Gyrotropic: Example collisionless plasma in confining magnetic field

$$\underline{B} = B\hat{z}$$

$$\underline{\underline{\epsilon}}_r = \begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$ relative dielectric tensor

$$\underline{\underline{\epsilon}}^{*,t} = \underline{\underline{\epsilon}}$$ Hermitian

$$\epsilon_\perp = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2 - \Omega_q^2} \qquad \epsilon_\parallel = 1 - \sum_q \frac{\omega_{pq}^2}{\omega^2}$$

$$\epsilon_x = \sum_q \frac{i\Omega_q}{\omega} \frac{\omega_{pq}^2}{\omega^2 - \Omega_q^2}$$

$$\omega_{pq}^2 = \frac{nq^2}{m\epsilon_0}$$ plasma frequency of specie q

$$\Omega_q = \frac{qB}{m}$$ cyclotron frequency

How to Calculate $\underline{\underline{\epsilon}}$

Newton's Law $\quad \dfrac{\partial \underline{V}}{\partial t} = \dfrac{q}{m_q}\underline{E} + \dfrac{q}{m_q}\underline{V}\times\underline{B} \qquad \underline{B} = B\hat{\underline{z}}$

Pass to Phasor Notation

$$-i\omega\hat{\underline{V}}_\perp = \frac{q}{m_q}\hat{\underline{E}}_\perp + \Omega_q \hat{\underline{V}}_\perp \times \hat{\underline{z}} \qquad -i\omega\hat{V}_z = \frac{q}{m_q}\hat{E}_z$$

$$\Omega_q = qB/m_q$$

Solve for $\hat{\underline{V}} = \hat{\underline{V}}_\perp + \hat{V}_z\hat{\underline{z}}$

$$\hat{\underline{V}}_\perp = \frac{q/m}{\omega^2 - \Omega^2}\left[i\omega\hat{\underline{E}}_\perp - \Omega\hat{\underline{E}}_\perp\times\hat{\underline{z}}\right] \qquad \hat{V}_z = \frac{q/m}{\omega}i\hat{E}_z$$

$$\hat{\underline{J}} = \sum_q q n_0 \hat{\underline{V}}$$

$$-i\omega\epsilon_0\hat{\underline{E}} + \hat{\underline{J}} = -i\omega\epsilon_0\,\underline{\underline{\epsilon}}\cdot\hat{\underline{E}}$$

From this the expression follows

Propagation $\perp$ to $\underline{B} = B\hat{z}$

Take $\underline{k} = k_x \hat{x}$

$$\underline{k} \times (\underline{k} \times \underline{E}) + \frac{\omega^2}{c^2} \underline{\underline{\epsilon}} \cdot \underline{E} = 0$$

$$\frac{\omega^2}{c^2} \begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp - n_x^2 & 0 \\ 0 & 0 & \epsilon_\parallel - n_x^2 \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \qquad n_x = \frac{k_x c}{\omega}$$

Two solutions for n_x^2

$n_x^2 = \epsilon_\parallel$ $\quad E_x, E_y = 0 \quad E_z \neq 0$ $\quad$ Ordinary mode

$\epsilon_\perp(\epsilon_\perp - n_x^2) - \epsilon_x^2 = 0$ $\quad E_z = 0 \quad E_x = i\frac{\epsilon_x}{\epsilon_\perp} E_y$ $\quad$ x-mode

<u>ORDinary mode</u> $\quad n_x^2 = \frac{k_x^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$

$k_x^2 c^2 = \omega^2 - \omega_p^2$ $\quad$ B plays no role!

Propagates if $\omega^2 > \omega_p^2$

X-mode $\quad n_x^2 = \dfrac{(\epsilon_\perp - \epsilon_x)(\epsilon_\perp + \epsilon_x)}{\epsilon_\perp}$

$$\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2}$$

$$\epsilon_\perp \pm \epsilon_x = 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2}\left(1 \mp \frac{\Omega}{\omega}\right)$$

$n_x^2 \to 0$ if $\epsilon_\perp \pm \epsilon_x = 0$

$$\epsilon_\perp \pm \epsilon_x = 1 - \frac{\omega_p^2}{\omega}\,\frac{1}{\omega \pm \Omega}$$

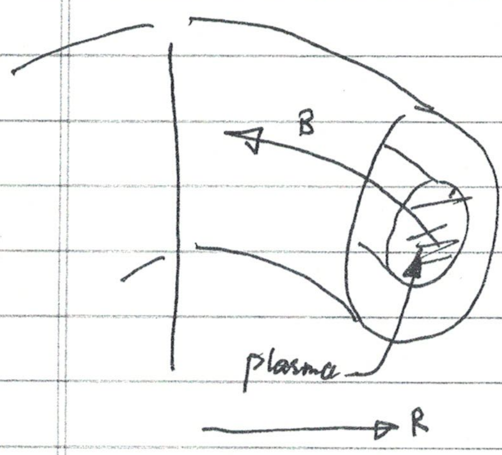
note $\dfrac{E_x}{E_y} = \dfrac{i\epsilon_x}{\epsilon_\perp}$ so, circularly polarized

cut-off

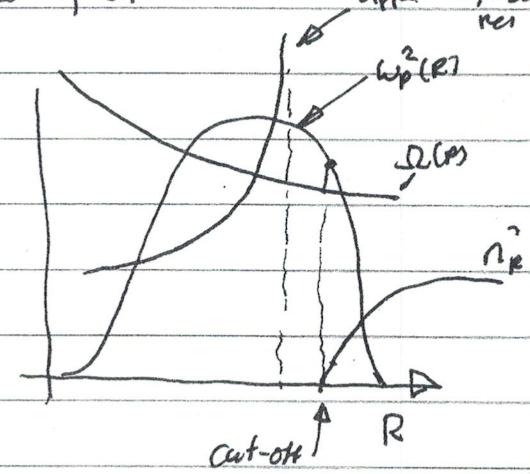
~~Note~~ $n_x^2 \to \infty \qquad \epsilon_\perp = 0$ resonance

$\epsilon_\perp = 1 - \dfrac{\omega_p^2}{\omega^2 - \Omega^2} \qquad \omega^2 = \Omega^2 + \omega_p^2$ upper hybrid resonance

Tokamak



profiles



Propagation $\parallel$ to $\underline{B} = B\hat{z}$

$$\begin{bmatrix} \epsilon_\perp - n_z^2 & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp - n_z^2 & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

solutions $\quad \epsilon_\parallel = 0 \quad$ plasma wave

$$(\epsilon_\perp - n_z^2)^2 - \epsilon_x^2 = 0$$

$$n_z^2 = \epsilon_\perp \pm \epsilon_x = \begin{cases} 1 - \dfrac{\omega_p^2}{\omega(\omega+\Omega)} \\ 1 - \dfrac{\omega_p^2}{\omega(\omega-\Omega)} \end{cases}$$

$$\frac{E_x}{E_y} = \frac{-i\epsilon_x}{\epsilon_\perp - n_z^2} = \begin{cases} i \\ -i \end{cases} \qquad \text{Circularly Polarized}$$

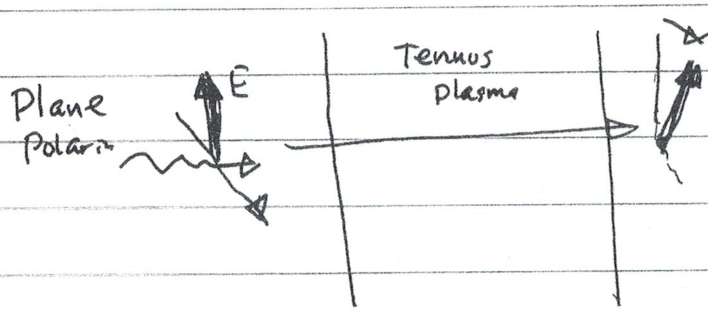
suppose $\quad \omega \gg \omega_p, \Omega \quad$ Low density

$$n_z^2 = 1 - \frac{\omega_p^2}{\omega^2}\left(1 \mp \frac{\Omega}{\omega}\right) \qquad n_z = \frac{k_z c}{\omega}$$

$$k_z = \bar{k}_z \pm \delta k_z$$

$$\bar{n}_z^2 = 1 - \frac{\omega_p^2}{\omega^2} \qquad 2\delta n_z n_z = \pm \frac{\omega_p^2}{\omega^3}\Omega$$

Faraday Rotation



Emerging wave
is plane polarized
but Vector $\underline{E}$
is rotated $\propto B$

Break into
$\pm$ circular
polarized

circularly
polarized
waves propagate with different k_z's