1$ (the units of $A$ depend upon $\alpha$, being volt$^{\alpha+1}$ sec).

- INVERTER HIGH-LOW (HL) TRANSITION DEPENDS UPON THE NMOSFET
  $V_{th,n} > 0$ and $V_{dd} > V_{th,n}$ required for proper gate operation at $V_{GS,n} = V_{dd}$
  \[ \tau_{\Delta, HL} = A \left[ V_{dd}^2 / (V_{dd} - V_{th,n})^{\alpha_n+1} \right] \]
  $V_{dd} > V_{th,n}$
  
  - To capture improper inverter operation in case $V_{dd} \leq V_{th,n}$ we supplement above with
    \[ \tau_{\Delta, HL} = \infty \quad V_{dd} \leq V_{th,n} \]

- INVERTER LOW-HIGH (LH) TRANSITION DEPENDS UPON THE PMOSFET
  $V_{th,p} < 0$, and $V_{dd} < V_{th,p}$ required for proper gate operation at $V_{SG,p} = |V_{dd}|$
  \[ \tau_{\Delta, LH} = A \left[ V_{dd}^2 / (|V_{dd}| - |V_{th,p}|)^{\alpha_p+1} \right] |V_{dd}| > |V_{th,p}| \]
  supplemented by
  \[ \tau_{\Delta, LH} = \infty \quad |V_{dd}| \leq |V_{th,p}| \]

- WE USE
  $\tau_\Delta = V_{dd}^2 / (V_{dd} - V_{th})^{\alpha+1}$

- Model Assumes that $V_{dd}$ is constant during the entire logic transition time of the inverter (the so-called static case)
FUNDAMENTAL IDEA

- Treat $V_{dd}$ as a random variable having an arbitrary piecewise continuous pdf and $\tau_\Delta$ as the random variable that is the function of $V_{dd}$ given previously:

$$\tau_\Delta(V_{dd}) = \begin{cases} 
\frac{V_{dd}^2}{(V_{dd} - V_{th})^{\alpha+1}} & \text{if } V_{dd} > V_{th} \\
\infty & \text{if } V_{dd} \leq V_{th}
\end{cases}$$

- Compute the pdf and cdf of $\tau_\Delta$ in terms of the pdf of $V_{dd}$:

$$f_{V_{dd}} \rightarrow F_{V_{dd}} \rightarrow F_{\tau_\Delta} \rightarrow f_{\tau_\Delta}$$
PROBABILISTIC MODEL - II

- **Probability Space**
  \[ S = (\Omega, \mathcal{A}, \mathbb{P}) \quad \Omega = [0, \infty) \]

- **Random Variables**
  - Density \( f \) is *given* as the basis for the pdf for the random variable \( \nu_{dd}(v) = v, \; v \in \Omega \), representing \( V_{dd} \) in \( S \):
    \[ \begin{align*}
    f_{\nu_{dd}}(v) & = f \\
    & \text{if } v < 0 \\
    & \text{if } v \geq 0 \\
    \end{align*} \]
  - Distribution function (cdf) \( F_{\nu_{dd}} : \mathbb{R} \rightarrow [0, 1] \) for \( V_{dd} \) is given by
    \[ F_{\nu_{dd}}(v) = \begin{cases} 
    \int_{-\infty}^{v} f_{\nu_{dd}}(v') dv' & \text{if } v \geq 0 \\
    \int_{0}^{v} f_{\nu_{dd}}(v') dv' & \text{if } v < 0 
    \end{cases} \]
  - Random variable \( \tau_{\Delta} \) representing \( \tau_{\Delta} \) in \( S \):
    \[ \tau_{\Delta}(v) = \begin{cases} 
    \frac{V_{dd}^{2}(v)}{[V_{dd}(v) - V_{th}(v)]^{\alpha+1}} & \text{if } V_{dd}(v) > V_{th} \\
    \infty & \text{if } V_{dd}(v) \leq V_{th} 
    \end{cases} \]
    \[ = \begin{cases} 
    v^{2}/(v - V_{th})^{\alpha+1} & \text{if } v > V_{th} \\
    \infty & \text{if } v \leq V_{th} 
    \end{cases} \]
  - Distribution function \( F_{\tau_{\Delta}} : \mathbb{R}^{\#} \rightarrow [0, 1] \) for random variable \( \tau_{\Delta} \) is given by
    \[ F_{\tau_{\Delta}}(\tau) = \mathbb{P}(\tau_{\Delta} \leq \tau) = \begin{cases} 
    \mathbb{P}(\tau_{\Delta} \leq \tau) & \text{if } \tau \leq 0 \\
    \mathbb{P}(\nu \in \Omega \mid \tau_{\Delta}(\nu) \leq \tau) = \mathbb{P}(\nu > V_{th} \text{ and } v^{2}/(v - V_{th})^{\alpha+1} \leq \tau) & \text{if } 0 < \tau < \infty 
    \end{cases} \]
SUMMARY EXPRESSIONS FOR $F_{\Delta T}(\tau)$ AND $f_{\Delta T}(\tau)$

$$F_{\Delta T}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 0 \\
\mathcal{P}(v > V_{th} \& \ v^2 / (v - V_{th})^{\alpha+1} \leq \tau) & \text{if } 0 < \tau < \infty \\
1 & \text{if } \tau = \infty
\end{cases}$$

$$f_{\Delta T}(\tau) = \begin{cases} 
\frac{d}{d\tau}[\mathcal{P}(v > V_{th} \& \ v^2 / (v - V_{th})^{\alpha+1} \leq \tau)] & \text{if } 0 < \tau < \infty \\
F_{\mathcal{V}dd}(V_{th}) \delta(\tau - \infty) & \text{if } \tau = \infty
\end{cases}$$

$\mathcal{P}(\mathcal{T}_\Delta = \infty) = F_{\mathcal{V}dd}(V_{th})$ so

$$\lim_{\tau \to \infty} F_{\Delta T}(\tau) = 1 - \mathcal{P}(\mathcal{T}_\Delta = \infty) = 1 - \mathcal{P}(\mathcal{V}_{dd} \leq V_{th}) = 1 - F_{\mathcal{V}dd}(V_{th}) < 1 \text{ if } \mathcal{P}(\mathcal{T}_\Delta = \infty) > 0$$

FURTHER ELUCIDATE THE EXPRESSION

$$\mathcal{P}(v > V_{th} \& \ v^2 / (v - V_{th})^{\alpha+1} \leq \tau)$$

Distinguish three separate but totally inclusive cases for velocity-saturation-index: $\alpha = 1, \alpha = 2, 1 < \alpha < 2$. 

6/11/2016
**PROBABILISTIC MODEL - IV: COMPUTATION FOR \( \alpha = 1 \)**

- \( \nu > V_{th} \) & \( \nu^2/(\nu - V_{th})^2 \leq \tau \) \iff \( \nu > V_{th} \) & \( \nu - V_{th} \geq \nu / \tau^{1/2} \)
  - Straight line functions of \( \nu \) on the two sides of the inequality:
    - \( g_0(\nu) = \nu - V_{th} \)
    - \( g_1(\nu) = \nu / \tau^{1/2} \)
  - Intersect when \( \nu(1 - \tau^{-1/2}) = V_{th} \), i.e,
    - Not at any \( \nu \geq 0 \) when \( \tau \leq 1 \) (since \( V_{th} > 0 \))
    - At \( \nu = \nu_1^+(\tau, V_{th}) \equiv V_{th}/(1 - \tau^{-1/2}) > V_{th} > 0 \) when \( 1 < \tau < \infty \)

- \( \nu > V_{th} \) & \( \nu - V_{th} \geq \nu / \tau^{1/2} \) \iff \( \nu \in \left[ V_{th}/(1 - \tau^{-1/2}), \infty \right) \) if \( 1 < \tau < \infty \)
  - \( \emptyset \) if \( 0 < \tau \leq 1 \)

- \( P(\nu > V_{th} \) \& \( \nu^2/(\nu - V_{th})^{\alpha+1} \leq \tau) = \)
  \[
  \int_{V_{th}}^{\infty} \left[ \int_{v_1^+(\tau, V_{th})} f_{\nu_{dd}}(\nu) d\nu \right] = 1 - F_{\nu_{dd}}(\nu_1^+(\tau, V_{th}))
  \] if \( 1 < \tau < \infty \)
  
  \[
  0 \quad \text{if} \quad 0 < \tau \leq 1
  \]

The functions \( g_0(\nu) \) and \( g_1(\nu) \) for the case \( \alpha = 1 \).
PROBABILISTIC MODEL - V: FINAL CDF AND PDF FOR $\alpha = 1$

- **CDF**

$$F_{\tau_{\Delta}}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 1 \\
1 - F_{V_{dd}}(V_1^*(\tau, V_{th})) & \text{if } 1 < \tau < \infty \\
1 & \text{if } \tau = \infty 
\end{cases}$$

- **PDF**

\[
\begin{align*}
\frac{d}{d\tau} F_{\tau_{\Delta}}(\tau) &= - \frac{d}{dV} F_{V_{dd}}(V_1^*(\tau, V_{th})) \left( \frac{\partial V_1^*/\partial \tau}{\partial (V_{th})} \right)(\tau, V_{th}) \\
&= f_{V_{dd}}(V_1^*(\tau, V_{th}))(V_{th}/2)/[(1 - \tau^{-1/2})^2 \tau^{3/2}] \\
f_{\tau_{\Delta}}(\tau) &= \begin{cases} 
0 & \text{if } -\infty < \tau \leq 1 \text{ or } V_1^*(\tau, V_{th}) \in D_{f_{V_{dd}}} \\
f_{V_{dd}}(V_1^*(\tau, V_{th}))(V_{th}/2)/[(1 - \tau^{-1/2})^2 \tau^{3/2}] & \text{if } 1 < \tau < \infty \text{ and } V_1^*(\tau, V_{th}) \notin D_{f_{V_{dd}}} \\
F_{V_{dd}}(V_{th}) \delta(\tau - \infty) & \text{if } \tau = \infty 
\end{cases}
\]
PROBABILISTIC MODEL - VI: COMPUTATION FOR $\alpha = 2$

- $v > V_{th}$ & $v^2/(v - V_{th})^3 \leq \tau$ \iff $v > V_{th}$ & $v - V_{th} \geq v^{2/3}/\tau^{1/3}$

- Straight line and power functions of $v$ on the two sides of the inequality
  
  $g_0(v) = v - V_{th}$
  
  $g_1(v) = v^{2/3}/\tau^{1/3}$

- Intersections given by roots of cubic equation
  
  $v^3 - (3V_{th} + \tau^{-1})v^2 + 3V_{th}^2v - V_{th}^3 = 0$.

- In general three solutions but Figure reveals that in this instance exactly one of the solutions is real and the other two are complex (since there is clearly exactly one intersection of $g_0$ and $g_2$ in the $v$-$g(v)$ plane for any of our allowable values of $\tau$ and $V_{th}$).

  $v = v_2^*(\tau, V_{th}) \equiv S_+(\tau, V_{th}) + S_-(\tau, V_{th}) + [V_{th} + (3\tau)^{-1}] > V_{th} > 0$

where

- $S_+(\tau, V_{th}) = \{r(\tau, V_{th}) + [q^3(\tau, V_{th}) + r^2(\tau, V_{th})]^{1/2}\}^{1/3}$

- $S_-(\tau, V_{th}) = \{r(\tau, V_{th}) - [q^3(\tau, V_{th}) + r^2(\tau, V_{th})]^{1/2}\}^{1/3}$

- $q(\tau, V_{th}) = -(2/3)\tau^{-1}[V_{th} + (6\tau)^{-1}]$

- $r(\tau, V_{th}) = (2\tau)^{-1}\{[V_{th} + (3\tau)^{-1}]^2 - (27\tau^2)^{-1}\}$

- $S_\pm(\tau, V_{th}) = (2\tau)^{-1/3}\{[V_{th} + (3\tau)^{-1}]^2 - (27\tau^2)^{-1} \pm V_{th}^{3/2}[V_{th} + 2 \pm (27\tau)^{-1}]^{1/2}\}^{1/3}$
PROBABILISTIC MODEL - VII: FINAL CDF AND PDF FOR $\alpha = 2$

- $v > V_{th}$ & $v - V_{th} \geq v^{2/3} / \tau^{1/3}$ iff $v > V_{th}$ & $v \in [v_2^*(\tau, V_{th}), \infty)$ $(0 < \tau < \infty)$
- $\mathbb{P}(v > V_{th} \& v^2 / (v - V_{th})^3 \leq \tau) = \int_{v_2(\tau, V_{th})}^{\infty} f_{\nu_{dd}}(v) dv = 1 - F_{\nu_{dd}}(v_2^*(\tau, V_{th}))$ $(0 < \tau < \infty)$

**CDF**

$$F_{\nu_0}(\tau) = \begin{cases} 0 & \text{if } -\infty < \tau \leq 0 \\ 1 - F_{\nu_{dd}}(v_2^*(\tau, V_{th})) & \text{if } 0 < \tau < \infty \\ 1 & \text{if } \tau = \infty \end{cases}$$

**PDF**

$$f_{\nu_0}(\tau) = \begin{cases} 0 & \text{if } -\infty < \tau \leq 0 \text{ or } v_2^*(\tau, V_{th}) \in D_{f_{\nu_{dd}}} \\ -f_{\nu_{dd}}(v_2^*(\tau, V_{th})) (\partial v_2^*/\partial \tau)(\tau, V_{th}) & \text{if } 0 < \tau < \infty \text{ and } v_2^*(\tau, V_{th}) \notin D_{f_{\nu_{dd}}} \\ F_{\nu_{dd}}(V_{th}) \delta(\tau - \infty) & \text{if } \tau = \infty \end{cases}$$
PROBABILISTIC MODEL - VIII: COMPUTATION FOR $1 < \alpha < 2$

- $\nu > \nu_{th} \& \frac{\nu^2}{(\nu - \nu_{th})^{\alpha+1}} \leq \tau$ iff $\nu > \nu_{th} \& \nu - \nu_{th} \geq \frac{\nu^2(1+\alpha)}{\tau^{1/(1+\alpha)}}$

- Straight line and power functions of $\nu$ on the two sides of the inequality
  
  $g_0(\nu) = \nu - \nu_{th} \quad g_\alpha(\nu) = \frac{\nu^2(1+\alpha)}{\tau^{1/(1+\alpha)}}$

- Intersections cannot be done analytically in general for $1 < \alpha < 2$ but may easily be done computationally

- Graph $g_0$ and $g_\alpha$ to determine general behavior: Since $2/3 < \frac{2}{1 + \alpha} < 1$ then $g_\alpha(\nu)$ passes through the origin, is strictly increasing and concave down, and is unbounded on $[0, \infty)$—behavior identical to that of $g_2$.

The functions $g_0(\nu)$ and $g_\alpha(\nu)$ for the case $1 < \alpha < 2$. 

PROBABILISTIC MODEL - IX: CDF AND PDF FOR FOR $1 < \alpha < 2$

• CDF

$$F_{T\Delta}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 0 \\
1 - F_{V_{dd}}(v^*(\tau, V_{th})) & \text{if } 0 < \tau < \infty. 1 \\
1 & \text{if } \tau = \infty 
\end{cases}$$

• PDF

$$f_{T\Delta}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 0 \text{ or } v^*(\tau, V_{th}) \not\in D_{f_{V_{dd}}} \\
-f_{V_{dd}}(v^*(\tau, V_{th}))(\partial v^*/\partial \tau)(\tau, V_{th}) & \text{if } 0 < \tau < \infty \text{ and } v^*(\tau, V_{th}) \not\in D_{f_{V_{dd}}} \\
F_{V_{dd}}(V_{th}) \delta(\tau - \infty) & \text{if } \tau = \infty 
\end{cases}$$
WANT TO DETERMINE PROBABILITY OF MALFUNCTION OF ANY CIRCUIT \( \mathcal{C} \) IN WHICH ONLY ONE INVERTER RESIDES

Here, malfunction caused only by some unacceptable inverter propagation delay times resulting from some “non-conforming” values of \( V_{dd} \)

In general, given the circuit \( \mathcal{C} \) there is a set \( \Theta^+\!\!\!\!\!\mathrm{[C]} \) of inverter propagation delay times which produce no malfunctions in proper operation of \( \mathcal{C} \)

Complementary set \( \Theta^\ominus\!\!\!\!\!\mathrm{[C]} = (0, \infty) \setminus \Theta^+\!\!\!\!\!\mathrm{[C]} \) contains those inverter propagation delay times which do produce malfunctions in proper operation of \( \mathcal{C} \)

For example, most commonly there exists a \( \tau^*\!\!\!\!\!\mathrm{[C]} > 0 \) such that

\[
\Theta^+\!\!\!\!\!\mathrm{[C]} = (0, \tau^*\!\!\!\!\!\mathrm{[C]}] \quad \text{and} \quad \Theta^\ominus\!\!\!\!\!\mathrm{[C]} = (\tau^*\!\!\!\!\!\mathrm{[C]}, \infty)
\]

In the general case
\[
P(T_\Delta \in \Theta^+\!\!\!\!\!\mathrm{[C]}) = P(T_\Delta^{-1}[\Theta^+\!\!\!\!\!\mathrm{[C]}]) = P(\{v \in \Omega \mid T_\Delta(v) \in \Theta^+\!\!\!\!\!\mathrm{[C]}\})
\]
\[
P(T_\Delta \in \Theta^\ominus\!\!\!\!\!\mathrm{[C]}) = P(T_\Delta^{-1}[\Theta^\ominus\!\!\!\!\!\mathrm{[C]}]) = 1 - P(T_\Delta \in \Theta^+\!\!\!\!\!\mathrm{[C]}).
\]

In the typical case above
\[
P(T_\Delta \in \Theta^+\!\!\!\!\!\mathrm{[C]}) = F_{T_\Delta}(\tau^*\!\!\!\!\!\mathrm{[C]}]
\]
\[
P(T_\Delta \in \Theta^\ominus\!\!\!\!\!\mathrm{[C]}) = 1 - F_{T_\Delta}(\tau^*\!\!\!\!\!\mathrm{[C]}).
\]
EXAMPLE—GAMMA DISTRIBUTION

\[ f(\nu) = \nu e^{-\nu}, \quad \nu \in [0, \infty) \]

\[ f_{V_{th}}(\nu) = \begin{cases} 
\nu e^{-\nu} & \text{if } \nu \geq 0 \\
0 & \text{if } \nu < 0 
\end{cases} \]

For \( \alpha = 1 \):

\[ f_{\tau}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 1 \\
(\nu_{th}^2/2)\left\{ \left[ 1 - \tau^{-1/2} \right]^{3/2} \exp\left[ -\nu_{th}/(1 - \tau^{-1/2}) \right] \right\} & \text{if } 1 < \tau < \infty \\
1 - (1 + \nu_{th})e^{-\nu_{th}} \delta(\tau - \infty) & \text{if } \tau = \infty 
\end{cases} \]

\[ F_{\tau}(\tau) = \begin{cases} 
0 & \text{if } -\infty < \tau \leq 1 \\
(1 + \nu_{th}/(1 - \tau^{-1/2}))e^{-\nu_{th}/(1 - \tau^{-1/2})} & \text{if } 1 < \tau < \infty \\
1 & \text{if } \tau = \infty 
\end{cases} \]

\[ P_{\tau_{dd}}(\tau_{\Delta} \in \Theta^{-}[C]) = 1 - F_{\tau}(\tau^*\Delta[C]) \]

\[ = \begin{cases} 
1 & \text{if } -\infty < \tau^*\Delta[C] \leq 1 \\
1 - (1 + \nu_{th}/(1 - \tau^*\Delta[C]^{-1/2})) \exp\left[ -\nu_{th}/(1 - \tau^*\Delta[C]^{-1/2}) \right] & \text{if } 1 < \tau^*\Delta[C] < \infty \\
0 & \text{if } \tau^*\Delta[C] = \infty 
\end{cases} \]

The gamma distribution \( \nu e^{-\nu} \).

The area \( (1 + \nu_{th})e^{-\nu_{th}} \).

The pdf \( f_{\tau}(\tau) \) for the gamma distribution \( \nu e^{-\nu} \).
SUMMARY AND FUTURE WORK

- SUMMARY
  - We have derive analytically expressions for the probability distribution of the propagation delay time of logic signals through a CMOS inverter when the inverter power supply voltage $V_{dd}$ is itself probabilistic for some reason (e.g., system noise or a random external electromagnetic disturbance), where the pdf is any piecewise continuous function (non-negative and normalized).
  - We have shown results for this methodology applied this to a gamma distribution (continuous pdf).
  - We also have analogous complete results ($\alpha = 1$) for a uniform distribution on a finite interval (discontinuous pdf) and a $\delta$-function distribution (answer immediate by inspection so a good test).

- TO BE SUBMITTED (IEEE Trans EMC)
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- FUTURE WORK
  - Treating circuit containing $n \geq 2$ inverters can be treated by considering the $n$-dimensional joint probability distribution of the inverters’ $V_{dd}$’s and the resulting $n$-dimensional joint probability distribution of the inverters’ propagation delay times; circuit malfunction may be inferred from the max of the entries of the latter.
  - Investigating deterministic propagation delay time models for other gates and render them probabilistic.
  - Formulating a treatment for a small collection of logic gates.