Anisotropic Media

$$\underline{D} = \underline{\underline{\epsilon}} \cdot \underline{E}$$

$$\underline{\underline{\epsilon}} = \begin{bmatrix} & & \\ & \epsilon_{ij} & \\ & & \end{bmatrix} \quad 3\times3 \text{ matrix}$$

Examples

uniaxial crystal (one direction is different from the other 2)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

biaxial (all directions are differ)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

NOTE $\underline{\underline{\epsilon}}$ is still diagonal gyrotropic (magnetized plasma)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

Plane Waves

$$\nabla \times \underline{H} = \frac{\partial}{\partial t} \underline{\underline{\epsilon}} \cdot \underline{E} \qquad \underline{k} \times \hat{\underline{H}} = -\omega \underline{\underline{\epsilon}} \cdot \hat{\underline{E}}$$

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t} \mu_0 \underline{H} \qquad \underline{k} \times \hat{\underline{E}} = \omega \mu_0 \hat{\underline{H}}$$

Poynting Flux

$$\overline{\underline{S}} = \frac{1}{2} \mathrm{Re} \left\{ \hat{\underline{E}}^* \times \hat{\underline{H}} \right\} = \frac{1}{2} \mathrm{Re} \left\{ \hat{\underline{E}}^* \times \frac{1}{\omega \mu_0} \left(\underline{k} \times \hat{\underline{E}} \right) \right\}$$

$$\overline{\underline{S}} = \frac{1}{2 \omega \mu_0} \mathrm{Re} \left\{ \underline{k} |\underline{E}|^2 - \hat{\underline{E}}\, \underline{k} \cdot \hat{\underline{E}}^* \right\}$$

Direction of Power flow and wavevector are different unless $\underline{k} \cdot \hat{\underline{E}}^* = 0$

DISPERSION RELATION

$$\underline{k} \times (\omega \mu_0 \hat{\underline{H}}) = -(\omega^2 \mu_0 \epsilon_0) \frac{1}{\epsilon_0} \underline{\underline{\epsilon}} \cdot \underline{E}$$

$$\underline{k} \times (\underline{k} \times \hat{\underline{E}}) = -k_0^2 \, \underline{\underline{\epsilon}}_r \cdot \underline{E}$$

$$\underline{k}\,\underline{k} \cdot \underline{E} - k^2 \underline{E} + k_0^2 \underline{\underline{\epsilon}}_r \cdot \underline{E} = 0$$

$$\left[\underline{k}\,\underline{k} - \underline{\underline{1}} k^2 + k_0^2 \underline{\underline{\epsilon}}_r \right] \cdot \underline{E} = 0$$

dispersion relation $\quad \mathrm{Det}\left[\underline{k}\,\underline{k} - \underline{\underline{1}} k^2 + k_0^2 \underline{\underline{\epsilon}}_r \right] = 0$

Consider uni-axial case

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

take k to be in x-z plane

$$\begin{bmatrix} k_x^2 - k^2 + k_0^2 \epsilon_\perp & 0 & k_x k_z \\ 0 & -k^2 + k_0^2 \epsilon_\perp & 0 \\ k_x k_z & 0 & k_z^2 - k^2 + k_0^2 \epsilon_\parallel \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

DET { above } — mess

Alternatively write

$$\underline{\underline{\epsilon}}_r = \underline{\underline{1}}\,\epsilon_\perp + \hat{z}\hat{z}(\epsilon_\parallel - \epsilon_\perp)$$

$$\begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\perp \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_\parallel - \epsilon_\perp \end{bmatrix}$$

$$\underline{k}(\underline{k}\cdot\hat{\underline{E}}) - k^2\hat{\underline{E}} + k_0^2\epsilon_\perp\hat{\underline{E}} + \hat{z}k_0^2(\epsilon_\parallel - \epsilon_\perp)\hat{z}\cdot\hat{\underline{E}} = 0$$

Consider THREE nonorthogonal components of $\underline{E}$

$$\underline{k}\cdot\hat{\underline{E}}, \quad \hat{z}\cdot\hat{\underline{E}}, \quad \underline{k}\times\hat{z}\cdot\hat{\underline{E}}$$

$\underline{k}\times\hat{z}\cdot\{\text{above}\} \;=\; (k_0^2\epsilon_\perp - k^2)\,\underline{k}\times\hat{z}\cdot\underline{E} = 0$

$\hat{z}\cdot\{\text{above}\} \qquad k_z(\underline{k}\cdot\underline{E}) - (k^2 - k_0^2\epsilon_\parallel)\hat{z}\cdot\underline{E} = 0$

$\underline{k}\cdot\{\text{above}\} \qquad k_0^2\epsilon_\perp\,\underline{k}\cdot\hat{\underline{E}} + k_z k_0^2(\epsilon_\parallel - \epsilon_\perp)\hat{z}\cdot\underline{E} = 0$

FROM FIRST EQUATION

either $(k_0^2\epsilon_\perp - k^2) = 0$ $\quad \underline{k}\times\hat{z}\cdot\underline{E} \neq 0$

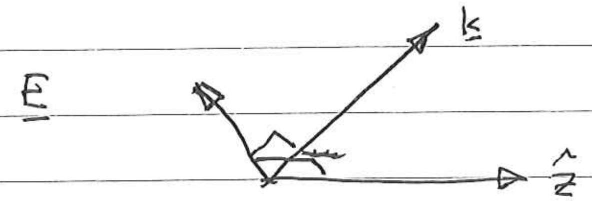
or $(k_0^2\epsilon_\perp - k^2) \neq 0$ $\quad \underline{k}\times\hat{z}\cdot\underline{E} = 0$

second and third equation give

$$\begin{bmatrix} k_z & -(k^2 - k_0^2\epsilon_\parallel) \\ k_0^2\epsilon_\perp & k_z k_0^2(\epsilon_\parallel - \epsilon_\perp) \end{bmatrix} \begin{pmatrix} \underline{k}\cdot\underline{E} \\ \hat{z}\cdot\underline{E} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

IF $k_0^2\epsilon_\perp - k^2 = 0$ then $\underline{k}\cdot\underline{E}, \hat{z}\cdot\underline{E} = 0$

$\underline{k}\times z\cdot\underline{E} \neq 0$



ORDINARY MODE

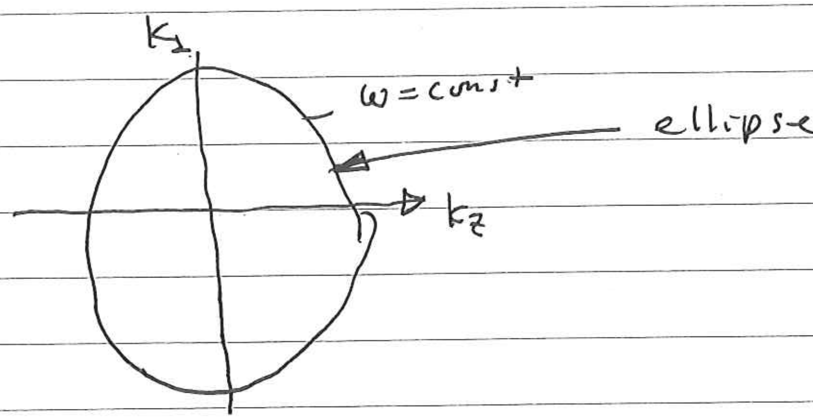
$$\mathrm{DET}\begin{bmatrix} k_z & -(k^2 - k_0^2\epsilon_\parallel) \\ k_0^2\epsilon_\perp & k_z k_0^2(\epsilon_\parallel - \epsilon_\perp) \end{bmatrix} = 0$$

$$k_z^2 k_0^2(\epsilon_\parallel - \epsilon_\perp) + (k^2 - k_0^2\epsilon_\parallel)k_0^2\epsilon_\perp = 0$$

$$k_z^2(\epsilon_\parallel - \epsilon_\perp) + \epsilon_\perp(k_\perp^2 + k_z^2 - k_0^2\epsilon_\parallel) = 0$$

$$k_z^2\epsilon_\parallel + k_\perp^2\epsilon_\perp - k_0^2\epsilon_\parallel\epsilon_\perp = 0$$

Solution $$k_0^2 = \frac{k_z^2}{\epsilon_\perp} + \frac{k_\perp^2}{\epsilon_\parallel} = \frac{\omega^2}{c^2}$$



Polarization

$$k_z \underline{k}\cdot\underline{E} = (k_z^2 + k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot E$$

$$k_z \underline{k}_\perp\cdot\underline{E}_\perp + k_z^2\,\hat{z}\cdot\underline{E} = (k_z^2 + k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot\underline{E}$$

$$k_z \underline{k}_\perp\cdot\underline{E}_\perp = (k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot\hat{\underline{E}}$$

$$k_0^2\epsilon_\parallel = k_\perp^2 + \frac{\epsilon_\parallel}{\epsilon_\perp}k_z^2$$

$$k_z \underline{k}_\perp\cdot\underline{E}_\perp = -\frac{\epsilon_\parallel}{\epsilon_\perp}k_z^2\,\underline{z}\cdot\underline{E}$$

$$\underline{k}_\perp\cdot\underline{E}_\perp = -\frac{\epsilon_\parallel}{\epsilon_\perp}k_z\,\hat{z}\cdot\underline{E}$$

$$\boxed{\frac{E_\perp}{E_z} = -\frac{\epsilon_\parallel}{\epsilon_\perp}\frac{k_z}{k_\perp}}$$