Lecture 21

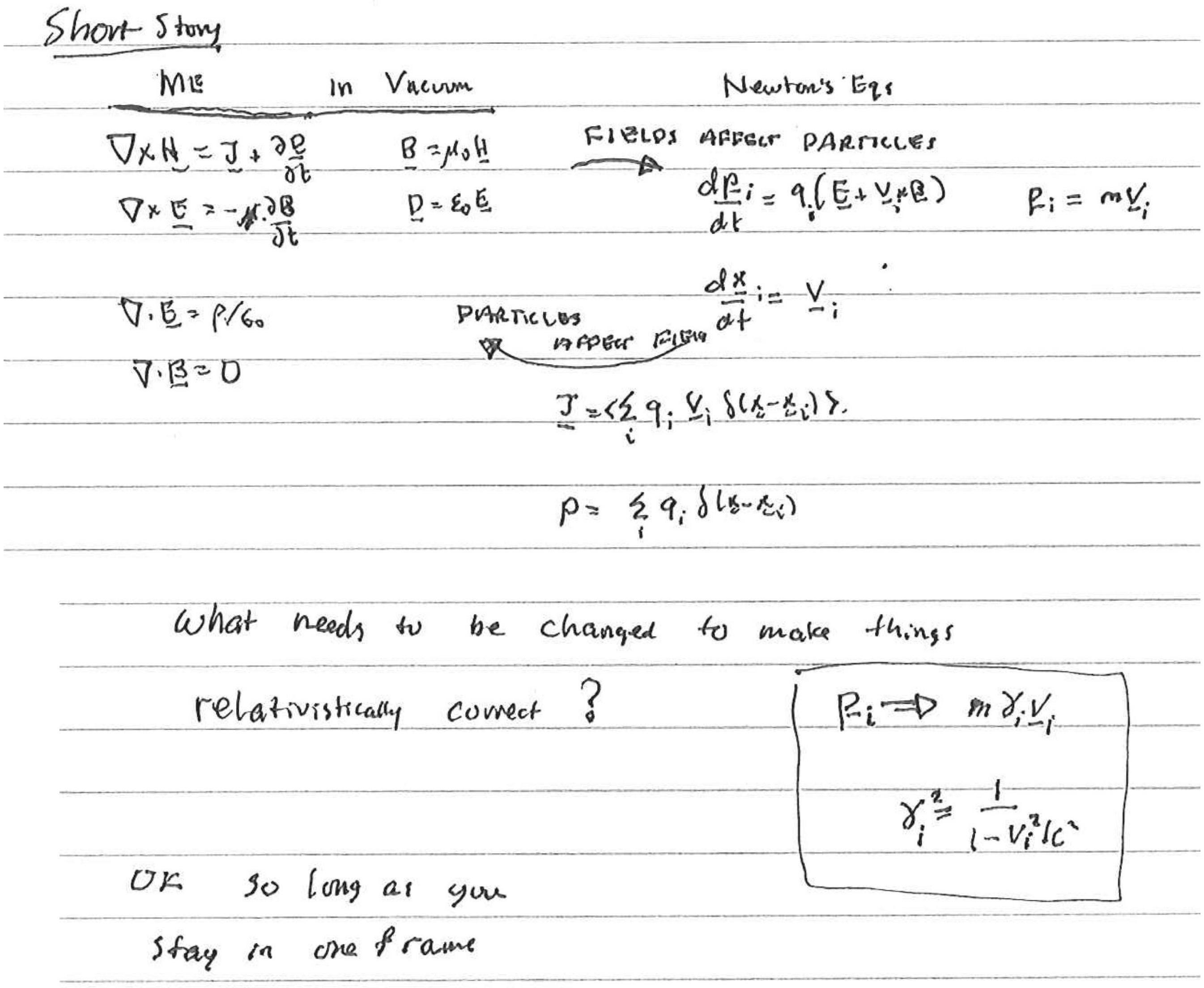
Special Relativity Continued

Outline

- Review relativistic classical mechanics

- Lorentz transformation in 4D
    - invariant separation
    - proper time
    -
- Four vectors

Short Story



Short Story

ME in Vacuum

Newton's Eqs

$\nabla\times\underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$ $\qquad \underline{B} = \mu_0 \underline{H}$

$\nabla\times\underline{E} = -\cancel{\mu}\frac{\partial \underline{B}}{\partial t}$ $\qquad \underline{D} = \varepsilon_0 \underline{E}$

FIELDS AFFECT PARTICLES

$\frac{d\underline{P}_i}{dt} = q_i(\underline{E} + \underline{V}_i\times\underline{B})$ $\qquad \underline{P}_i = m\underline{V}_i$

$\frac{d\underline{x}_i}{dt} = \underline{V}_i$

$\nabla\cdot\underline{E} = \rho/\varepsilon_0$

$\nabla\cdot\underline{B} = 0$

PARTICLES AFFECT FIELDS

$\underline{J} = \langle \sum_i q_i \underline{V}_i \delta(\underline{x}-\underline{x}_i)\rangle.$

$\rho = \sum_i q_i \delta(\underline{x}-\underline{x}_i)$

What needs to be changed to make things relativistically correct?

$$\boxed{\begin{array}{c}\underline{P}_i \Rightarrow m\gamma_i\underline{V}_i \\ \gamma_i^2 = \frac{1}{1-V_i^2/c^2}\end{array}}$$

OK so long as you stay in one frame

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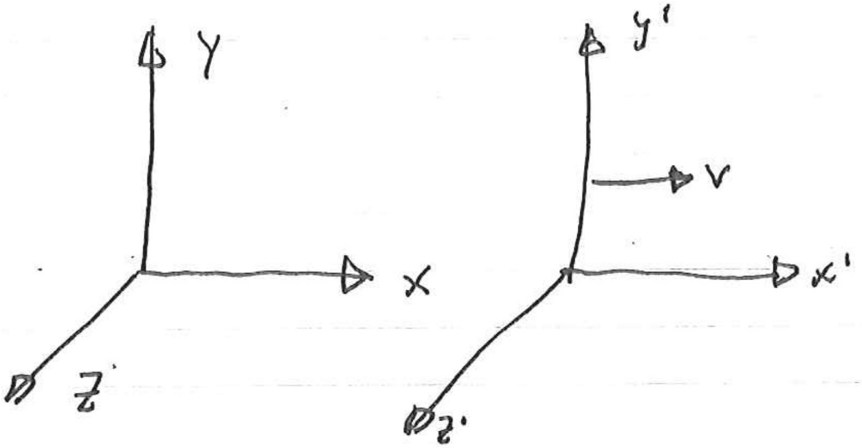
Lorentz Trans



$$\left.\begin{array}{l} x' = \gamma(x - vt) \\ t' = \gamma(t - xv/c^2) \\ y' = y \\ z' = z \end{array}\right\}$$

TRANSformation of
~~an~~ coordinates of an event

Generalize to arbitrary dim

$$x_{\parallel}' = \gamma(x_{\parallel} - \beta ct)$$

$$ct' = \gamma(ct - \underline{\beta} \cdot \underline{x})$$

$$\underline{x}_{\perp}' = \underline{x}_{\perp}$$

$\underline{\beta} = \underline{V}/c$

$\parallel$ & $\perp$ Refer to $\parallel$ & $\perp$ to $\underline{V}$

Four Vectors

INTRODUCE 4-VECTR

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad A_0 \ A_1 \ A_2 \ A_3 \\ \underline{A} = (ct, \underline{x}) = (ct, x, y, z) = (A_0, \underline{A}) \end{array}$$

$$A_0' = \gamma(A_0 - \underline{\beta} \cdot \underline{A})$$

$$A_{\parallel}' = \gamma(A_{\parallel} - \beta A_0)$$

$$\underline{A}_{\perp}' = \underline{A}_{\perp}$$

"Scalar Product" $(A_0, \underline{A})$ $(B_0, \underline{B})$

$$SP = (A_0 B_0 - \underline{A} \cdot \underline{B}) \qquad A \circ B$$

What is $SP' = A_0' B_0' - \underline{A}' \cdot \underline{B}'$?

$$= (A_0 B_0 - \underline{A} \cdot \underline{B}) = SP$$

Four Vector Examples

Space-time coordinate $\left(ct, z, x, y\right) \equiv X$

Space-time wave vector $\left(\frac{\omega}{c}, k_z, k_x, k_y\right) \equiv K$

Invariant product

$\mathrm{K} \circ \mathrm{X} = \omega t - \mathbf{k} \cdot \mathbf{x} = \Phi$ wave phase

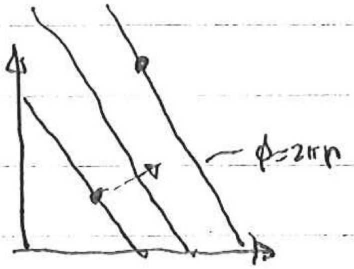
Same for all observers

Differentiation

$$\left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x_{\parallel}}, -\frac{\partial}{\partial x_{\perp}} \right)$$ is a 4-vector

Wave Phase $\Phi = \omega t - \mathbf{k} \cdot \mathbf{x}$ $$\left(\frac{\partial}{c\partial t}, -\nabla \right) \Phi = \left(\frac{\omega}{c}, \mathbf{k} \right)$$

$\Phi = 2\pi \overset{\swarrow \text{ integer}}{n}$ denotes the crests of the wave



In an other frame

Four Vectors

THIS means if $(A_0, A_{\parallel}, \underset{\sim}{A}_{\perp})$ is a fourvecto Field

Then

$$\frac{\partial A_0}{\partial ct} - \left(-\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underline{A}\right)$$ is a Lorentz invariant

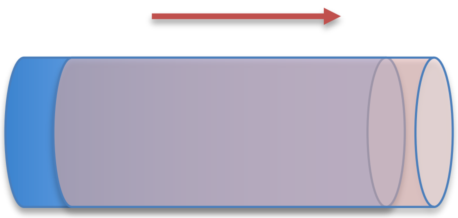
$$\frac{\partial}{\partial ct} A_0 + \nabla \cdot \underline{A}$$ is a Lorentz invariant

Contuity of charge

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \underset{\sim}{J} = 0$$

$$\frac{\partial}{\partial ct} c\rho + \nabla \cdot \underline{J} = 0 \qquad (c\rho, \underset{\sim}{J})$$ is a 4 vector

Field Transformations



Current carrying wire

In lab frame:

$$\rho_e = -\rho_i$$

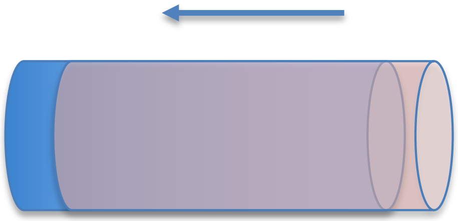
$$J_e = v_e \rho_e$$

Fields

$$E = 0$$

$$B = B_\theta = \frac{\mu_0 I}{2\pi r}$$

In frame co-moving with electrons:



$$\rho'_e = \gamma\left(\rho_e - \frac{v_e}{c^2} J_e\right) = \gamma\left(1 - \frac{v_e^2}{c^2}\right)\rho_e = \gamma^{-1}\rho_e$$

$$J'_e = \gamma\left(J_e - v_e \rho_e\right) = 0$$

$$\rho'_i = \gamma\left(\rho_i - \frac{v_e}{c^2} J_i\right) = \gamma\rho_i = -\gamma\rho_e$$

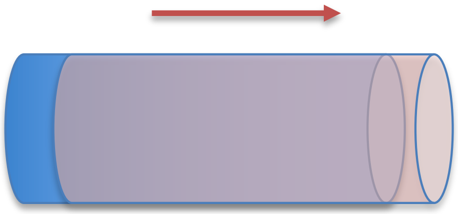
$$J'_i = \gamma\left(J_i - v_e \rho_i\right) = -\gamma v_e \rho_i = \gamma v_e \rho_e$$

Fields

$$E = E_r = \left(\gamma - \gamma^{-1}\right)\frac{I / v_e}{2\pi\varepsilon_0 r}$$

$$B = B_\theta = \gamma \frac{\mu_0 I}{2\pi r}$$

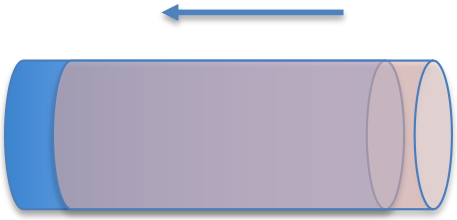
Fields Transform



Fields

$$E = 0$$

$$B = B_\theta = \frac{\mu_0 I}{2\pi r}$$



Fields

$$E' = E'_r = \left(\gamma - \gamma^{-1}\right)\frac{I / v_e}{2\pi\varepsilon_0 r} = \gamma\left(1 - \gamma^{-2}\right)\frac{1}{v_e \varepsilon_0 \mu_0}\frac{\mu_0 I}{2\pi r} = \gamma v_e B_\theta$$

$$B' = B'_\theta = \gamma \frac{\mu_0 I}{2\pi r} = \gamma B_\theta$$

Transformation of fields

$$\left(\Phi, \mathbf{A}\right) \Leftrightarrow \left(\Phi', \mathbf{A}'\right)$$

4-vector

$$\mathbf{B}' = \nabla' \times \mathbf{A}'$$

$$\mathbf{E}' = -\frac{\partial}{\partial t'}\mathbf{A}' - \nabla'\Phi'$$

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}'_{\perp} + \mathbf{v} \times \mathbf{B}'_{\perp}\right)$$

$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}'_{\perp} - \frac{\mathbf{v} \times \mathbf{E}'_{\perp}}{c^2}\right)$$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

Transformation of Maxwell’s Equations

WHAT ABOUT

$$\left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}\right) \circ \left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}\right)$$

$$= \left(\frac{\partial}{\partial ct}\right)^2 - \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \frac{\partial}{\partial x_2}\cdot\frac{\partial}{\partial x_2}\right) = \text{wave operator}$$

Lorenz Gauge condn

$$\nabla\cdot\underline{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

$$= \left(\frac{\partial}{c\partial t}, -\nabla\right) \circ \left(\phi_{/c}, \underline{A}\right) = 0$$

Thus, $(\phi/c, \underline{A})$ is a 4-vector

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A = \mu_0 \underline{J}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \phi_{/c} = \rho/\epsilon_0 c \qquad \epsilon_0 c = ?$$

$$= c\rho/(\epsilon_0 c^2) = \mu_0 (c\rho)$$

∴ Same Equation in all frames

Particle velocity is not part of a four vector

addition of velocities

$$\underset{\sim}{u} = (u_1, u_2, u_3)$$

(not part of a four vector

$$u'_{\parallel} = \frac{u_{\parallel} - V}{1 - \underset{\sim}{V}\cdot\underset{\sim}{u}/c^2}$$

* $u_{\parallel}$ refers to $\parallel$ to $\underset{\sim}{\beta}$

$$\underset{\sim}{u}'_{\perp} = \frac{\underset{\sim}{u}_{\perp}}{\gamma_V(1-\underset{\sim}{V}\cdot\underset{\sim}{u}/c^2)}$$

Relation between Energy and momentum and proper time

consider the trajectory of a particle defined by the 4 vector

$(ct, \underset{\sim}{x}(t))$

The proper time for this particle is defined by the equation

$$d\tau = \frac{dt}{\gamma(u(t))} \qquad \underset{\sim}{u}(t) = \frac{d\underset{\sim}{x}(t)}{dt}$$

differentiat the positim four vector

$(ct, \underset{\sim}{x}(t))$ with respect to

proper time

$$\frac{d\,(ct, \underset{\sim}{x}(t))}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx(t)}{d\tau}\right)$$

also a four vector. why

$(c\,dt, dx(t))$ is a four vector

$d\tau$ is a lorentz invariant
same is calculated in any frame

THUS THEIR RATIO IS A FOUR vector

$$\left(c\frac{dt}{d\tau}, \frac{d\underset{\sim}{x}(t)}{d\tau}\right) = \left(c\gamma, \gamma\underset{\sim}{u}\right) = \left(\frac{cE, \underset{\sim}{P}}{m}\right)$$

m is a Lorentz invariant //

Energy – Momentum four vector

$$\left(\frac{cdt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right) = \left(\gamma c, \gamma\mathbf{u}\right) = \frac{1}{m}\left(\frac{\gamma mc^2}{c}, \gamma m\mathbf{u}\right) = \frac{1}{m}\left(\frac{E}{c}, \mathbf{p}\right)$$

$$E = \gamma mc^2$$

$$\mathbf{p} = \gamma m\mathbf{u}$$

Relativistic Energy

The **total energy** E of a particle is

$$E = \gamma_p mc^2 = E_0 + K = \text{rest energy} + \text{kinetic energy}$$

This total energy consists of a **rest energy**

$$E_0 = mc^2$$

and a relativistic expression for the *kinetic energy*

$$K = (\gamma_p - 1)mc^2 = (\gamma_p - 1)E_0$$

This expression for the kinetic energy is very nearly $mu^2/2$ when $u << c$.

Where does this definition of energy come from?

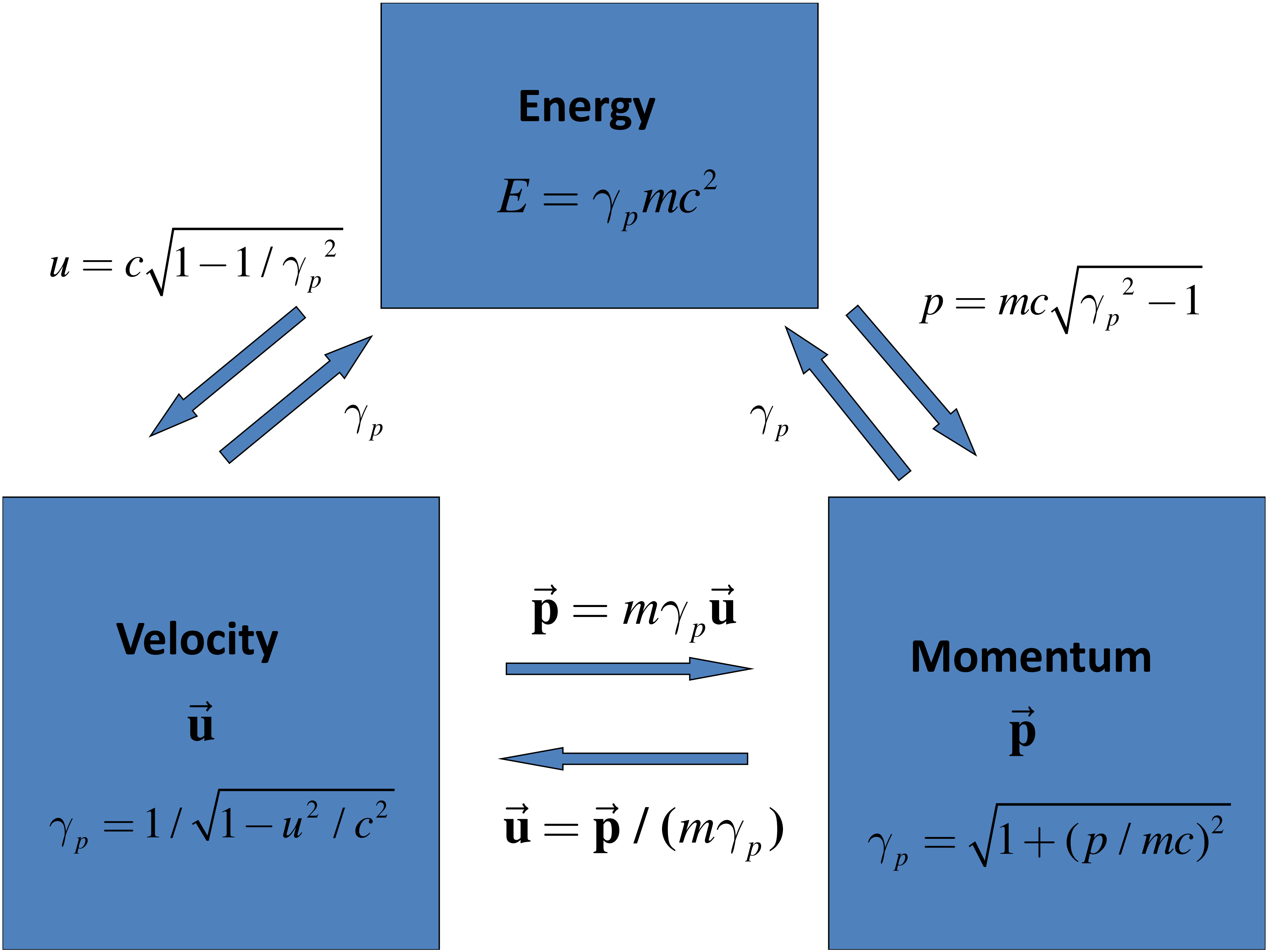
$$\frac{d}{dt}\gamma_p mc^2 = mc^2 \frac{d}{dt}\sqrt{1+(p/mc)^2} = \frac{p}{m\sqrt{1+(p/mc)^2}}\frac{dp}{dt}$$

Thus,

$$\frac{d}{dt}\gamma_p mc^2 = \frac{p}{m\gamma_p}\frac{dp}{dt} = u\frac{dp}{dt} = uF$$

Rate at which work is done

Replaces kinetic energy



Energy

$$E = \gamma_p mc^2$$

$$u = c\sqrt{1 - 1/\gamma_p{}^2}$$

γ_p

$$p = mc\sqrt{\gamma_p{}^2 - 1}$$

γ_p

Velocity

$$\vec{\mathbf{u}}$$

$$\gamma_p = 1/\sqrt{1 - u^2/c^2}$$

$$\vec{\mathbf{p}} = m\gamma_p\vec{\mathbf{u}}$$

$$\vec{\mathbf{u}} = \vec{\mathbf{p}}/(m\gamma_p)$$

Momentum

$$\vec{\mathbf{p}}$$

$$\gamma_p = \sqrt{1 + (p/mc)^2}$$

Energy of EM waves and particles now given by the same formula

Energy in an EM wave

$$E = pc$$

p = momentum in an EM wave

For particles:

$$E = \gamma_p mc^2 = mc^2\sqrt{1 + (p / mc)^2} = c\sqrt{(mc)^2 + (p)^2}$$

Let $m \to 0$ $E \to pc$

Revised Newton’s Laws

#1 $$\frac{d}{dt}\vec{\mathbf{p}}_{\mathbf{i}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}}_{\mathbf{i}} \times \vec{\mathbf{B}}\right)$$

#2 $$\vec{\mathbf{p}}_{\mathbf{i}} = m\gamma_i \vec{\mathbf{v}}_{\mathbf{i}}$$

$$\gamma_i = 1 / \sqrt{1 - \left|\vec{\mathbf{v}}_{\mathbf{i}}\right|^2 / c^2}$$

Momentum can become large, but particle speed is always less than c

#3 $$\frac{d}{dt}\vec{\mathbf{x}}_{\mathbf{i}} = \vec{\mathbf{v}}_{\mathbf{i}}$$

Relativistic Charged Particle Motion

Relativistic Momentum Equation (esu)

$$\frac{d}{dt}\mathbf{p} = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) = q\left(-\nabla\Phi - \frac{\partial}{c\,\partial t}\mathbf{A} + \frac{\mathbf{v}\times\nabla\times\mathbf{A}}{c}\right)$$

$$\mathbf{v}\times\nabla\times\mathbf{A} = \nabla\left(\mathbf{v}\cdot\mathbf{A}\right) - \mathbf{v}\cdot\nabla\mathbf{A}$$

Vector identity

$$\frac{d}{dt}\mathbf{p} = q\left(-\frac{1}{c}\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{A} - \nabla\left(\Phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right)\right)$$

$$\frac{d}{dt}\mathbf{P} = -q\nabla\left(\Phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right)$$

$$\mathbf{P} = \mathbf{p} + \frac{q}{c}\mathbf{A}$$

Canonical Momentum

Hamiltonian - Energy

$$H(\mathbf{P},\mathbf{x},t) = mc^2\gamma + q\Phi$$

$$\gamma = \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}$$

$$\gamma = \left(1 + \frac{\left(\mathbf{P} - q\mathbf{A}/c\right)^2}{m^2c^2}\right)^{1/2}$$

Write relativistic factor in terms of P

Differentiate H w.r.t. time

$$\frac{d}{dt}H = \frac{1}{m\gamma}\left(\mathbf{P} - q\mathbf{A}/c\right)\cdot\frac{d}{dt}\left(\mathbf{P} - q\mathbf{A}/c\right) + q\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\Phi$$

$$\frac{d}{dt}H = \mathbf{v}\cdot\frac{d}{dt}\mathbf{p} + q\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\Phi = q\mathbf{v}\cdot E + q\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\Phi = \frac{\partial}{\partial t}H$$

$$\boxed{\frac{d}{dt}H = \frac{\partial}{\partial t}H}$$

Time independent H is constant

Hamilton's Equations - Conservation Laws

$$\frac{d}{dt}\mathbf{P} = -q\nabla\left(\Phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) = -\nabla H \qquad \frac{d}{dt}\mathbf{x} = \frac{\mathbf{p}}{\gamma m} = \frac{\partial}{\partial \mathbf{p}}H$$

$$\frac{d}{dt}P_z = -q\frac{\partial}{\partial z}\left(\Phi - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right) = -\frac{\partial H}{\partial z}$$

If fields only depend on z-ct (plane wave) then:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$P_x = \text{constant}$$

$$P_y = \text{constant}$$

$$H - cP_z = \text{constant}$$

- Plane Wave Laser Field: $\boldsymbol{E} = -\frac{\partial}{c\partial t}\boldsymbol{A}_\perp = \left[i\frac{\omega}{c}\hat{\boldsymbol{A}}_\perp \exp[-i\omega(t - z/c)] + c.c.\right]/2$

- Electron Hamiltonian: $H = mc^2\gamma = H(\boldsymbol{P}, t - z/c), \qquad \boldsymbol{P} = \boldsymbol{p} + \frac{q\boldsymbol{A}}{c}$

- Relativistic Factor: $\gamma = \sqrt{1 + \left(\frac{P_z}{mc}\right)^2 + \left(\frac{P_\perp - qA_\perp/c}{mc}\right)^2}$

- Hamilton's Equations:

$$\frac{dP_\perp}{dt} = -\frac{\partial H}{\partial x_\perp} \quad \Rightarrow P_\perp = \text{constant} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t}, \quad \frac{dP_z}{dt} = -\frac{\partial H}{\partial z} \quad \Rightarrow H - cP_z = \text{constant}$$

- **Laser Field:** $$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x}_\perp, z, t)$$

Weak dependence on transverse coordinate

- **Transverse Canonical Momentum:** $$\boldsymbol{p}_\perp + \frac{q}{c}\boldsymbol{A}_\perp = const. = 0$$

- **Quiver velocity:** $$\frac{\boldsymbol{p}_\perp}{mc} = \frac{\gamma \boldsymbol{v}_\perp}{c} = -\frac{q\boldsymbol{A}_\perp}{mc^2} = -\boldsymbol{a}$$

Normalized vector potential

- **Example:** $I=10^{18}$ *watts/cm²*, $\lambda = 10^{-4}$ *cm*, $|a| = .86$

Assume electrons originate in field-free region: $P_{\perp}=0, \quad P_{z0}=0, \quad \gamma_0=1$

Constants of motion imply: $p_{\perp}=-\frac{qA_{\perp}}{c} \propto -\cos\omega t\,, \quad p_z=\frac{p_{\perp}^2}{2mc} \propto \cos^2\omega t = \frac{1+\cos 2\omega t}{2}$

Mean drift in propagation direction

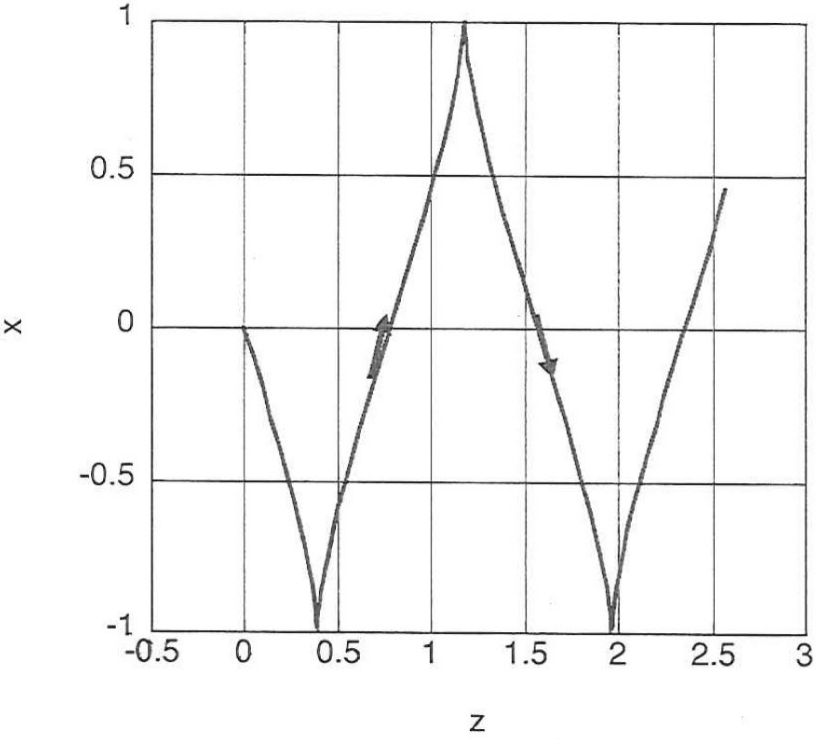
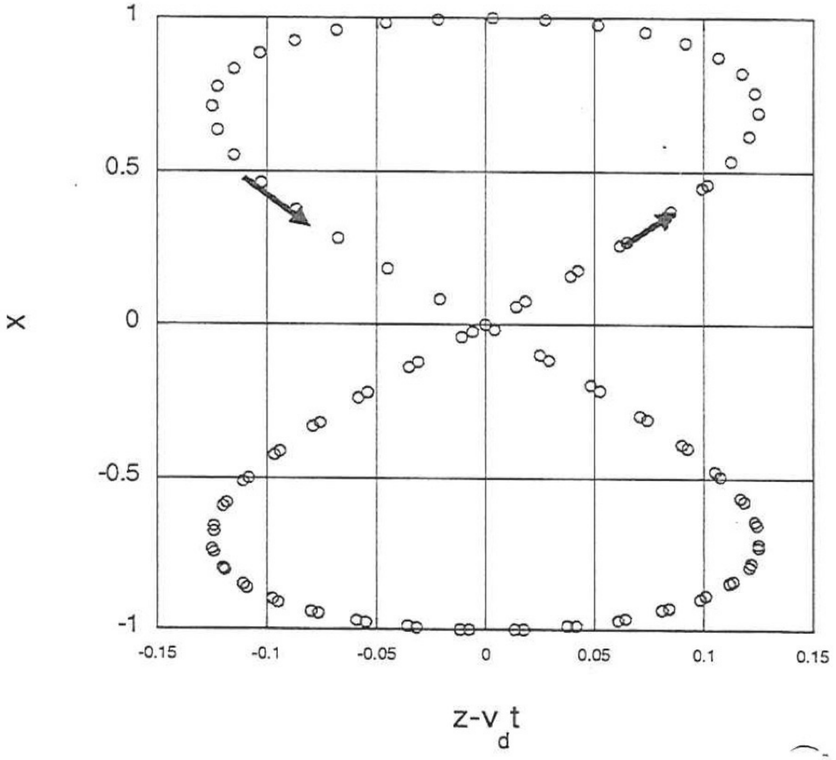


Figure-8 in drifting frame



- **Ponderomotive Force:** low frequency force , quadratic in field strength

- **Lorentz force:** $$\boldsymbol{F} = q(\boldsymbol{E}(\boldsymbol{x},t) + \frac{\boldsymbol{v} \times \boldsymbol{B}(x,t)}{c})$$

- **Trajectory:** $$\boldsymbol{x}(t) = \boldsymbol{x}_0(t) + \tilde{\boldsymbol{x}}(t)$$

Rapid quiver

Slowly varying

- **Slowly varying component:** $$\boldsymbol{F}_p = q\left\langle \tilde{\boldsymbol{x}} \cdot \nabla \boldsymbol{E}(\boldsymbol{x}_0,t) + \frac{\tilde{\boldsymbol{v}} \times \boldsymbol{B}(\boldsymbol{x}_0,t)}{c} \right\rangle_{\text{Laser period}}$$

- **Ponderomotive Potential:** $$\boldsymbol{F}_p = -\frac{mc^2}{2\gamma}\nabla\left\langle |\boldsymbol{a}|^2 \right\rangle$$

Proportional to laser intensity