Anisotropic Media

$$\underline{D} = \underline{\underline{\epsilon}} \cdot \underline{E}$$

$$\underline{\underline{\epsilon}} = \begin{bmatrix} & & \\ & \epsilon_{ij} & \\ & & \end{bmatrix} \quad 3 \times 3 \text{ matrix}$$

Examples

uniaxial crystal (one direction is different from the other 2)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{\perp} & 0 & 0 \\ 0 & \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix}$$

biaxial (all directions are differen)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

NOTE $\underline{\underline{\epsilon}}$ is still diagonal gyrotropic (magnetized plasma)

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

Plane Waves

$$\nabla \times \underline{H} = \frac{\partial}{\partial t} \underline{\underline{\epsilon}} \cdot \underline{E} \qquad \underline{k} \times \underline{\hat{H}} = -\omega \underline{\underline{\epsilon}} \cdot \underline{\hat{E}}$$

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t} \mu_0 \underline{H} \qquad k \times \underline{\hat{E}} = \omega \mu_0 \underline{\hat{H}}$$

Poynting Flux

$$\bar{\underline{S}} = \frac{1}{2} Re\left\{ \underline{\hat{E}}^* \times \underline{\hat{H}} \right\} = \frac{1}{2} Re\left\{ \underline{\hat{E}}^* \times \frac{1}{\omega\mu_0} (\underline{k} \times \underline{\hat{E}}) \right\}$$

$$\bar{\underline{S}} = \frac{1}{2\omega\mu_0} Re\left\{ \underline{k} |\underline{E}|^2 - \underline{\hat{E}}\, \underline{k} \cdot \underline{\hat{E}}^* \right\}$$

Direction of Power flow and wavevector are different unless $\underline{k} \cdot \underline{\hat{E}}^* = 0$

DISPERSION RELATION

$$\underline{k} \times (\omega \mu_0 \hat{\underline{H}}) = -(\omega^2 \mu_0 \epsilon_0) \frac{1}{\epsilon_0} \underline{\underline{\epsilon}} \cdot \underline{E}$$

$$\underline{k} \times (\underline{k} \times \hat{\underline{E}}) = -k_0^2 \, \underline{\underline{\epsilon}}_r \cdot \underline{E}$$

$$\underline{k}\,\underline{k} \cdot \underline{E} - k^2 \underline{E} + k_0^2 \underline{\underline{\epsilon}}_r \cdot \underline{E} = 0$$

$$\left[\underline{k}\,\underline{k} - \underline{\underline{1}} k^2 + k_0^2 \underline{\underline{\epsilon}}_r \right] \cdot \underline{E} = 0$$

dispersion relation $\quad \text{Det}\left[\underline{k}\,\underline{k} - \underline{\underline{1}} k^2 + k_0^2 \underline{\underline{\epsilon}}_r \right] = 0$

Consider uni-axial case

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

take k to be in x-z plane

$$\begin{bmatrix} k_x^2 - k^2 + k_0^2 \epsilon_\perp & 0 & k_x k_z \\ 0 & -k^2 + k_0^2 \epsilon_\perp & 0 \\ k_x k_z & 0 & k_z^2 - k^2 + k_0^2 \epsilon_\parallel \end{bmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

DET {above} — mess

Alternatively write

$$\underline{\underline{\epsilon}}_r = \underline{\underline{1}}\epsilon_\perp + \hat{z}\hat{z}(\epsilon_\parallel - \epsilon_\perp)$$

$$\begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\perp \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_\parallel - \epsilon_\perp \end{bmatrix}$$

$$\underline{k}(\underline{k}\cdot\hat{\underline{E}}) - k^2\hat{\underline{E}} + k_0^2\epsilon_\perp\hat{\underline{E}} + \hat{z}k_0^2(\epsilon_\parallel - \epsilon_\perp)\hat{z}\cdot\hat{\underline{E}} = 0$$

Consider THREE nonorthogonal components of $\underline{E}$

$$\underline{k}\cdot\hat{\underline{E}}, \quad \hat{z}\cdot\hat{\underline{E}}, \quad \underline{k}\times\hat{z}\cdot\hat{\underline{E}}$$

$\underline{k}\times\hat{\underline{z}}\cdot\{\text{above}\}$ $\quad = \quad (k_0^2\epsilon_\perp - k^2)\,\underline{k}\times\hat{z}\cdot\underline{E} = 0$

$\hat{\underline{z}}\cdot\{\text{above}\}$ $\quad k_z(\underline{k}\cdot\underline{E}) - (k^2 - k_0^2\epsilon_\parallel)\hat{z}\cdot\underline{E} = 0$

$\underline{k}\cdot\{\text{above}\}$ $\quad k_0^2\epsilon_\perp\,\underline{k}\cdot\hat{\underline{E}} + k_z k_0^2(\epsilon_\parallel - \epsilon_\perp)\hat{z}\cdot\underline{E} = 0$

FROM FIRST EQUATION

either $(k_0^2 \epsilon_\perp - k^2) = 0 \qquad \underline{k} \times \hat{z} \cdot \underline{E} \neq 0$

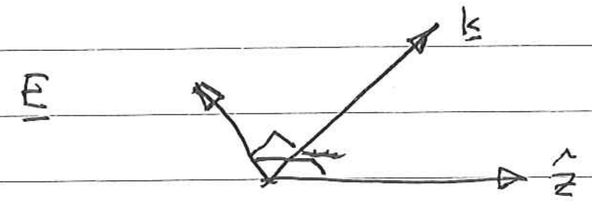
or $(k_0^2 \epsilon_\perp - k^2) \neq 0 \qquad \underline{k} \times \hat{z} \cdot \underline{E} = 0$

second and third equation give

$$\begin{bmatrix} k_z & -(k^2 - k_0^2 \epsilon_\parallel) \\ k_0^2 \epsilon_\perp & k_z k_0^2 (\epsilon_\parallel - \epsilon_\perp) \end{bmatrix} \begin{pmatrix} \underline{k} \cdot \underline{E} \\ \hat{z} \cdot \underline{E} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

IF $k_0^2 \epsilon_\perp - k^2 = 0$ then $\underline{k} \cdot \underline{E},\ \hat{z} \cdot \underline{E} = 0$

$\underline{k} \times z \cdot \underline{E} \neq 0$



ORDINARY MODE

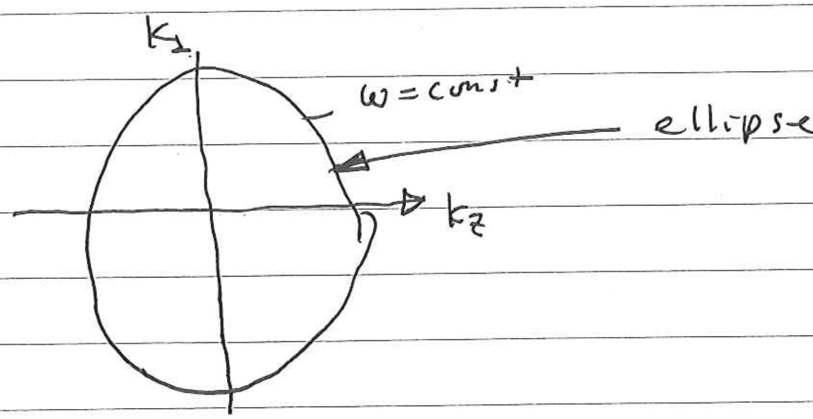
$$\text{DET} \begin{bmatrix} k_z & -(k^2 - k_0^2 \epsilon_\parallel) \\ k_0^2 \epsilon_\perp & k_z k_0^2 (\epsilon_\parallel - \epsilon_\perp) \end{bmatrix} = 0$$

$$k_z^2 k_0^2 (\epsilon_\parallel - \epsilon_\perp) + (k^2 - k_0^2 \epsilon_\parallel) k_0^2 \epsilon_\perp = 0$$

$$k_z^2 (\epsilon_\parallel - \epsilon_\perp) + \epsilon_\perp (k_\perp^2 + k_z^2 - k_0^2 \epsilon_\parallel) = 0$$

$$k_z^2 \epsilon_\parallel + k_\perp^2 \epsilon_\perp - k_0^2 \epsilon_\parallel \epsilon_\perp = 0$$

Solution $\quad k_0^2 = \frac{k_z^2}{\epsilon_\perp} + \frac{k_\perp^2}{\epsilon_\parallel} = \frac{\omega^2}{c^2}$



Polarization

$$k_z \mathbf{k}\cdot\mathbf{E} = (k_z^2 + k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot\mathbf{E}$$

$$k_z \mathbf{k}_\perp\cdot\mathbf{E}_\perp + k_z^2\,\hat{z}\cdot\mathbf{E} = (k_z^2 + k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot\mathbf{E}$$

$$k_z \mathbf{k}_\perp\cdot\mathbf{E}_\perp = (k_\perp^2 - k_0^2\epsilon_\parallel)\,\hat{z}\cdot\mathbf{E}$$

$$k_0^2\epsilon_\parallel = k_\perp^2 + \frac{\epsilon_\parallel}{\epsilon_\perp}k_z^2$$

$$k_z \mathbf{k}_\perp\cdot\mathbf{E}_\perp = -\frac{\epsilon_\parallel}{\epsilon_\perp}k_z^2\,\mathbf{z}\cdot\mathbf{E}$$

$$\mathbf{k}_\perp\cdot\mathbf{E}_\perp = -\frac{\epsilon_\parallel}{\epsilon_\perp}k_z\,\hat{z}\cdot\mathbf{E} \qquad \boxed{\frac{E_\perp}{E_z} = -\frac{\epsilon_\parallel}{\epsilon_\perp}\frac{k_z}{k_\perp}}$$

Lect. 14

Dielectric Properties of a Magnetized Plasma: An Example of an Anisotropic Medium

$$\underline{B}_0 = B_0 \hat{\underline{z}}$$

Neglect ion motion

Electron motion

$$m \frac{d^2 \underline{r}_e}{dt^2} = -e \left[\underline{E} + \frac{d\underline{r}_e}{dt} \times (B_0 \hat{\underline{z}}) \right] - m\Gamma \frac{d\underline{r}_e}{dt}$$

friction with ions

really e-i collisions

Introduce phasors. Proceed as in previous example.

$\omega(\omega + i\Gamma)\tilde{\underline{r}}_e + i\omega\omega_c \hat{z} \times \tilde{\underline{r}}_e = \frac{e}{m}\tilde{\underline{E}}$; solve for $\underline{r}_e$ in terms of $\tilde{\underline{E}}$; ....

Get (see book)

$$\underline{\underline{\epsilon}}(\omega) = \epsilon_0 \begin{bmatrix} 1 - \frac{\omega_p^2 \bar{\omega}^2}{\bar{\omega}^4 - \omega^2\omega_c^2} & i\frac{\omega_p^2 \omega_c \omega}{\bar{\omega}^4 - \omega^2\omega_c^2} & 0 \\ -i\frac{\omega_p^2 \omega_c \bar{\omega}}{\bar{\omega}^4 - \omega^2\omega_c^2} & 1 - \frac{\omega_p^2 \bar{\omega}^2}{\bar{\omega}^4 - \omega^2\omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\bar{\omega}^2} \end{bmatrix}$$

where $\boxed{\bar{\omega}^2 = \omega(\omega + i\Gamma)}$

Note: In lossless case $\Gamma = 0 \rightarrow \bar{\omega}^2 = \omega^2$ $\quad \underline{\underline{\epsilon}}(\omega) = \underline{\underline{\epsilon}}^{\dagger}(\omega)$ ✓.

$$\underline{\underline{\epsilon}}(\omega) = \epsilon_0 \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & \frac{i\omega_p \omega_c}{\omega(\omega^2 - \omega_c^2)} & 0 \\ -i\frac{\omega_p \omega_c}{\omega(\omega^2 - \omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{bmatrix} = \begin{bmatrix} A & iB & 0 \\ -iB & A & 0 \\ 0 & 0 & C \end{bmatrix}$$

A, B, C are real.

Unmagnetized: $\omega_c \to 0$ $\quad \Gamma \neq 0$

$$\underline{\underline{\epsilon}}(\omega) = \epsilon_0 \begin{bmatrix} 1 - \omega_p^2/\bar{\omega}^2 & 0 & 0 \\ 0 & 1 - \omega_p^2/\bar{\omega}^2 & 0 \\ 0 & 0 & 1 - \omega_p^2/\bar{\omega}^2 \end{bmatrix} = \epsilon_0 (1 - \omega_p^2/\bar{\omega}^2)\, \underline{\underline{1}}$$

$$\Rightarrow \epsilon = \epsilon_0 (1 - \omega_p^2/\bar{\omega}^2)$$

'Strongly Magnetized':

$$\omega_c = \frac{eB_0}{m} \gg \omega, \omega_p, T$$

Take $\omega_c \to \infty$:

$$\underline{\underline{\epsilon}}(\omega) = \begin{bmatrix} \epsilon_0 & 0 & 0 \\ 0 & \epsilon_0 & 0 \\ 0 & 0 & \epsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right) \end{bmatrix} \leftarrow \text{A uniaxial medium}$$

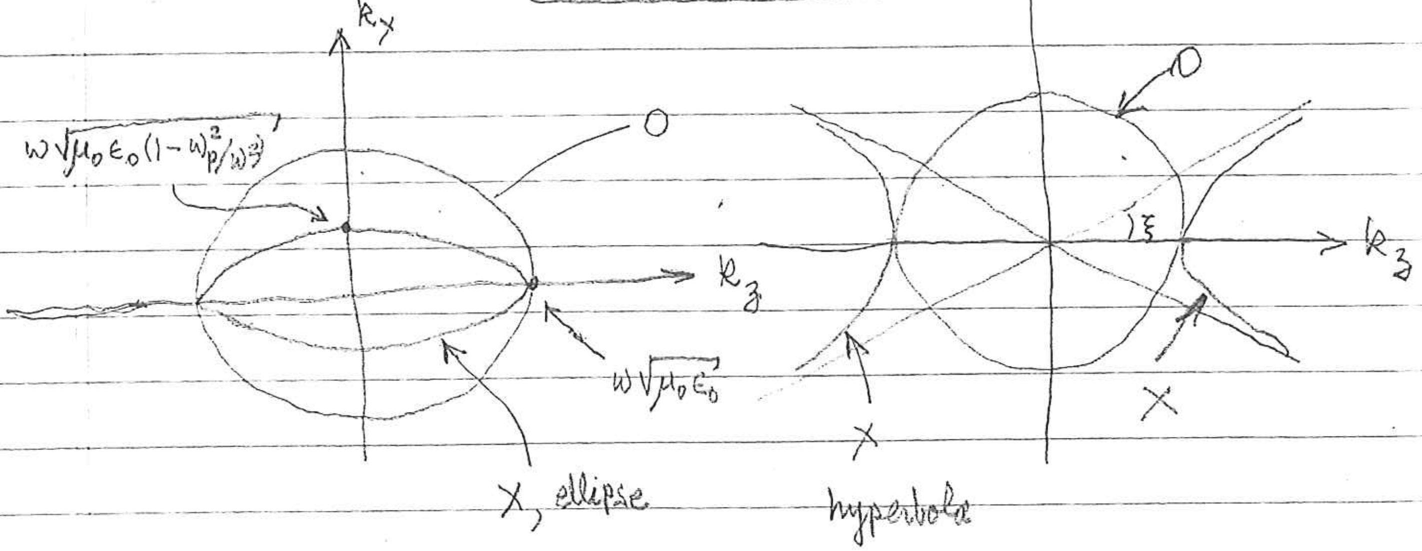
Physical explaination: $D_{x,y} = \epsilon_0 E_{x,y}$ $\quad D_z = \epsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)E_z$

e

$B_0 \hat{z}$ ...

$\omega > \omega_p$

k_x vrs. k_z at const ω



Gyrotropic

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix}$$

$$\left[\underline{k}\,\underline{k} - \underline{\underline{1}}\,k^2 + k_0^2\underline{\underline{\epsilon}}\right]\cdot\hat{\underline{E}} = 0$$

Special case: take k in $\hat{z}$ dir

$$k_0^2\begin{bmatrix} \epsilon_\perp & i\epsilon_x & 0 \\ -i\epsilon_x & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix} + \begin{bmatrix} -k_z^2 & 0 & 0 \\ 0 & -k_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$E_x, E_y = 0$

$$k_0^2\begin{bmatrix} \cancel{\epsilon_\perp} & \cancel{i\epsilon_x} \\ \cancel{-i\epsilon_x} & \cancel{\epsilon_\perp} \end{bmatrix}$$

$$\begin{bmatrix} k_0^2\epsilon_\perp - k_z^2 & ik_0^2\epsilon_x \\ -ik_0^2\epsilon_x & k_0^2\epsilon_\perp - k_z^2 \end{bmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

DISPERSION Reln

$$(k_0^2\epsilon_\perp - k_z^2)^2 - (k_0^2\epsilon_x)^2 = 0$$

$$\cancel{k_z^2 - k_0^2\epsilon_\perp = \pm k_0^2\epsilon_x}$$

$$k_z^2 = k_0^2(\epsilon_\perp \pm \epsilon_x)$$ two k_z^2 values

$$k_0^2\epsilon_\perp - k_z^2 = \pm k_0^2\epsilon_x \qquad k_z^2 = k_0^2(\epsilon_\perp \mp \epsilon_x)$$

$$\pm k_0^2\epsilon_x E_x + i k_0^2 \epsilon_x E_y = 0$$

$$\boxed{\frac{E_y}{E_x} = \pm i}$$ circularly polarized

Faraday Rotation

Part #1 Ponderomotive force

Cold fluid Equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = \frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{u} \times \underset{\sim}{B}}{c}\right)$$

$$\underset{\sim}{E} = \underset{\sim}{E}_f + \underset{\sim}{E}_s \qquad \underset{\sim}{E}_f = Re\left\{ \hat{\underset{\sim}{E}}_f(\underset{\sim}{x}, t)\, e^{-i\omega_f t} \right\}$$

$$\underset{\sim}{u} = \underset{\sim}{u}_f + \underset{\sim}{u}_s \qquad \underset{\sim}{u}_f = Re\left\{ \hat{\underset{\sim}{u}}_f(\underset{\sim}{x}, t)\, e^{-i\omega_f t} \right\}$$

fast ~~field~~ flow is ~~ess~~ the linear high frequency response

$$\frac{\partial \underset{\sim}{u}_f}{\partial t} = \frac{q}{m} \underset{\sim}{E}_f$$

Slow flow contains nonlinear terms

$$\frac{\partial u_s}{\partial t} = \frac{q}{m}\left(\underset{\sim}{E}_s\right) - \left\langle \underset{\sim}{u}_f \cdot \nabla \underset{\sim}{u}_f \right\rangle_{\omega_f} + \frac{q}{m}\left\langle \frac{\underset{\sim}{u}_f \times \underset{\sim}{B}_f}{c} \right\rangle_{\omega_f}$$

this means average ~~of~~ over fast time $T_f = \frac{2\pi}{\omega_f}$

Note $\quad \underset{\sim}{u}_f \cdot \nabla \underset{\sim}{u}_f = \nabla \frac{1}{2} \underset{\sim}{u}_f \cdot \underset{\sim}{u}_f - \underset{\sim}{u}_f \times \nabla \times \underset{\sim}{u}_f$

$$\frac{\partial \underset{\sim}{u}_s}{\partial t} = \frac{q}{m} \underset{\sim}{E}_s - \nabla \frac{1}{2} \langle \underset{\sim}{u}_f \cdot \underset{\sim}{u}_f \rangle_{\omega_f}$$

$$+ \left\langle \underset{\sim}{u}_f \times \left[\frac{q \underset{\sim}{B}_f}{mc} + \nabla \times \underset{\sim}{u}_f \right] \right\rangle_{\omega_f}$$

This term aveages to zero

$$\frac{\partial \underset{\sim}{u}_f}{\partial t} = \frac{q}{m} \underset{\sim}{E}_f$$

$$-\frac{1}{c} \frac{\partial \underset{\sim}{B}_f}{\partial t} = \nabla \times \underset{\sim}{E}_f$$

$$\frac{\partial}{\partial t} \left[\frac{q \underset{\sim}{B}_f}{mc} + \nabla \times \underset{\sim}{u}_f \right] = \frac{q}{mc} \frac{\partial \underset{\sim}{B}_f}{\partial t} + \nabla \times \frac{\partial \underset{\sim}{u}_f}{\partial t}$$

$$= -\frac{q}{m} \nabla \times \underset{\sim}{E}_f + \nabla \times \frac{q}{m} \underset{\sim}{E}_f = 0$$

Ponderomotive force

$$m \frac{\partial \underset{\sim}{u}_s}{\partial t} = q \underset{\sim}{E}_s + \underset{\sim}{F}_p$$

Ponderomotive potential

$$\underset{\sim}{F}_p = -\nabla V_p$$

$$\underset{\sim}{V}_p = \frac{1}{2} m \langle \underset{\sim}{u}_f \cdot \underset{\sim}{u}_f \rangle$$

$$\underset{\sim}{u}_f = \frac{1}{2}\left(\hat{\underset{\sim}{u}}_f e^{-i\omega_f t} + \hat{\underset{\sim}{u}}_f^{*} e^{i\omega_f t}\right) = \mathrm{Re}\{\hat{\underset{\sim}{u}}_f e^{-i\omega_f t}\}$$

$$V_p = \frac{m}{4} |\hat{\underset{\sim}{u}}_f|^2$$

$$\hat{\underset{\sim}{u}}_f = \frac{q}{m(-i\omega_f)} \cdot \hat{\underset{\sim}{E}}_f \qquad \text{FROM EQ. of motion}$$

$$\boxed{V_p = \frac{q^2 |\hat{\underset{\sim}{E}}_f|^2}{4\omega_f^2 m}} = \frac{1}{4}\left|\frac{q\hat{\underset{\sim}{A}}}{mc^2}\right|^2$$

* PONDEROMOTIVE FORCE same sign for e, i
* much bigger for electrons
* pushes particles out of high field region.