Lecture 15

Periodic Structures

Periodic Structures

Filters
Gratings
Slow Wave Structures
- particle accelerators
- Cherenkov microwave generators

Metamaterials

Floquet Theory

Floquet Theory – periodic medium

$$E(z,t)=\mathrm{Re}\left\{\hat{E}(z)e^{-i\omega t}\right\}$$

$$\frac{\partial^2}{\partial z^2}\hat{E}(z)+\frac{\omega^2}{c^2}\varepsilon_{rel}(z)\hat{E}(z)=0$$

$$\varepsilon_{rel}(z)=\varepsilon_{rel}(z+L)$$

-Time harmonic, spatially dependent field
-Inhomogeneous relative dielectric
-Dielectric is spatially periodic

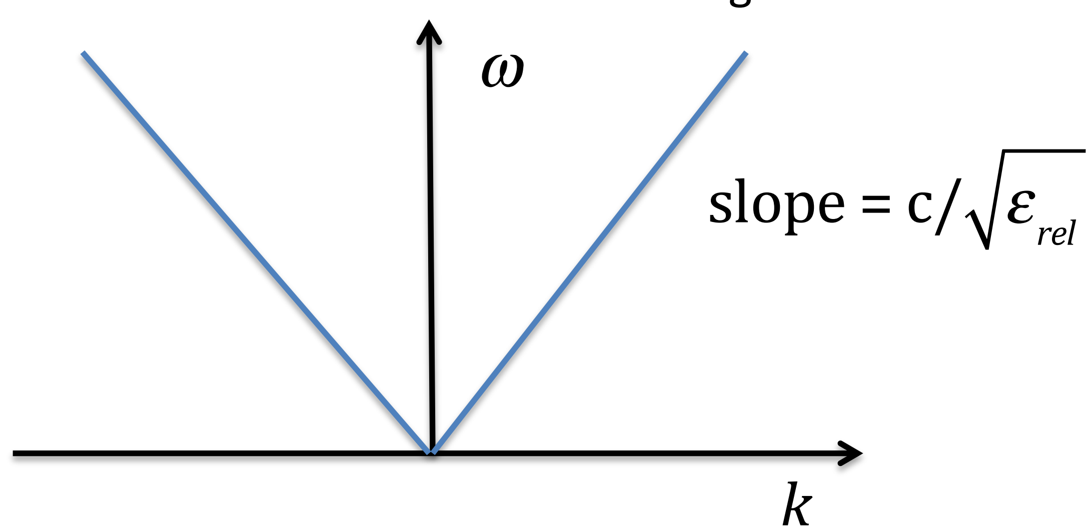
Special case - homogeneous

$$\partial\varepsilon_{rel}/\partial z=0$$

$$\hat{E}(z)=\hat{E}_0\exp\left(ikz\right)$$

$$\omega(k)=\pm kc/\sqrt{\varepsilon_{rel}}$$

Homogeneous case



Spatially Inhomogeneous Case

$$E(z,t)=\mathrm{Re}\left\{\hat{E}(z)e^{-i\omega t}\right\}$$

$$\frac{\partial^2}{\partial z^2}\hat{E}(z)+\frac{\omega^2}{c^2}\varepsilon_{rel}(z)\hat{E}(z)=0$$

$$\varepsilon_{rel}(z)=\varepsilon_{rel}(z+L)$$

$$\hat{E}(z)=\hat{E}_0(k,z)\exp\left(ikz\right)$$

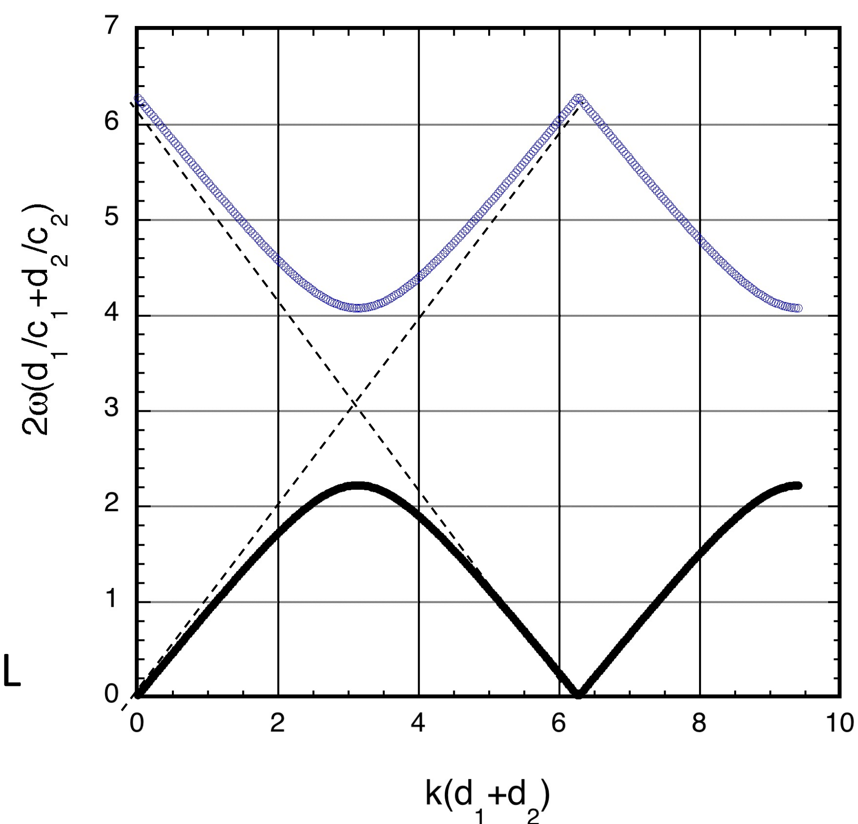
$$\hat{E}_0(k,z)=\hat{E}_0(k,z+L)$$

Periodic in z, period L

$$\omega(k)=\omega(k+k_0)$$

Periodic in k, period k_0

$$k_0=2\pi/L$$



$\omega(k)$

Stop Band

Smith Island Cake

The Smith Island Cake is the official dessert of the State of Maryland. It consists of alternating layers of two dielectric materials as pictured at right. Suppose the dielectric constants and the thicknesses of the two alternating layers are $\varepsilon_1, \varepsilon_2$ and d_1, d_2, respectively. In this sense the cake is a metamaterial.

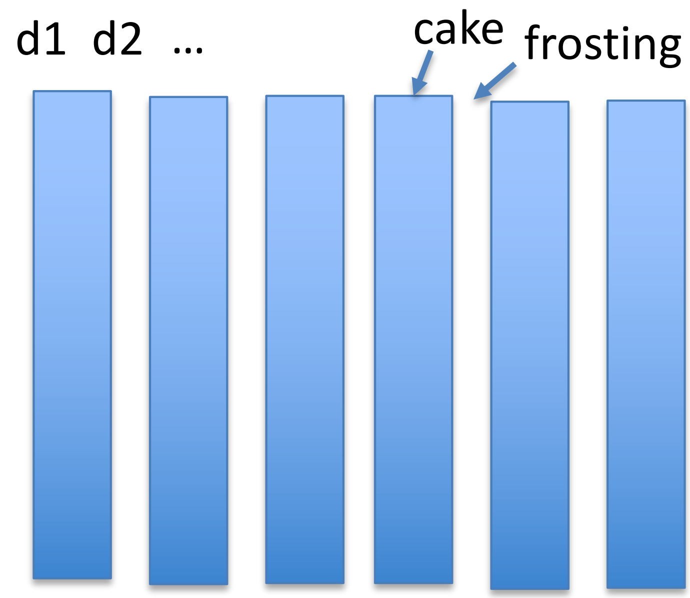
In a layer

$$E(z) = \left[E_+ e^{ikz} + E_- e^{-ikz} \right]$$

$$H(z) = Z^{-1} \left[E_+ e^{ikz} - E_- e^{-ikz} \right] \qquad Z = \sqrt{\mu_0 / \varepsilon_{1,2}}$$

$$E(0) = E_+ + E_-$$

$$H(0) = Z^{-1} \left[E_+ - E_- \right] \qquad k_{1,2} = \omega \sqrt{\varepsilon_{1,2} \mu_0}$$



In a layer

$$E(z) = \left[E_+ e^{ikz} + E_- e^{-ikz} \right]$$

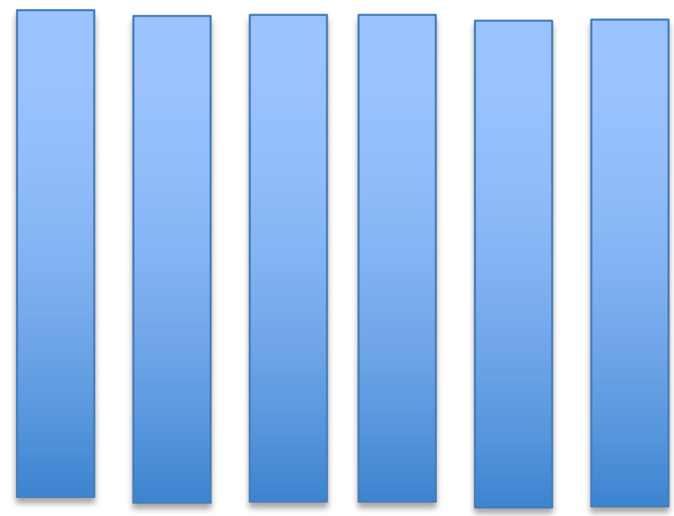
$$H(z) = Z^{-1} \left[E_+ e^{ikz} - E_- e^{-ikz} \right]$$

Fields at z=0

$$E(0) = E_+ + E_-$$

$$H(0) = Z^{-1} \left[E_+ - E_- \right]$$

d1 d2 ...



Solve for E+/-

$$E_+ = \frac{1}{2} \left[E(0) + ZH(0) \right]$$

$$E_+ = \frac{1}{2} \left[E(0) - ZH(0) \right]$$

$$Z = \sqrt{\mu_0 / \varepsilon}$$

$$\theta = kd = \omega \sqrt{\varepsilon \mu_0} d$$

Find fields at z=d

$$\begin{pmatrix} E(d) \\ H(d) \end{pmatrix} = \begin{bmatrix} \cos\theta & iZ\sin\theta \\ iZ^{-1}\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

Smith Island Cake

One Layer

$$\begin{pmatrix} E(d) \\ H(d) \end{pmatrix} = \begin{bmatrix} \cos\theta & iZ\sin\theta \\ iZ^{-1}\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

$$Z = \sqrt{\mu_0/\varepsilon}$$

$$\theta = kd = \omega\sqrt{\varepsilon\mu_0}\,d$$

Two layers of different material

$$\begin{pmatrix} E(d_2+d_1) \\ H(d_2+d_1) \end{pmatrix} = \begin{bmatrix} \cos\theta_1 & iZ_1\sin\theta_1 \\ iZ_1^{-1}\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & iZ_2\sin\theta_2 \\ iZ_2^{-1}\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

$$\begin{pmatrix} E(d_2+d_1) \\ H(d_2+d_1) \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

After Multiple layers

$$\begin{pmatrix} E(n(d_2+d_1)) \\ H(n(d_2+d_1)) \end{pmatrix} = \left(\underline{\underline{M}}\right)^n \begin{pmatrix} E(0) \\ H(0) \end{pmatrix}$$

Find eigenfunctions and eigenvalues of M

$$\begin{pmatrix} E(z+d_1+d_2) \\ H(z+d_1+d_2) \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} E(z) \\ H(z) \end{pmatrix} = \lambda \begin{pmatrix} E(z) \\ H(z) \end{pmatrix}$$

Express results in terms of eigenfunctions, eigenvalues of M

$$\underline{\underline{M}} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \frac{Z_2}{Z_1}\sin\theta_1\sin\theta_2 & i\left(Z_1\sin\theta_1\cos\theta_2 + Z_2\cos\theta_1\sin\theta_2\right) \\ i\left(Z_1^{-1}\sin\theta_1\cos\theta_2 + Z_2^{-1}\cos\theta_1\sin\theta_2\right) & \cos\theta_1\cos\theta_2 - \frac{Z_1}{Z_2}\sin\theta_1\sin\theta_2 \end{bmatrix}$$

$$\det\left[\underline{\underline{M}} - \lambda \underline{\underline{1}}\right] = 0$$

$$\lambda^2 + b\lambda + 1 = 0$$

Det[M]=1

$$b = \left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2}\right)\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2$$

$$\theta_{1,2} = k_{1,2} d_{1,2} = \omega\sqrt{\varepsilon_{1,2}\mu_0}\, d_{1,2}$$

$$\lambda = \lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

$$b = -2\cos\left(\theta_1 + \theta_2\right) + \Delta \sin\theta_1 \sin\theta_2$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2}$$

$$\lambda_+ \lambda_- = 1$$

$$\lambda_\pm = \exp(\pm i\phi) \quad \text{or} \quad \lambda_\pm - \text{real}$$

Propagating or evanescent

Fields advance by phase on each layer
Or decay exponentially

$$\cos\phi = \cos(\theta_1 + \theta_2) - \frac{\Delta}{2}\sin(\theta_1)\sin(\theta_2)$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2} \qquad \Delta = 0 \quad \text{no stop band}$$

Special case: $\theta_1 = \theta_2$

$$\cos(\theta_1 + \theta_2) = \frac{\cos\phi + \Delta/4}{1 + \Delta/4}$$

$$\theta_1 + \theta_2 = \frac{\omega}{c}\left(d_1\sqrt{\varepsilon_1} + d_2\sqrt{\varepsilon_2}\right)$$

Solutions

$$\begin{pmatrix} E(z+d_1+d_2) \\ H(z+d_1+d_2) \end{pmatrix} = \lambda \begin{pmatrix} E(z) \\ H(z) \end{pmatrix}$$

$$\lambda = e^{i\phi}$$

$$\theta = kd = \omega\sqrt{\varepsilon\mu_0}\,d$$

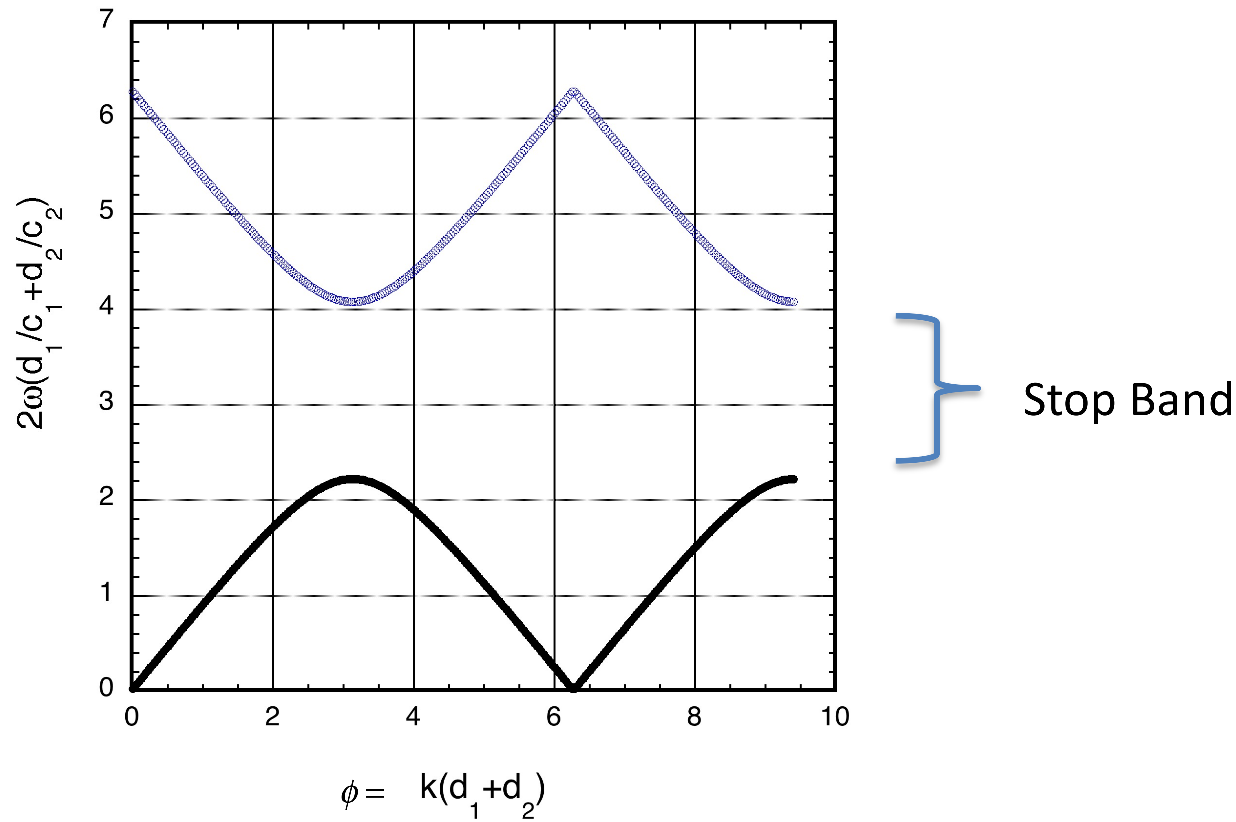
$$\cos\phi = \cos(\theta_1+\theta_2) - \frac{\Delta}{2}\sin(\theta_1)\sin(\theta_2)$$

$$\Delta = \frac{\left(Z_1 - Z_2\right)^2}{Z_1 Z_2}$$

Special case: $\theta_1 = \theta_2$

$$\cos(\theta_1+\theta_2) = \frac{\cos\phi + \Delta/4}{1+\Delta/4}$$

$$\theta_1+\theta_2 = \frac{\omega}{c}\left(d_1\sqrt{\varepsilon_1} + d_2\sqrt{\varepsilon_2}\right)$$



Continuous Variations

Mathieu Equation

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}(1+\delta\cos k_0 z)\right]\hat{E}(z) = 0 \qquad k_0 = \frac{2\pi}{L}$$

periodic

Write $\hat{E}(z) = e^{ikz}\,\hat{E}_0(z,k)$

$$\hat{E}_0 = \sum_{n=-\infty}^{\infty} \hat{E}_n \exp(ik_n z) \qquad k_n = nk_0$$

Fourier Series

Solution by Fourier Series

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2}(1+\delta\cos k_0 z)\right]\hat{E}(z) = 0$$

$$\int_0^L \frac{dz}{L} \exp(-i(k+nk_0)z) \cdot \{\text{Mathieu Equation}\} = 0$$

$$\left[-(k+nk_0)^2 + \frac{\omega^2}{c^2}\right]E_n + \frac{\omega^2}{c^2}\delta \sum_m \int_0^L \frac{dz}{L} e^{-ink_0 z} \cos k_0 z \, E_m e^{ik_m z} = 0$$

note

$$\int_0^L \frac{dz}{L} \cos k_0 z \exp[ik_0 z(m-n)] = \begin{cases} \frac{1}{2} & \text{if } m = n \pm 1 \\ 0 & \text{otherwize} \end{cases}$$

$$\left[\frac{\omega^2}{c^2} - (k+nk_0)^2\right] E_n + \frac{\omega^2}{c^2}\frac{\delta}{2}\left[E_{n+1} + E_{n-1}\right] = 0$$

note if $\omega(k)$ is a solution with E_n

Then $k \to k+k_0$ $n \to n-1$ is also a solution

$$\omega(k) = \omega(k+k_0)$$

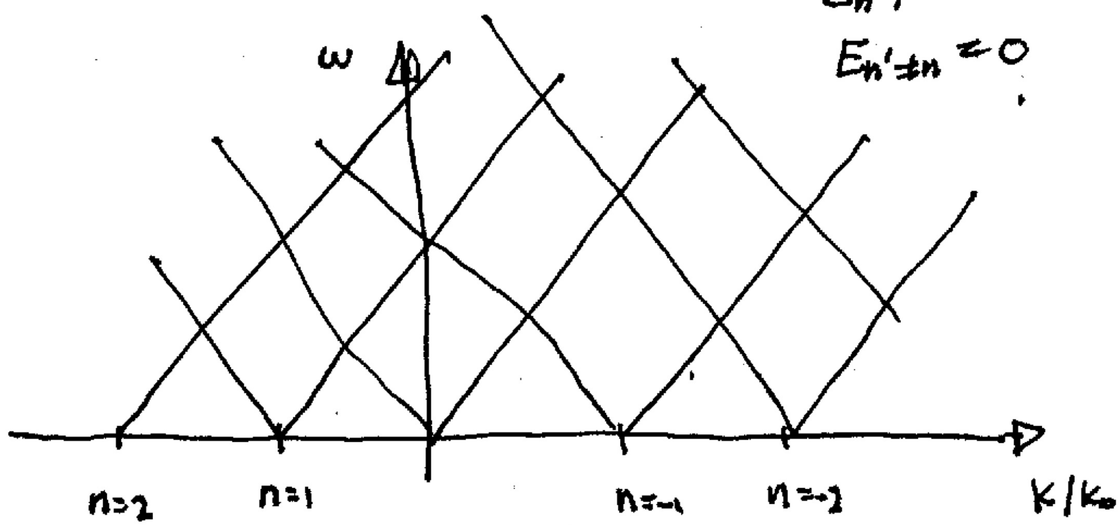
$$\left[\frac{\omega^2}{c^2} - (k+nk_0)^2\right] E_n + \frac{\omega^2}{c^2}\frac{\delta}{2}\left[E_{n+1} + E_{n-1}\right] = 0$$

Approximat solutions for $|\delta| << 1$

For each n $\frac{\omega^2}{c^2} = (k+nk_0)^2$ is a solution

$E_n \neq 0$

$E_{n' \neq n} \approx 0$



Pattern repeats with period k_0

Solution breaks down where lines cross

n = 0

$$\left[\frac{\omega^2}{c^2} - k^2\right] E_0 + \frac{\omega^2}{c^2}\frac{\delta}{2} E_{-1} = 0$$

n = -1

$$\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] E_{-1} + \frac{\omega^2}{c^2}\frac{\delta}{2} E_0 = 0$$

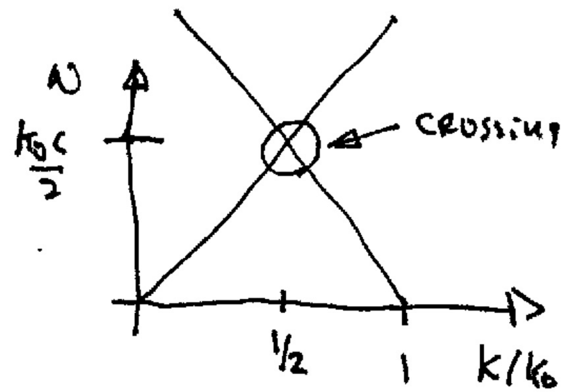
Combine

$$\left[\frac{\omega^2}{c^2} - k^2\right]\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] = \left(\frac{\omega^2}{c^2}\frac{\delta}{2}\right)^2$$

Combine

$$\left[\frac{\omega^2}{c^2} - k^2\right]\left[\frac{\omega^2}{c^2} - (k-k_0)^2\right] = \left(\frac{\omega^2}{c^2}\frac{\delta}{2}\right)^2$$

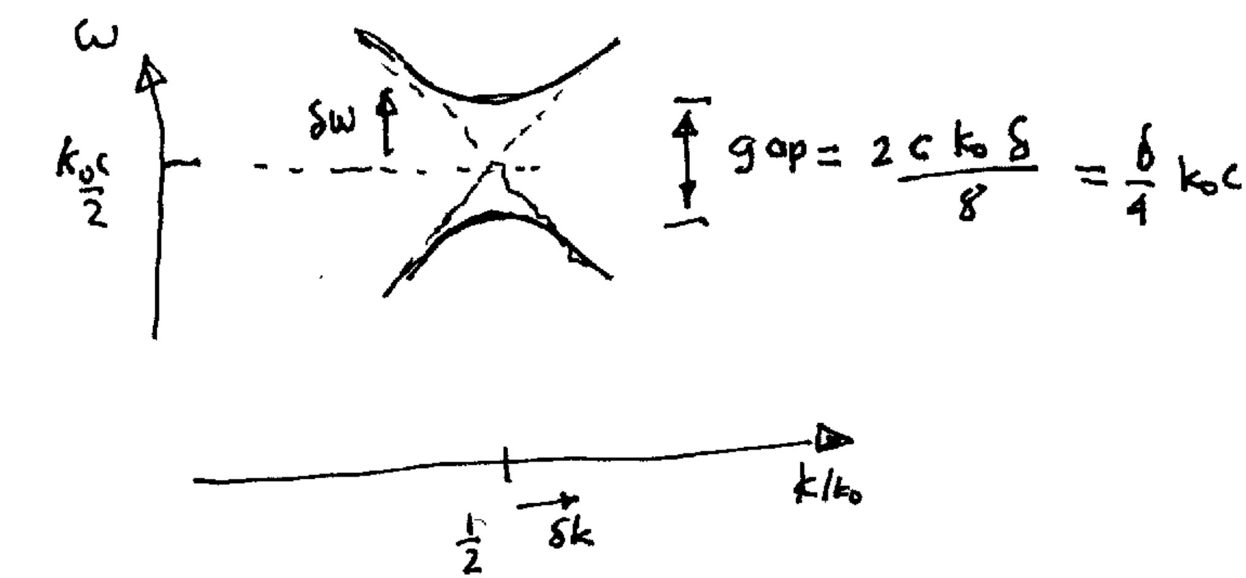
Let $\omega = \frac{k_0}{2}c + \delta\omega \qquad k = \frac{k_0}{2} + \delta k$



$$\left[\frac{k_0 c\,\delta\omega}{c^2} - k_0\delta k\right]\left[\frac{k_0 c\,\delta\omega}{c^2} + k_0\delta k\right] = \frac{\delta^2}{2^2}\left(\frac{k_0 c}{c\,2}\right)^4$$

$$\left[\frac{\delta\omega}{c} - \delta k\right]\left[\frac{\delta\omega}{c} + \delta k\right] = \frac{\delta^2}{64}\,k_0^2$$

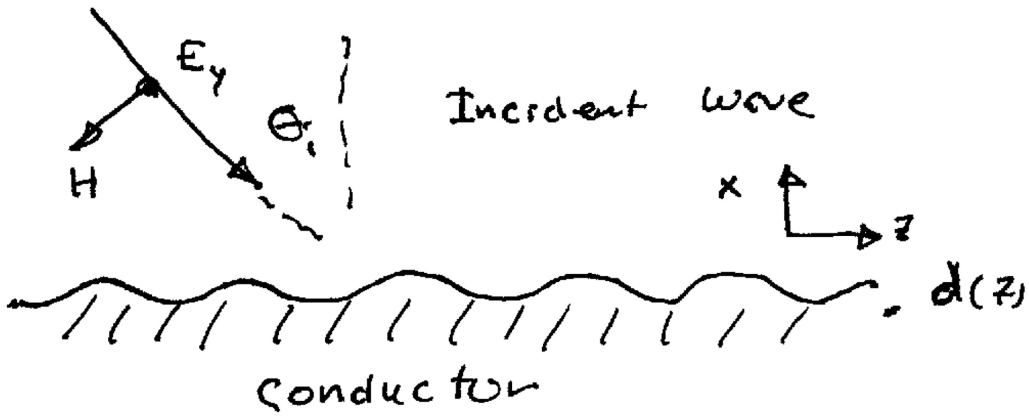
$$\left(\frac{\delta\omega}{c}\right)^2 = \delta k^2 + \frac{k_0^2}{64}\delta^2$$



Avoided Crossing

Gap is like band gap for electrons in a crystal structure

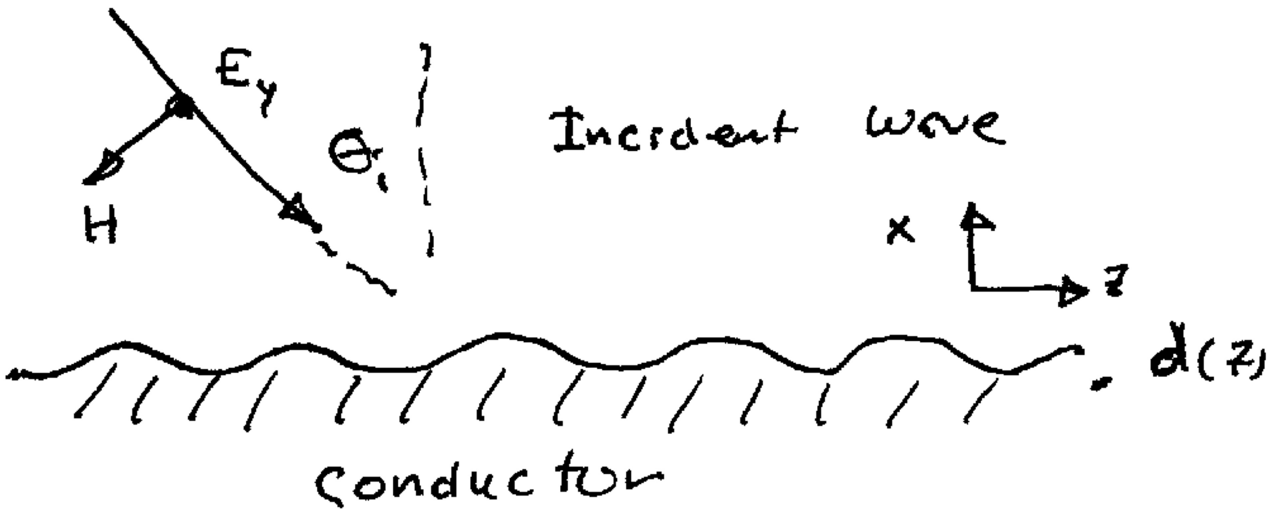
Grating



$$\nabla^2 \hat{E}(x,z) + \frac{\omega^2}{c^2} \hat{E}(x,z) = 0$$

Boundary Condition $\quad \hat{E}(x = d(z), z) = 0$

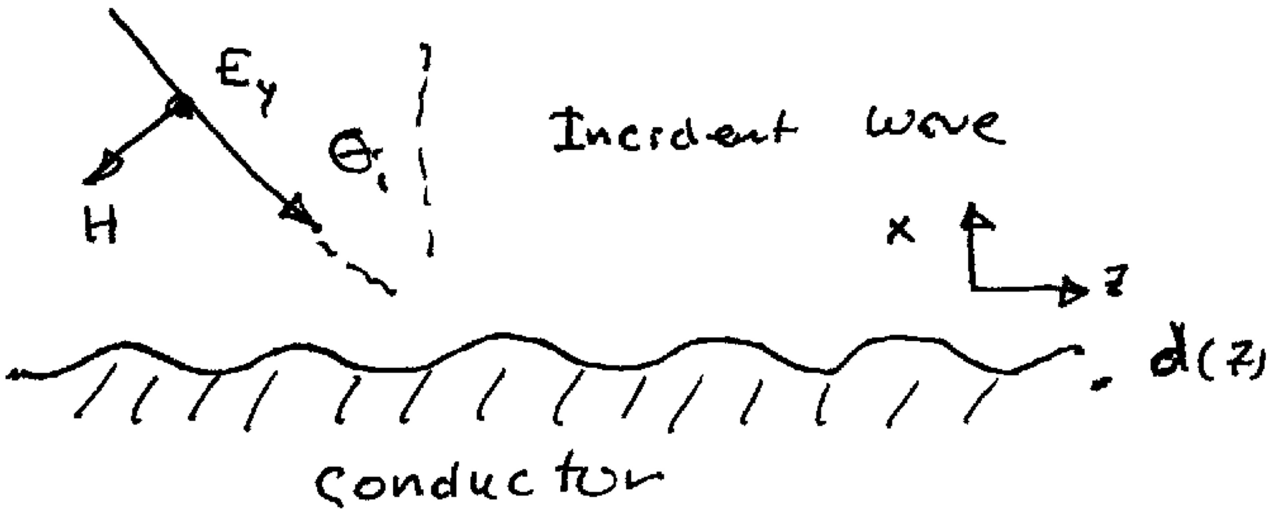
$$d(z) = \sum_m d_m \, e^{i m k_0 z}$$



Incident Wave

$$\hat{E} = \hat{E}_{inc} \exp[i k_z z - i k_x x]$$

$$k_z = \frac{\omega}{c} \sin\theta_i \qquad k_x = \frac{\omega}{c} \cos\theta_i$$



Reflected Waves

$$\hat{E}_{ref} = \sum_m \hat{E}_m \exp\left[i k_z z + i m k_0 z + i k_{xm} x \right]$$

$$k_{xm} = \begin{cases} \sqrt{\dfrac{\omega^2}{c^2} - (k_z + m k_0)^2} & |k_z + m k_0| < \omega/c \\ +i\sqrt{(k_z + m k_0)^2 - \dfrac{\omega^2}{c^2}} & |k_z + m k_0| > \omega/c \end{cases}$$

Boundary Condition

$$e^{ik_z z}\left[\hat{E}_{inc}\exp[-ik_x d(z)] + \sum_m \hat{E}_m \exp(imk_0 z + ik_{xm} d(z))\right] = 0$$

Specialize to small $d(z)$

Specularly reflected wave ($m=0$)

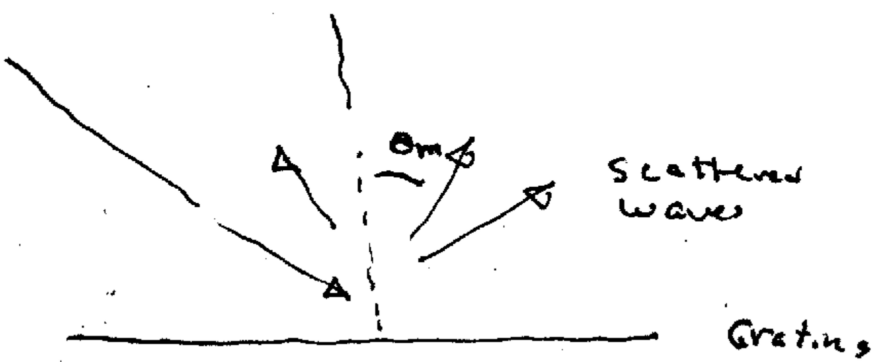
$$\hat{E}_0 = -\hat{E}_{inc}$$

OTHER waves

Substitute $d(z) = \sum_m d_m e^{i k_m z}$

$$\hat{E}_{inc} \left\{ (1 - i k_x d) - (1 + i k_x d) + \sum_{m \neq 0} E_m \exp(i m k_0 z) \right\} = 0$$

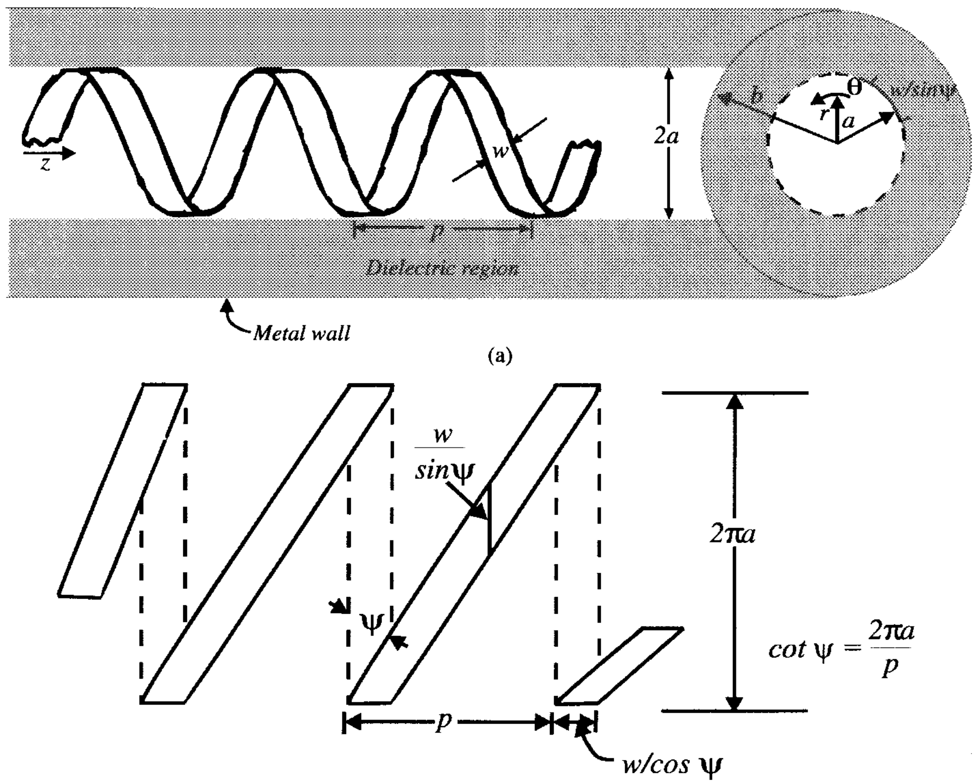
$$E_m = 2 i k_x d_m \hat{E}_{inc}$$



$$\frac{\omega}{c} \cos\theta_m = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\omega}{c} \sin\theta_i + m k_0\right)^2}$$

$$\cos\theta_m = \sqrt{1 - (\sin\theta_i + m k_0 c/\omega)^2}$$

Tape Helix

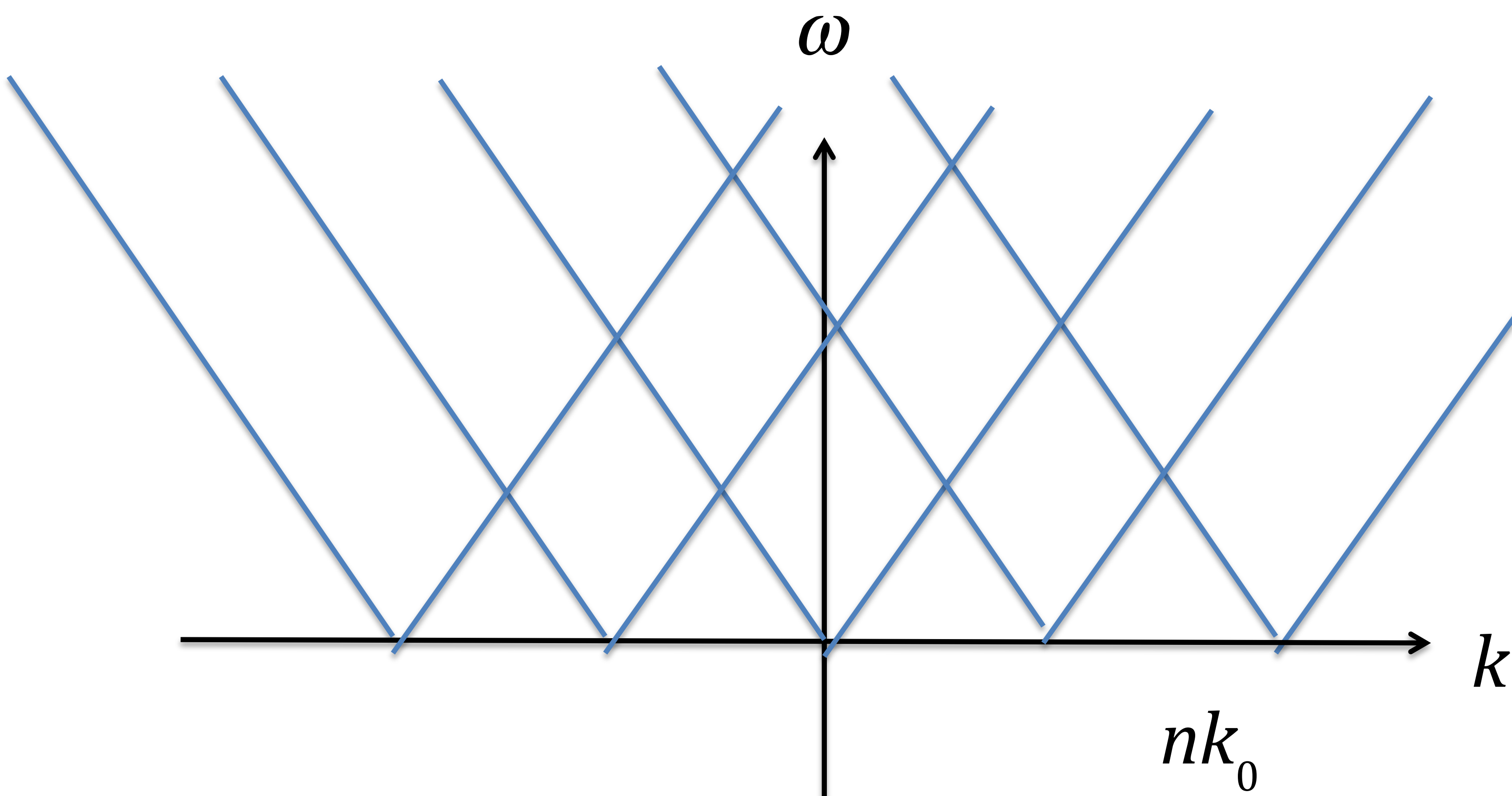


Approximate solution

$$\omega = k v_p$$

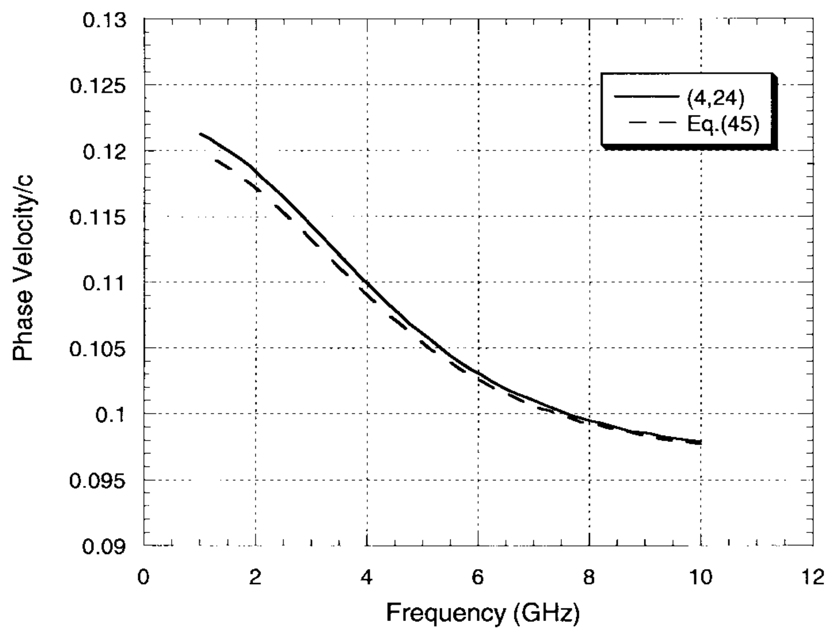
$$v_p = c \frac{p}{\sqrt{p^2 + (2\pi r)^2}}$$

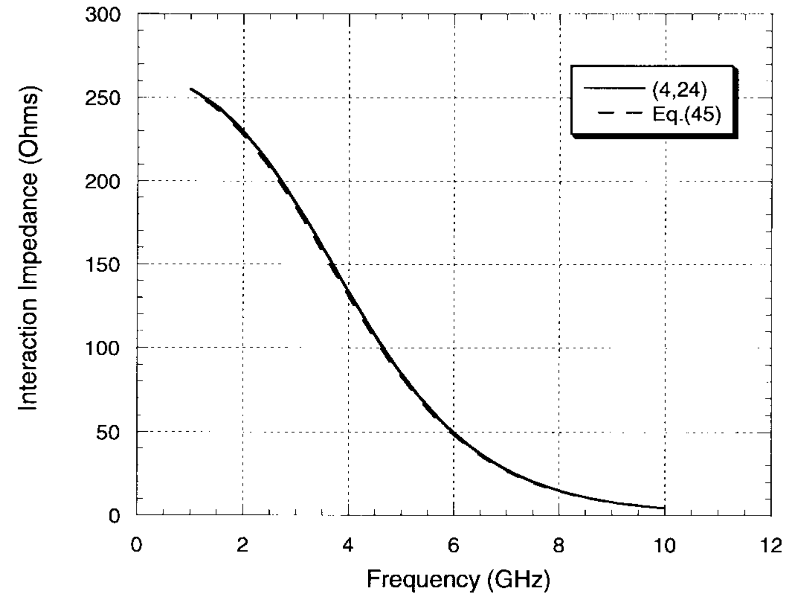
Crossings – not gaps



Consequence of helical symmetry

$$k_0 = 2\pi / p$$





$$Z_{Pierce} = \frac{\left|\overline{E}_z\right|^2}{2k_z^2 P}$$

Pierce: Vacuum electronics pioneer
Pulse code modulation
First communications satellite
Bohlen-Pierce musical scale
Coined name “Transistor”

Spatially Inhomogeneous Case

$$E(z,t) = \mathrm{Re}\left\{\hat{E}(z)e^{-i\omega t}\right\}$$

$$\frac{\partial^2}{\partial z^2}\hat{E}(z) + \frac{\omega^2}{c^2}\varepsilon_{rel}(z)\hat{E}(z) = 0$$

$$\varepsilon_{rel}(z) = \varepsilon_{rel}(z+L)$$

$$\hat{E}(z) = \hat{E}_0(k,z)\exp\left(ikz\right)$$

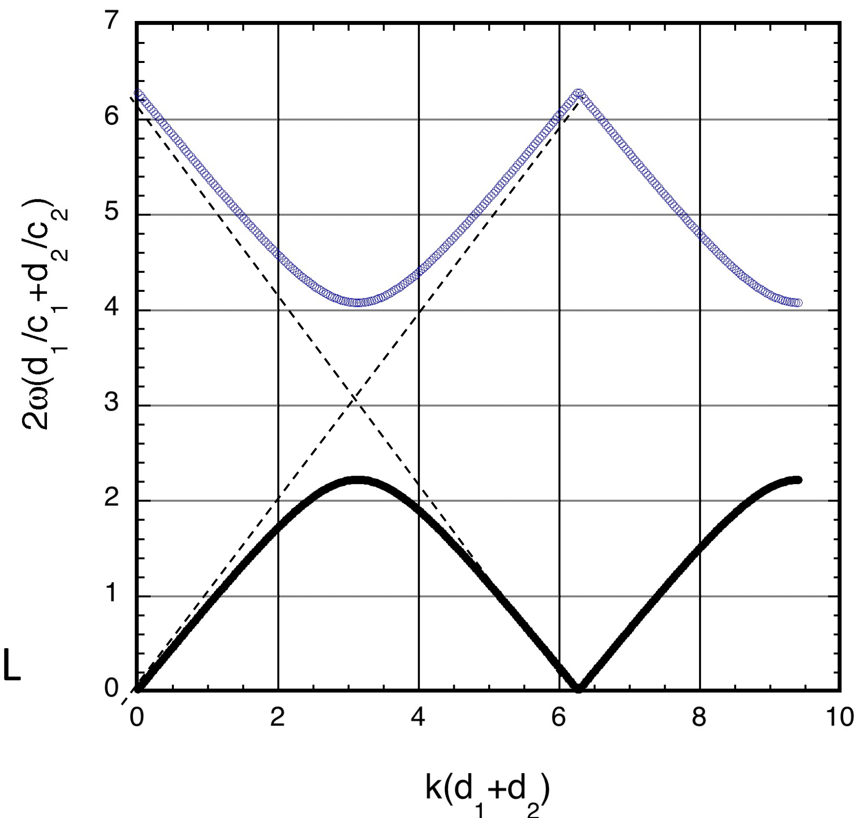
$$\hat{E}_0(k,z) = \hat{E}_0(k,z+L)$$

Periodic in z, period L

$$\omega(k) = \omega(k+k_0)$$

Periodic in k, period k_0

$$k_0 = 2\pi / L$$



$\omega(k)$

Stop Band

Solutions

$$\begin{pmatrix} E(z+d_1+d_2) \\ H(z+d_1+d_2) \end{pmatrix} = \lambda \begin{pmatrix} E(z) \\ H(z) \end{pmatrix}$$

$$\lambda = e^{i\phi}$$

$$\theta = kd = \omega\sqrt{\varepsilon\mu_0}\,d$$

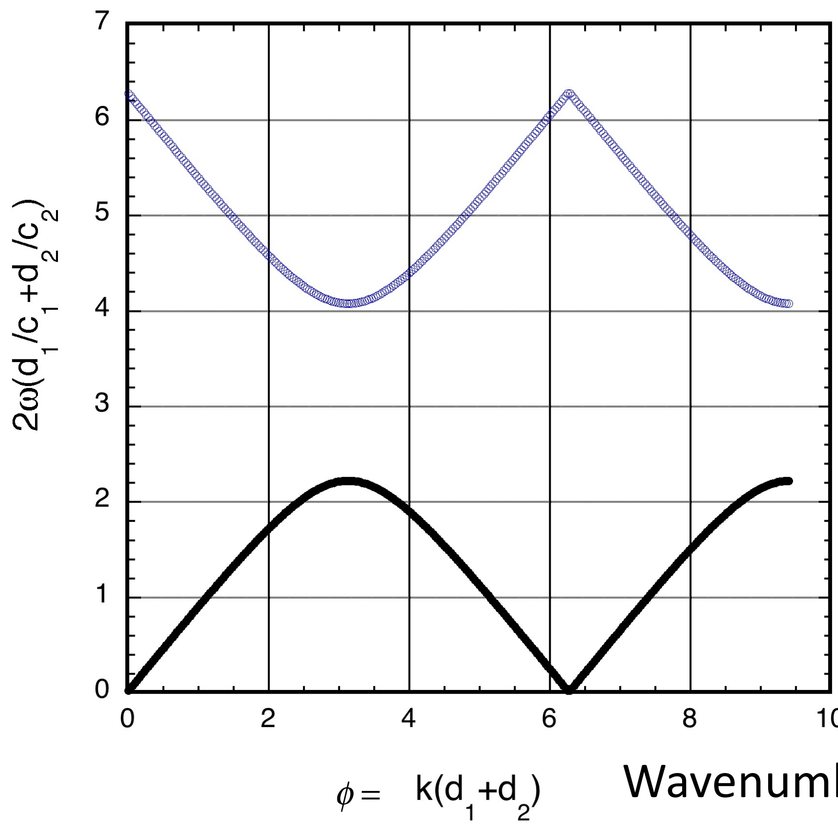
$$\cos\phi = \cos(\theta_1+\theta_2) - \frac{\Delta}{2}\sin(\theta_1)\sin(\theta_2)$$

$$\Delta = \frac{(Z_1 - Z_2)^2}{Z_1 Z_2}$$

Special case: $\theta_1 = \theta_2$

$$\cos(\theta_1+\theta_2) = \frac{\cos\phi + \Delta/4}{1+\Delta/4}$$

$$\theta_1 + \theta_2 = \frac{\omega}{c}\left(d_1\sqrt{\varepsilon_1} + d_2\sqrt{\varepsilon_2}\right)$$



Frequency

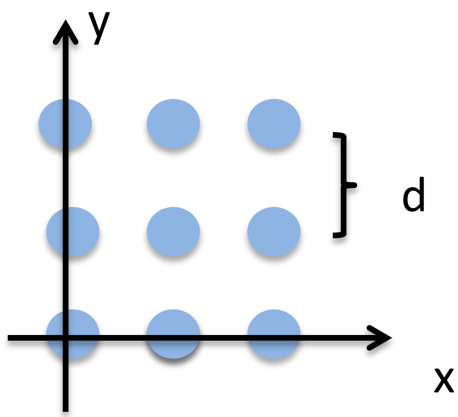
Stop Band

Wavenumber kL

Bragg Reflector

Reflects signals in a narrow frequency band

Higher Dimensions



$$\nabla^2 E(x,y) + \frac{\omega^2}{c^2}\left(1+\chi(x,y)\right)E(x,y) = 0$$

$$\chi(x,y) = \chi(x+d,y) = \chi(x,y+d)$$

$$E(x,y) = \sum_{m,n} \bar{E}_{m,n} \exp\left[i\left(k_x + nk_o\right)x + i\left(k_y + mk_o\right)\right]$$

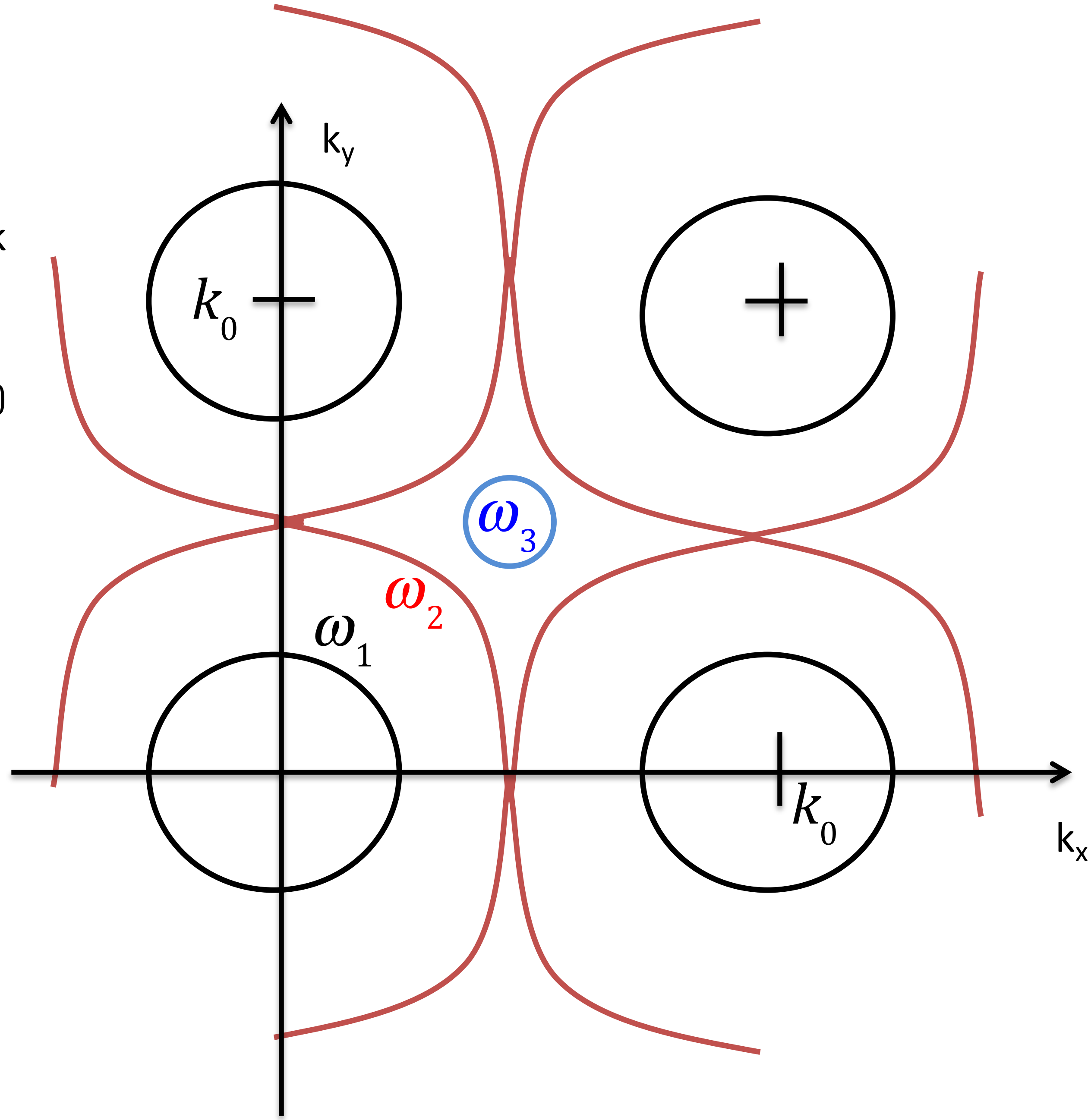
$$\omega(k_x,k_y) = \omega(k_x + qk_0, k_y + pk_0)$$

$$k_0 = 2\pi / d$$

Level curves of frequency in the k plane

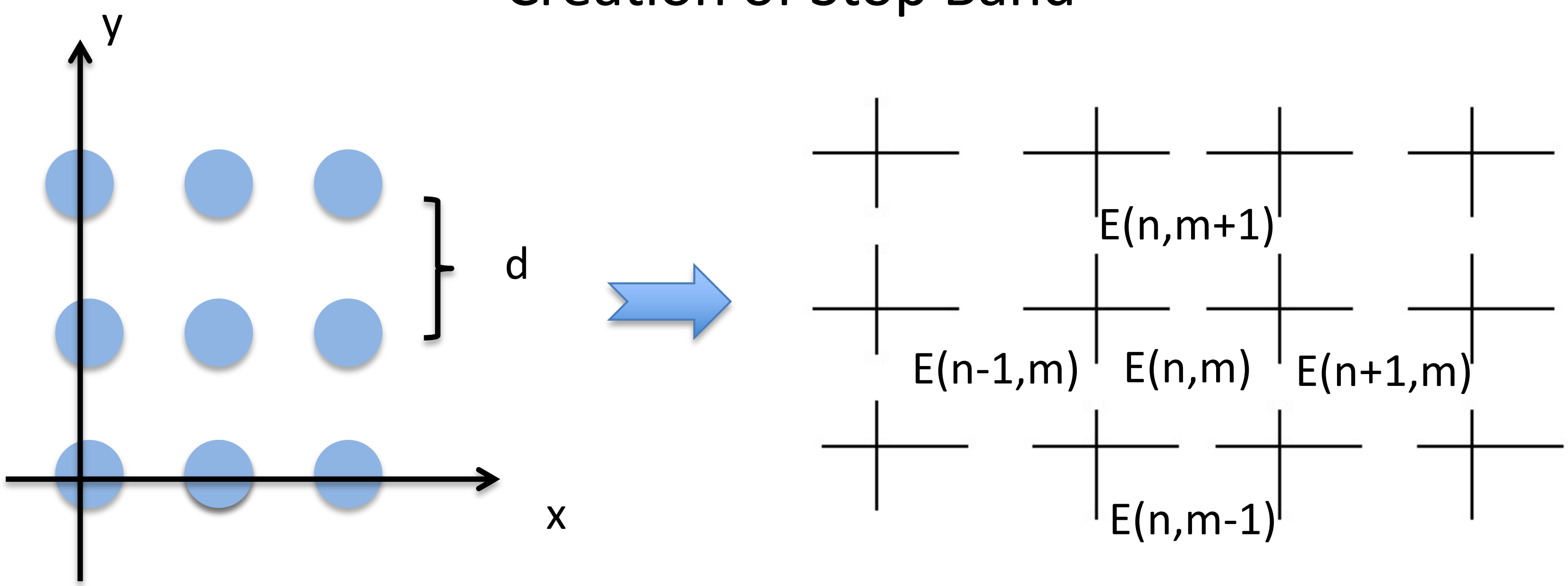
$$\omega(k_x,k_y)=\omega(k_x+qk_0,k_y+pk_0)$$

$$k_0=2\pi/d$$



Stop Band only for certain angles of k.

Creation of Stop Band



$$\left[\omega^2-\omega_c^2\right]E(n,m)=\frac{\delta}{2}\omega_c^2\left[E(n+1,m)+E(n-1,m)+E(n,m+1)+E(n,m-1)\right]$$

$$E(n,m)=E(0,0)\exp\left[i(k_x dn+k_y dm)\right]$$

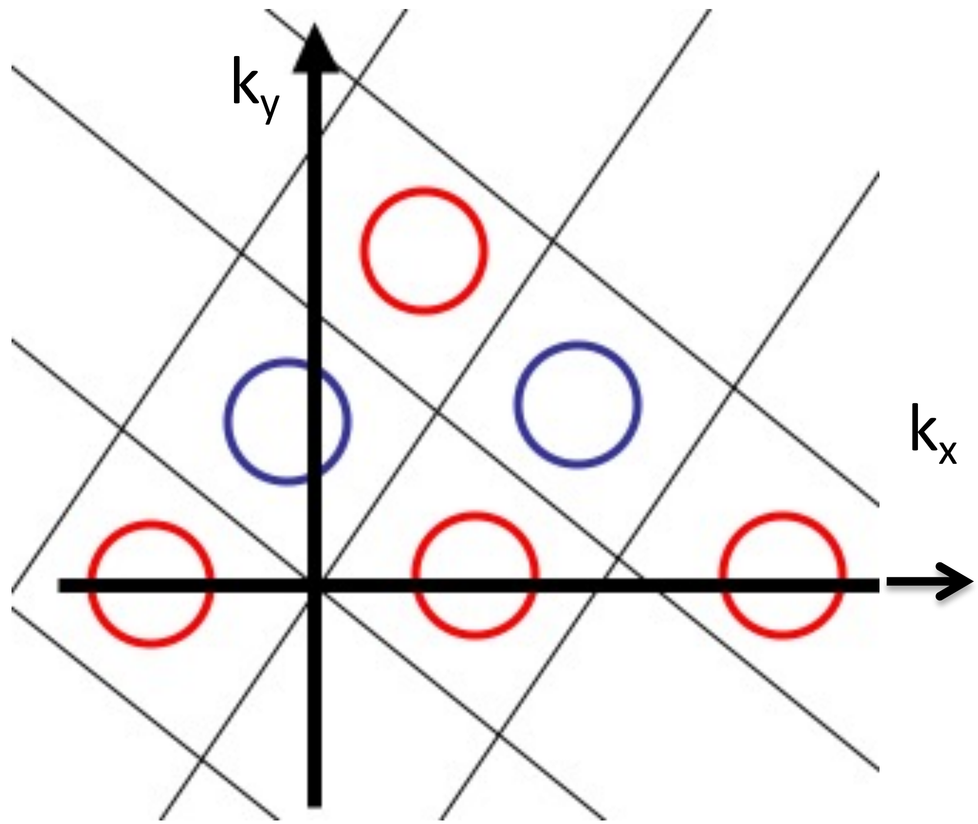
$$\left[\omega^2-\omega_c^2\right]=\delta\omega_c^2\left[\cos(k_x d)+\cos(k_y d)\right]=\delta\omega_c^2\cos\left[(k_x-k_y)d\right]\cos\left[(k_x+k_y)d\right]$$

$$\left[\omega^2-\omega_c^2\right]E(n,m)=\frac{\delta}{2}\omega_c^2\left[E(n+1,m)+E(n-1,m)+E(n,m+1)+E(n,m-1)\right]$$

$$E(n,m)=E(0,0)\exp\left[i(k_x dn+k_y dm)\right]$$

$$\left[\omega^2-\omega_c^2\right]=\delta\omega_c^2\left[\cos(k_x d)+\cos(k_y d)\right]$$

$$=\delta\omega_c^2\cos\left[(k_x-k_y)d\right]\cos\left[(k_x+k_y)d\right]$$



Individual cavities have a set of modes, $\omega_c^2 = \omega_{c1}^2, \omega_{c2}^2, \omega_{c3}^2$....

If the spacing between modes is greater that the frequency shift induced by coupling

$$\left|\omega_{cp}^2 - \omega_{cp+1}^2\right| < \delta\omega_c^2$$

then gaps in the spectrum with no propagating modes appear.

Metamaterials

Metamaterials are periodic structures that have engineered properties in the long wave length limit,

kd<<1

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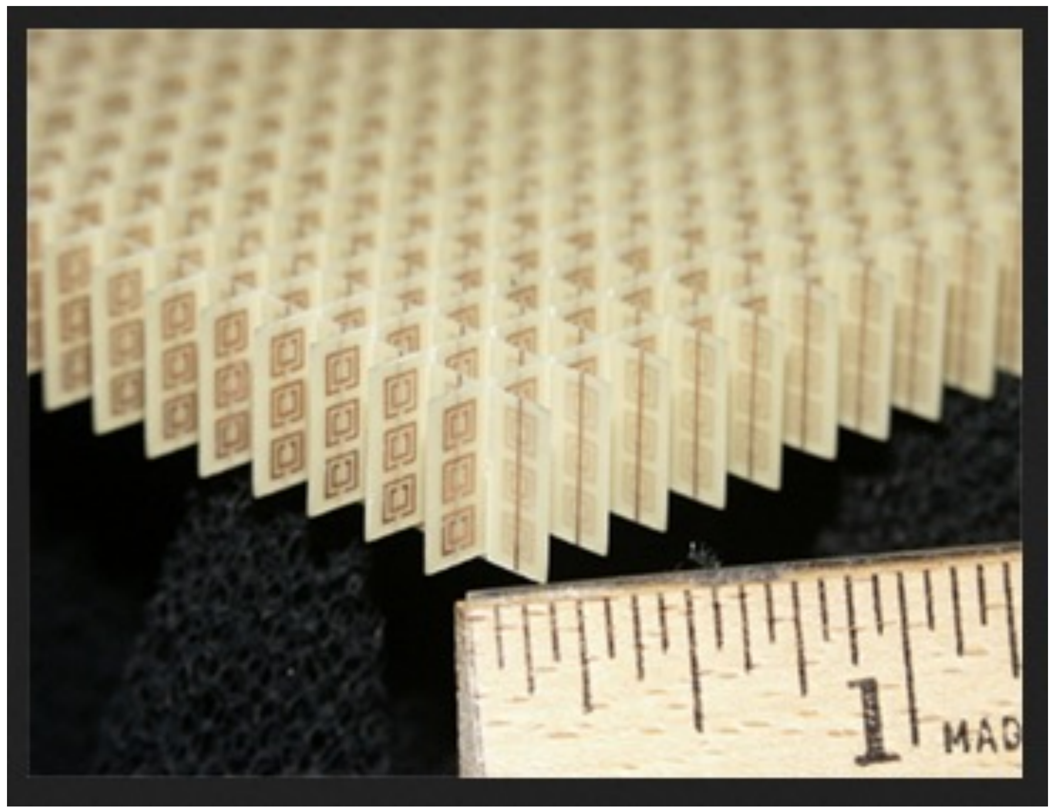


Smith Island Cake

MARYLAND AT A GLANCE

STATE SYMBOLS

Maryland State Dessert - Smith Island Cake

- Maryland Foods

Effective October 1, 2008, the Smith Island Cake became the State Dessert of Maryland (Chapters 164 & 165, Acts of 2008; Code General Provisions Article, sec. 7-313). Traditionally, the cake consists of eight to ten layers of yellow cake with chocolate frosting between each layer and slathered over the whole. However, many variations have evolved, both in the flavors for frosting and the cake itself.

Smith Island Cake, Smith Island, Somerset County, Maryland, 2008.

Smith Island, home to the State Dessert, is Maryland's last inhabited island, reachable only by boat. Straddling the Maryland - Virginia line, Smith Island is twelve miles west of Crisfield in Somerset County and 95 miles south of Baltimore.

Dielectric Tensor

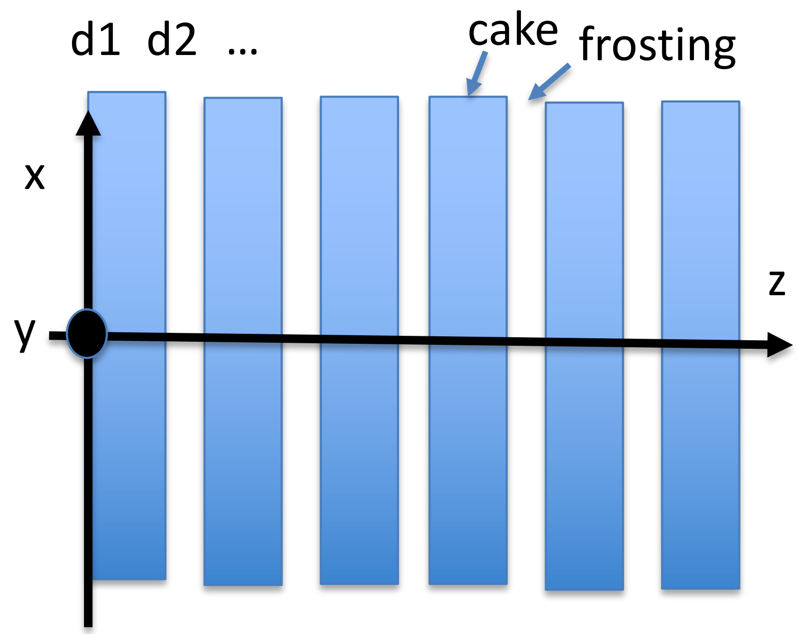
$$\mathbf{D} = \underline{\underline{\varepsilon}} \cdot \mathbf{E}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_a & 0 \\ 0 & 0 & \varepsilon_b \end{bmatrix}$$

$$\varepsilon_a = \frac{d_1\varepsilon_1 + d_2\varepsilon_2}{d_1 + d_2}$$

$$\bar{E} = \frac{d_1 E_1 + d_2 E_2}{d_1 + d_2} = \frac{d_1/\varepsilon_1 + d_2/\varepsilon_2}{d_1 + d_2}\bar{D}$$

$$\frac{1}{\varepsilon_b} = \frac{d_1 + d_2}{d_1/\varepsilon_1 + d_2/\varepsilon_2}$$



Negative epsilon and negative mu

In a restricted range of frequencies the effective constituitive parameters may be negative.

If both are positive or both are negative waves propagate.

$$k^2 = \omega^2 \varepsilon \mu > 0$$

If both are negative waves satisfy the left hand rule.

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}$$

For $\varepsilon<0$ or $\mu<0$ they must be functions of frequency.

Media are passive, stored energy is positive.

$$\mathrm{U}_E = \frac{1}{2}\frac{\partial}{\partial\omega}\left(\omega\varepsilon(\omega)\right)\left|E\right|^2 > 0, \qquad \frac{\partial}{\partial\omega}\left(\omega\varepsilon(\omega)\right) = \varepsilon(\omega) + \omega\frac{\partial}{\partial\omega}\varepsilon(\omega) > 0$$

If both $\varepsilon<0$ and $\mu<0$ group and phase velocities are opposite

If both ε<0 and μ<0 group and phase velocities are opposite

$$\frac{1}{v_g} = \frac{\partial}{\partial\omega}k = \frac{\partial}{\partial\omega}\left(\omega\sqrt{\varepsilon\mu}\right) = \sqrt{\varepsilon\mu} + \frac{\omega}{2\sqrt{\varepsilon\mu}}\frac{\partial}{\partial\omega}\left(\varepsilon\mu\right)$$

$$\frac{1}{v_g} = \frac{1}{2\sqrt{\varepsilon\mu}}\left[\mu\frac{\partial}{\partial\omega}\left(\omega\varepsilon(\omega)\right) + \varepsilon\frac{\partial}{\partial\omega}\left(\omega\mu(\omega)\right)\right] < 0 \quad \text{if both } \varepsilon \ \& \ \eta < 0$$

$$\frac{1}{v_p} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Backward Waves