Special Relativity

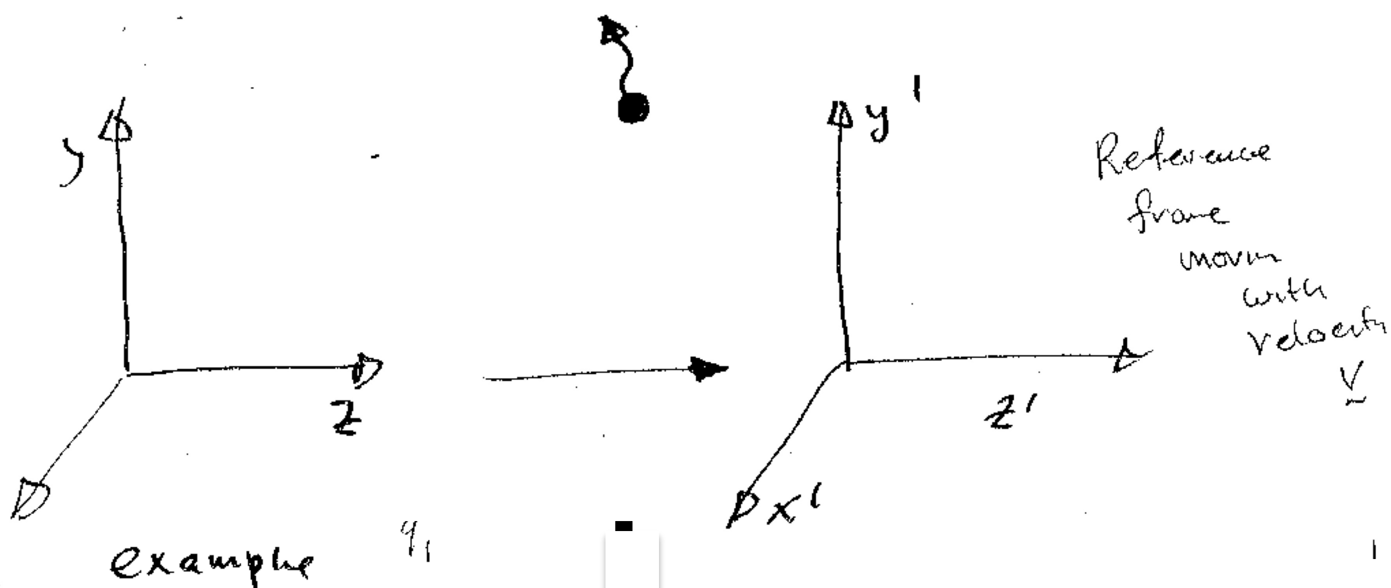
4/27/21

Classical Mechanics-Galilean Transfromations

1) The laws of Newtonian mechanics are invariant under Galilean transformations

2) Maxwell's equations are not invariant under galilean transformations

$\underset{\sim}{F} = m\underset{\sim}{a}$



$\underset{\sim}{F} = \underset{\sim}{F}'$

$m = m'$

$\underset{\sim}{u} = \underset{\sim}{u}' + \underset{\sim}{V}$

$t' = t$

example

$$\underset{\sim}{F} = +\frac{q_1 q_2 (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}{4\pi\epsilon_0 |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^3}$$

$$\frac{du}{dt} = \frac{du'}{dt} = \frac{F}{m}$$

Transformation of Wave Equation

Express the wave equation in terms of the coordinates in the moving frame,

$$\underset{\sim}{x} = \underset{\sim}{x}' + \underset{\sim}{v}t' \qquad \underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{v}t$$

$$t = t' \qquad t' = t$$

$$\frac{\partial}{\partial \underset{\sim}{x}} = \frac{\partial \underset{\sim}{x}'}{\partial \underset{\sim}{x}} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} + \frac{\partial t'}{\partial \underset{\sim}{x}} \frac{\partial}{\partial t'} = \frac{\partial}{\partial \underset{\sim}{x}'}$$

$$\frac{\partial}{\partial t} = \frac{\partial \underset{\sim}{x}'}{\partial t} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = -\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} + \frac{\partial}{\partial t'}$$

Wave Equation

$$\nabla^2 = \frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial}{\partial \underset{\sim}{x}} = \frac{\partial}{\partial \underset{\sim}{x}'} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}'}$$

THUS, IN TERMS OF PRIMED COORDINATES

$$\left[\frac{\partial}{\partial \underset{\sim}{x}'} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} - \frac{1}{c^2} \left(\frac{\partial}{\partial t'} - \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} \right)^2 \right] \cancel{\begin{pmatrix} \phi \\ \underset{\sim}{A} \end{pmatrix}} = 0$$

It should be

$$\left[\frac{\partial}{\partial \underset{\sim}{x}'} \cdot \frac{\partial}{\partial \underset{\sim}{x}'} - \frac{1}{c^2} \left(\frac{\partial}{\partial t'} \right)^2 \right] \begin{pmatrix} \phi' \\ \underset{\sim}{A}' \end{pmatrix} = 0$$

Explanations

assume ~~not~~ NEWTONIAN mechanics is correct

EITHER

1) Maxwell's equations wrong

or

2) Maxwell's equations are not supposed to be invariant

ETHER EXISTS MEDIUM supporting light waves

Alternatively

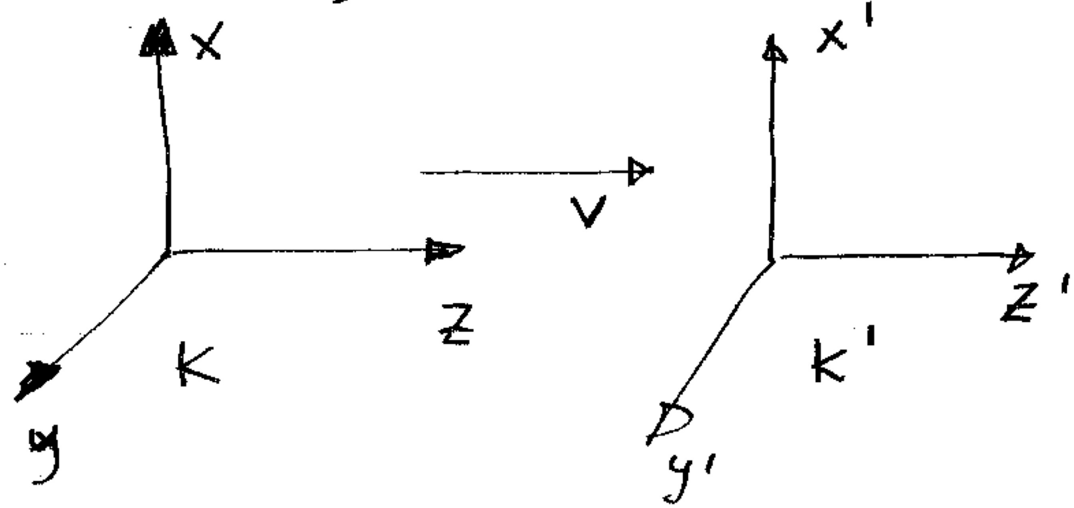
Assumes maxwell's equations are correct. & Lorentz transformations appropriate

1) THEN THE SPEED of LIGHT is THE same for any observer

2) Newtonian mechanics is wrong

but can be fixed

Lorentz Transformation



$$x = x'$$

$$y = y'$$

$$z = \gamma(z' + vt')$$

$$t = \gamma(t' + (v/c^2)z')$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - vt)$$

$$t' = \gamma(t - (v/c^2)z)$$

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}$$

THEN

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= \nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

"Derivation" of Lorentz Transformation

$x' = a_1 x + a_2 t$

$t' = a_3 t + a_4 x$

TRANSFORMATION must BE LINEAR. SPACE & TIME ARE HOMOGENEOUS

$x \to x + \Delta x \quad , \quad t \to t + \Delta t$

$x' \to x' + \Delta x' \quad , \quad t' \to t + \Delta t'$

$\Delta x'$ & $\Delta t'$ can not depend on x, t

Observers see each other at +-v

Requirement #1 — origin of x' is moving at velocity v in x, t

$x' = 0 = (a_1 v + a_2)t$

$a_2 = -a_1 v$

$x' = a_1 x + a_2 t$

$t' = a_3 t + a_4 x$

Requirement #2 — origin of x is moving at velocity $-v$ in x', t'

$x' = a_2 t = -a_1 v t = -v t'$

$t' = a_3 t$ $\qquad a_1 = a_3$

$x' = a_1 (x - vt)$ $\qquad$ call $a_4/a_1 = \hat{a}_4$

$t' = a_1 (t + a_4/a_1 x)$

Requirement #3 $\quad x^2 - c^2t^2 = (x')^2 - (ct')^2$

if $x = \pm ct$ then $x' = \pm ct'$

$$x'^2 - c^2t'^2 = a_1^2\left[(x-vt)^2 - c^2(t+\hat{a}_4 x)^2\right]$$

$$= a_1^2\left[x^2+v^2t^2-2xvt - c^2(t^2+2\hat{a}_4 xt+\hat{a}_4^2x^2)\right]$$

$$\hat{a}_4 = -\frac{v}{c^2}$$

$$x'^2-c^2t'^2 = a_1^2\left\{x^2\left(1-\frac{v^2}{c^2}\right) - c^2\left(1-\frac{v^2}{c^2}\right)t^2\right]$$

$$= x^2-c^2t^2 \qquad a_1^2\left(1-\frac{v^2}{c^2}\right)=1$$

$$a_1=\gamma=(1-v^2/c^2)^{-1/2}$$

What exactly does the Lorenz transformation mean, and how does one use it. Think of time as being a fourth coordinate. An event is characterized by its four coordinates. An event can be anything that occurs at a particular place and time.

Let $\underset{\sim}{x}, t$ be the coordinates of an event in some reference frame K. The same event as observed in an other reference frame will have a different set of coordinates $\underset{\sim}{x}', t'$

[redacted]

These coordinates are related to the coordinates in the unprimed frame by the Lorentz transformation

TRANSFORMATION $x \to x'$

$$x' = x$$
$$y' = y$$
$$z' = \gamma(z - vt)$$
$$t' = \gamma(t - (v/c^2)z)$$

INVERSE $x' \to x$

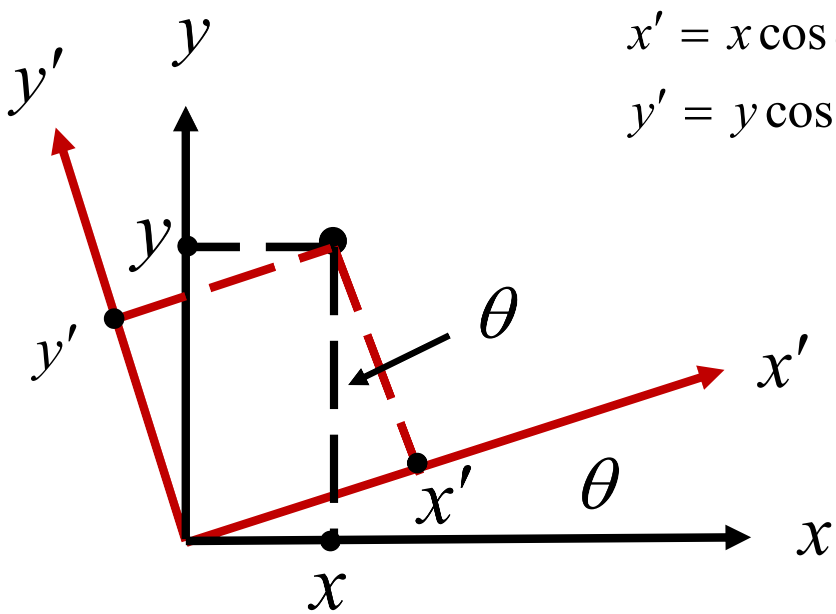
$$x = x'$$
$$y = y'$$
$$z = \gamma(z' + vt')$$
$$t = \gamma\left(t + \frac{v}{c^2} z\right)$$

Implicit is that we have picked our coordinate system such that an event at the origin in one frame occurs at the origin in the other

$$\underset{\sim}{x} = 0 \quad t = 0 \iff \underset{\sim}{x}' = 0 \quad t' = 0$$

Coordinate Rotation in a Plane

In ordinary space the x and y coordinates in a rotated coordinate system are given by



$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$
$$\rightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

length is perserved

$$(x')^2 + (y')^2 = x^2 + y^2$$

Lorentz transformations are analogous to a rotation of coordinates in **space – time**

Lorentz transformations

$$\begin{pmatrix} x' \\ c\,t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\,\beta \\ -\gamma\,\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ c\,t \end{pmatrix} = \begin{pmatrix} \cosh(\varsigma) & -\sinh(\varsigma) \\ -\sinh(\varsigma) & \cosh(\varsigma) \end{pmatrix} \begin{pmatrix} x \\ c\,t \end{pmatrix}$$

Invariant to a Lorentz transformation $\rightarrow$ like a rotation in space time coordinates

Length (in space-time) is conserved under a Lorentz transformation

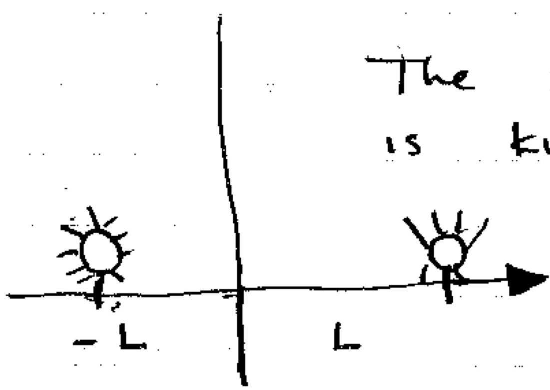
$$\gamma^2 - \gamma^2\,\beta^2 = 1 \quad \rightarrow \quad \cosh^2(\varsigma) - \sinh^2(\varsigma) = 1$$

$$\boxed{-c^2\,t^2 + x^2 + y^2 + z^2 = -c^2\,(t')^2 + (x')^2 + (y')^2 + (z')^2}$$

LACK OF simultaneity.

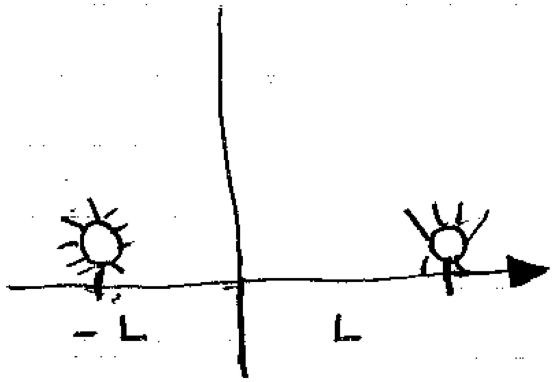
Two events occuring at the same time in one coordinate system may occur at different times in an other.

Example suppose that on observer in the unprimed frame determines that two ~~bulb~~ lights flashed on at $t=0$ at $Z=+L$ and $Z=-L$.



The speed of light is known to be c,

Thus, the precise time at which the bulbs flashed can be determined. Call this $t=0$



Coordinates of event #

$t = 0$

$x_1 = 0$

$y_1 = 0$

$z_1 = L$

~~Coor~~ Coordinates of event #2

$t = 0$

$x_2 = 0$

$y_2 = 0$

$z_2 = -L$

Observer moving with speed V in frame K' also sees two bulbs flash. Knowing the speed of light and the location of the flashes he can determine the precise times of the flashes

Event # 1

$x_1' = x_1 = 0$

$y_2' = y_2 = 0$

$z_1' = \gamma(z_1 - vt_1) = \gamma z_1 = \gamma L$

$t_1' = \gamma(t_1 - \frac{v}{c^2} z_1) = -\gamma \frac{v}{c^2} L$

Event #2

$x_2' = x_2 = 0$

$y_2' = y_2 = 0$

$z_2' = \gamma(z_2 - vt_2) = -\gamma L$

$t_2' = \gamma(t_2 - \frac{v}{c^2} z_2) = \gamma \frac{v}{c^2} L$

FOR THE PRIMED OBSERVER EVENT #1 occured before event #2

Could event #2 have caused event #1 ?

$$t_2' - t_1' = \frac{2(L\gamma)}{c}\left(\frac{\gamma v}{c}\right) < \frac{2(L\gamma)}{c} = \frac{\text{separation}}{c}$$

$$\frac{\gamma v}{c} = \frac{\beta}{\sqrt{1-\beta^2}} \qquad \beta = v/c$$

Separation between two events is said to be space like

Time and Space-like separations

Invariant separation $$s_{21}^2 = c^2(t_1 - t_2)^2 - \left|\mathbf{x}_1 - \mathbf{x}_2\right|^2$$

$$s_{21}^2 > 0 \quad \text{Time-like} \;, \qquad s_{21}^2 < 0 \quad \text{Space-like}$$

IF the separations are time like it is always possible to find a coordinate system where $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$ but $t_1 \neq t_2$

If the separation is space like it is possible to find a coordinate frame where $t_1 = t_2$

Time dilation and length contraction

An observer in reference frame K sees a meter stick moving at speed v in the z direction.

How long does it appear to be?

Construct events that give the relation between the length of the stick as measured in K and as measured in K' a frame co-moving with the stick

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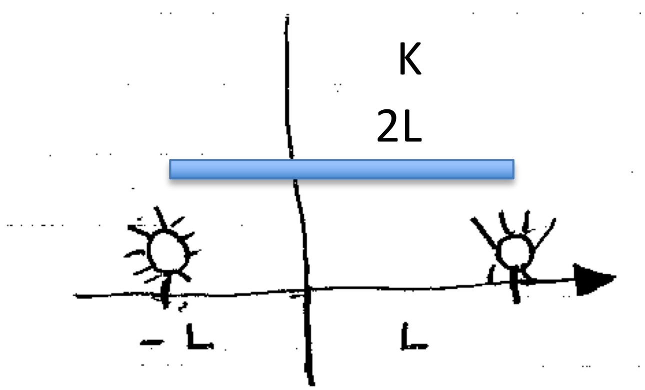


In K' Length is $2\gamma L = 1\text{m}$

Length in K is $2L = 1\text{m}/\gamma$

Time Dilation

A vehicle goes by at speed v, on the vehicle there is a light timed to flash periodically with period T'.

What is the period as observed in the frame K in which the vehicle is moving?

In co-moving frame K' $\quad z_1' = z_2' = 0, \quad t_1' = 0, \quad t_2' = T'$

Inverse transformation

$$z = \gamma\left(z' + vt'\right)$$

$$t = \gamma\left(t' + \frac{v}{c^2}z'\right)$$

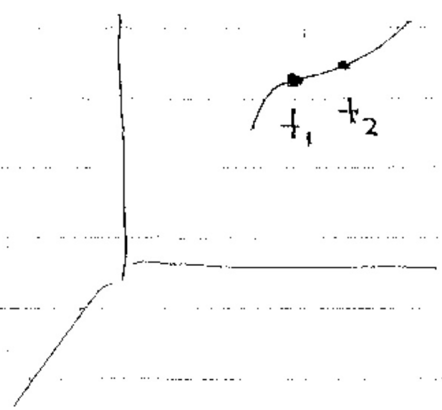
$$t_2 = \gamma T$$

Time between flashes as observed in K is longer than as observed in K'

Proper Time

<u>Proper time</u>

Consider an object moving with velocity $\underset{\sim}{u}(t)$. ~~Suppose the~~ consider ~~object~~ two events at the location of the object separated by small time difference dt



$$t_2 = t_1 + dt$$

$\underset{\sim}{x}_2 \cong \underset{\sim}{x}_1 + \underset{\sim}{u}\,dt$

S_{21}^2 is the same for all observers

$$S_{21}^2 = c^2 dt^2 - u^2 dt^2 = \frac{c^2 dt^2}{\gamma_u^2}$$

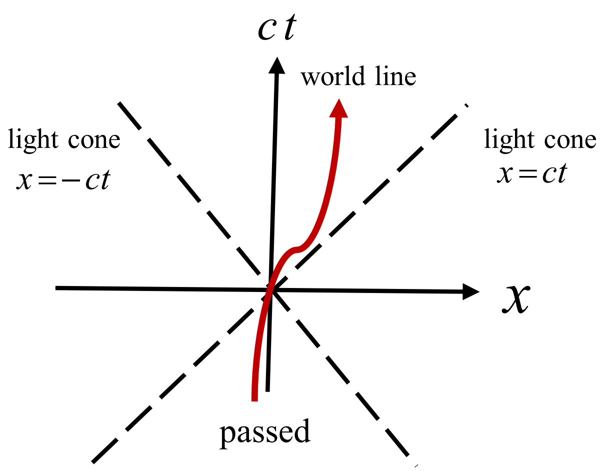
define $\quad d\tau = \sqrt{\frac{S_{21}^2}{c^2}} = \frac{dt}{\gamma_u(t)} \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$

$d\tau$ = interval of proper time

same for all inertial observers.

Proper Time is a Lorentz Invariant

Minkowski diagram



Consider the **four vector** $\mathbf{x}^{\mu} = (c\,t,\, \mathbf{x})$

All **four vectors** transform

the same way as $\mathbf{x}^{\mu} = (c\,t,\, \mathbf{x})$

Two points on the world line very close to each other

$$(\Delta s)^2 = -c^2\,(\Delta t)^2 + (\Delta x)^2 = -c^2\,(\Delta t)^2\,(1 - u^2/c^2)$$

$$\boxed{\Delta\tau = \frac{\Delta t}{\gamma} \quad \text{proper time interval}}$$

The quantity $\tau^2 = t^2 - (x^2 + y^2 + z^2)/c^2 \quad \rightarrow$ Lorentz invariant

The value of τ is the same in all inertial referrence frames, τ is the **proper time**

$$\tau^2 = t^2 - x^2/c^2 - y^2/c^2 - z^2/c^2 = (t')^2 - (x')^2/c^2 - (y')^2/c^2 - (z')^2/c^2$$

$$x' = \gamma\,(x - u\,t) \qquad y' = y \qquad z' = z \qquad t' = \gamma\,(t - u\,x/c^2)$$

Addition of Velocities

$x(t)$ trajectory in K

$x'(t')$ trajectory in K'

K' moves with velocity v in z direction in K

$$\mathrm{d}x'_\perp = \mathrm{d}x_\perp$$

$$dz' = \gamma\left(dz - v\,dt\right)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dz\right)$$

$$u'_\perp = \frac{\mathrm{d}x'_\perp}{dt'} = \frac{\mathrm{d}x_\perp}{\gamma\left(dt - \frac{v}{c^2}dz\right)} = \frac{\mathrm{d}x_\perp/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dz}{dt}\right)} = \frac{u_\perp}{\gamma\left(1 - vu_z/c^2\right)}$$

$$u'_\perp = \frac{u_\perp}{\gamma\left(1 - vu_z/c^2\right)}$$

$$u'_z = \frac{dz'}{dt'} = \frac{u_z - v}{\left(1 - vu_z/c^2\right)}$$

Four - Vectors

Quantities that transform from frame to frame according to Lorentz transformation

$$\left(A_0, \mathbf{A}\right) = \left(A_0, A_1, A_2, A_3\right)$$

$$A_0' = \gamma\left(A_0 - \beta A_1\right)$$
$$A_1' = \gamma\left(A_1 - \beta A_0\right)$$
$$A_2' = A_2$$
$$A_3' = A_3$$

Example: Space-time coordinate

$$\left(ct, z, x, y\right)$$

Invariant Product

$\mathrm{A} \circ \mathrm{B} = \mathrm{A}_0\mathrm{B}_0 - (\mathrm{A}_1\mathrm{B}_1 + \mathrm{A}_2\mathrm{B}_2 + \mathrm{A}_3\mathrm{B}_3)$

Same for all observers

Examples

Space-time coordinate $\left(ct, z, x, y\right) \equiv X$

Space-time wave vector $\left(\frac{\omega}{c}, k_z, k_x, k_y\right) \equiv K$

Invariant product

$\mathrm{K} \circ \mathrm{X} = \omega t - \mathbf{k} \cdot \mathbf{x} = \Phi$ wave phase

Same for all observers

Derivatives

Examples of 4-vectors

$$(ct, \underbrace{x_1}_{x_\parallel}, \underbrace{x_2, x_3}_{x_\perp})$$

<u>what about derivatives?</u>

$$\left.\begin{aligned} \underline{x}_\perp' &= \underline{x}_\perp \\ x_\parallel' &= \gamma(x_\parallel - \beta ct) \\ ct' &= \gamma(ct - \beta x_\parallel) \end{aligned}\right\} \qquad \left\{\begin{aligned} x_\perp &= x_\perp' \\ x_\parallel &= \gamma(x_\parallel' + \beta ct') \\ ct &= \gamma(ct' + \beta x_\parallel') \end{aligned}\right.$$

$$\left.\frac{\partial}{\partial x_\perp'}\right|_{t', x_\parallel'} = \underbrace{\frac{\partial x_\perp}{\partial x_\perp'}}_{=1} \frac{\partial}{\partial x_\perp} + \cancel{\frac{\partial x_\parallel}{\partial x_\perp'}}^{\,0} \frac{\partial}{\partial x_\parallel} + \cancel{\frac{\partial ct}{\partial x_\perp'}}^{\,0} \frac{\partial}{\partial ct}$$

$$\frac{\partial}{\partial x_\perp'} = \frac{\partial}{\partial x_\perp}$$

$$\left.\frac{\partial}{\partial x_\parallel'}\right|_{t', x_\perp'} = \cancel{\frac{\partial x_\perp}{\partial x_\parallel'}}^{\,0} \frac{\partial}{\partial x_\perp} + \frac{\partial x_\parallel}{\partial x_\parallel'} \frac{\partial}{\partial x_\parallel} + \frac{\partial ct}{\partial x_\parallel'} \frac{\partial}{\partial ct}$$

$$= \gamma \left(\frac{\partial}{\partial x_\parallel} + \beta \frac{\partial}{\partial ct} \right)$$

$$\frac{\partial}{c\,\partial t'} = \gamma \left(\frac{\partial}{\partial ct} + \beta \frac{\partial}{\partial x_\parallel} \right)$$

$\left(\frac{\partial}{\partial ct}, -\frac{\partial}{\partial x_\parallel}, -\frac{\partial}{\partial x_\perp} \right)$ is a 4-vector

THIS means if $(A_0, A_1, \underset{\sim}{A}_2)$ is a fourvector Field

Then

$$\frac{\partial A_0}{\partial ct} - \left(-\frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{A}\right)$$ is a Lorentz invariant

$$\frac{\partial}{\partial ct} A_0 + \nabla \cdot \underline{A}$$ is a Lorentz invariant

Contuity of charge

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \underset{\sim}{J} = 0$$

$$\frac{\partial}{\partial ct} c\rho + \nabla \cdot \underline{J} = 0 \qquad (c\rho, \underset{\sim}{J})$$ is a 4 vector