ENEE681

Networks

N-port Linear Systems

Wave Systems are characterized by scattering (or S) parameters. Actually a matrix

These give the amplitudes of waves leaving the system as a linear function of the amplitudes of waves entering the system.
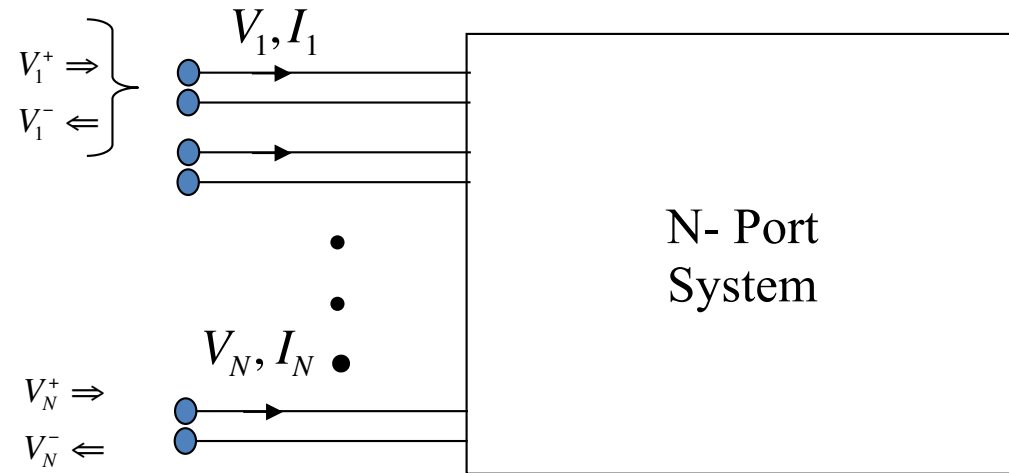
The simplest example is a reflection coefficient

$$V_{refl} / V_{inc} = \rho$$

Impedance (Z) and Scattering (S) Matrices

N ports
• voltages and currents,
• incoming and outgoing waves

Z matrix

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \mathbf{Z} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_2 \end{pmatrix}$$

voltage   current

S matrix

$$\begin{pmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_{N1}^- \end{pmatrix} = \mathbf{S} \begin{pmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_{N1}^+ \end{pmatrix}$$
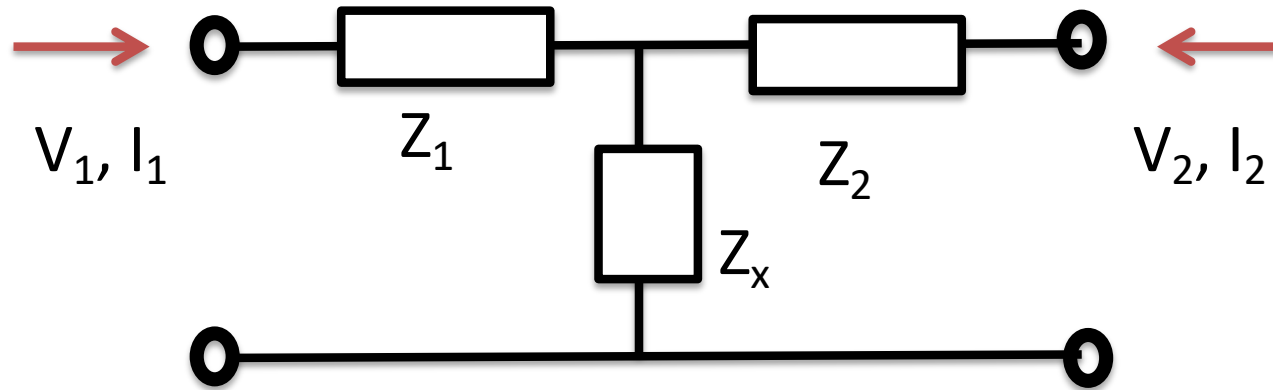
outgoing   incoming

$$S = (Z + Z_0)^{-1}(Z - Z_0)$$

$Z(\omega)$,
$S(\omega)$

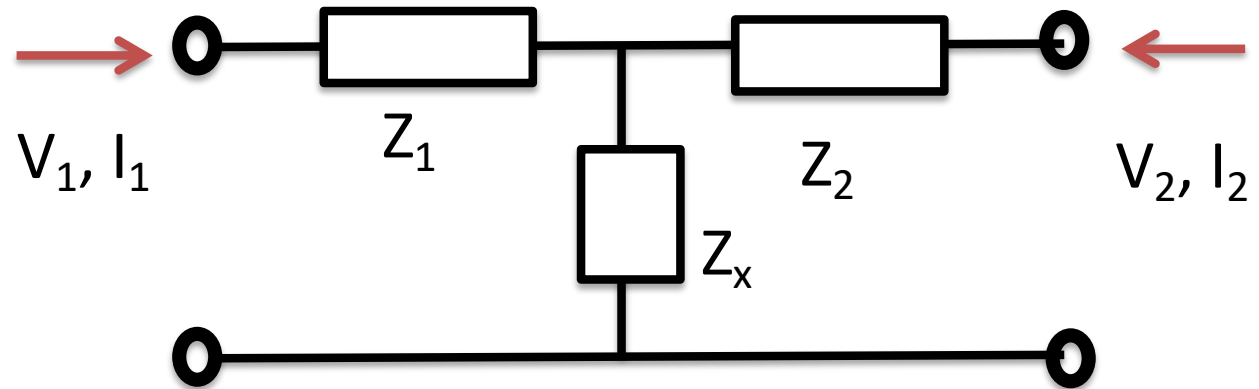# Example – T-circuit

What is the Impedance Matrix ?

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$Z_{11} = Z_1 + Z_x$$
$$Z_{22} = Z_2 + Z_x$$
$$Z_{12} = Z_{21} = Z_x$$

Find the Scattering Matrix

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \underline{\underline{Z}} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

* * * * * * * * * * * * *

$$V_1 = \left(V_{1,i} + V_{1,r} \right)$$

$$I_1 = Z_0^{-1} \left(V_{1,i} - V_{1,r} \right)$$

$$V_2 = \left(V_{2,i} + V_{2,r} \right)$$

$$I_2 = Z_0^{-1} \left(V_{2,i} - V_{2,r} \right)$$

* * * * * * * * * * * * *

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \underline{\underline{Z}}_0^{-1} \begin{pmatrix} \left(V_{1,i} - V_{1,r} \right) \\ \left(V_{2,i} - V_{2,r} \right) \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} + \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \underline{\underline{Z}} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} \right) \left\{ \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} - \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} \right\}$$

$$\left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right) \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix}$$

$$\underline{\underline{Z}}_0 = \begin{bmatrix} Z_{01} & 0 \\ 0 & Z_{02} \end{bmatrix}, \quad \underline{\underline{Z}} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

V_1, I_1 Z_1 Z_2 V_2, I_2 Z_x

Solve for Reflected Amplitude

in terms of Incident

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} + \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left(\underline{\underline{Z}}\ \underline{\underline{Z}}_0^{-1} \right) \left\{ \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} - \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} \right\}$$

$$\left(\underline{\underline{Z}}\ \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right) \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left(\underline{\underline{Z}}\ \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix}$$

$$\boxed{ \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left(\underline{\underline{Z}}\ \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right)^{-1} \left(\underline{\underline{Z}}\ \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} }$$

Not there yet

Define Normalized Wave Amplitudes

Power Entering the Network

$$P = \frac{1}{2}\mathrm{Re}\left\{ \underline{I}^{*T} \cdot \underline{V} \right\} = \frac{1}{2}\mathrm{Re}\left\{ \left(\underline{V}_i^* - \underline{V}_r^* \right)^T \cdot \underline{\underline{Z}}_0^{-1} \cdot \left(\underline{V}_i + \underline{V}_r \right) \right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{ \underline{V}_i^{*T} \cdot \underline{\underline{Z}}_0^{-1} \cdot \underline{V}_i - \underline{V}_r^{*T} \cdot \underline{\underline{Z}}_0^{-1} \cdot \underline{V}_r \right\}$$

Incident   Reflected

Normalized Amplitudes

$$\underline{A}_i = \underline{\underline{Z}}_0^{-1/2} \underline{V}_i, \quad \underline{A}_r = \underline{\underline{Z}}_0^{-1/2} \underline{V}_r$$

$$P = \frac{1}{2}\mathrm{Re}\left\{ \underline{A}_i^{*T} \cdot \underline{A}_i - \underline{A}_r^{*T} \cdot \underline{A}_r \right\}$$

$$\underline{V}_r = \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right)^{-1} \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \underline{V}_i$$

$$\underline{A}_r = \underline{\underline{Z}}_0^{-1/2} \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right)^{-1} \left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \underline{\underline{Z}}_0^{1/2} \underline{A}_i$$

$$\left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right)^{-1} = \left[\left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1} \right]^{-1} = \underline{\underline{Z}}_0 \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1}$$

$$\left(\underline{\underline{Z}}\, \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) = \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1}$$

$$\underline{A}_r = \underline{\underline{Z}}_0^{1/2} \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2} \underline{A}_i$$

Scattering Matrix

$$\underline{A}_r = \underline{\underline{S}} \cdot \underline{A}_i$$

$$\underline{\underline{S}} = \underline{\underline{Z}}_0^{1/2} \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2}$$

Scattering Matrix Properties

S and Z are often Symmetric – Reciprocity

Example:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$Z_{11} = Z_1 + Z_x$$
$$Z_{22} = Z_2 + Z_x$$
$$\color{red}{Z_{12} = Z_{21} = Z_x}$$

Unitary – Consequence of absence of losses and Time Reversal Symmetry

$$\underline{\underline{S}}^* \cdot \underline{\underline{S}} = \underline{\underline{1}}$$

If Network is lossless elements of Z are imaginary. $\quad \underline{\underline{Z}}^* = -\underline{\underline{Z}}$

$$\underline{\underline{S}} = \underline{\underline{Z}}_0^{1/2} \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2}$$

$$\underline{\underline{S}}^* = \underline{\underline{Z}}_0^{1/2} \left(-\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1} \left(-\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2} = \underline{\underline{Z}}_0^{1/2} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2}$$

$$\underline{\underline{S}}^* \underline{\underline{S}} = \underline{\underline{Z}}_0^{1/2} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right) \left(\underline{\underline{Z}} + \underline{\underline{Z}}_0 \right)^{-1} \left(\underline{\underline{Z}} - \underline{\underline{Z}}_0 \right) \underline{\underline{Z}}_0^{-1/2} = \underline{\underline{1}}$$

2 x 2 Unitary Scattering Matrix

$$\underline{\underline{S}} = \begin{bmatrix} R^{1/2}e^{i\theta_{11}} & T^{1/2}e^{i\theta_{12}} \\ T^{1/2}e^{i\theta_{12}} & R^{1/2}e^{i\theta_{22}} \end{bmatrix}$$

$$R + T = 1 \qquad\qquad \theta_{12} = \frac{\theta_{11} + \theta_{22}}{2} + \left(n + \frac{1}{2}\right)\pi$$

2 x 2 Unitary Scattering Matrix

$$\underline{\underline{S}}^* \cdot \underline{\underline{S}} = \underline{\underline{1}} = \begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}$$

$$\underline{\underline{1}} = \begin{bmatrix} \left|S_{11}\right|^2 + \left|S_{21}\right|^2 & S_{21}^* S_{22} + S_{11}^* S_{21} \\ S_{21}^* S_{11} + S_{22}^* S_{21} & \left|S_{22}\right|^2 + \left|S_{21}\right|^2 \end{bmatrix}$$

2 x 2 Unitary Scattering Matrix

$$\underline{\underline{1}} = \begin{bmatrix} \left|S_{11}\right|^2 + \left|S_{21}\right|^2 & S_{21}^* S_{22} + S_{11}^* S_{21} \\ S_{21}^* S_{11} + S_{22}^* S_{21} & \left|S_{22}\right|^2 + \left|S_{21}\right|^2 \end{bmatrix}$$

$$\left|S_{21}\right|^2 = T$$

$$\left|S_{11}\right|^2 = \left|S_{22}\right|^2 = R$$

$$\left|S_{11}\right|^2 + \left|S_{21}\right|^2 = 1 = \left|S_{22}\right|^2 + \left|S_{21}\right|^2$$

$$R + T = 1$$

$$\underline{\underline{S}} = \begin{bmatrix} R^{1/2} e^{i\theta_{11}} & T^{1/2} e^{i\theta_{12}} \\ T^{1/2} e^{i\theta_{12}} & R^{1/2} e^{i\theta_{22}} \end{bmatrix}$$

$$S_{21}^* S_{11} + S_{22}^* S_{21} = 0 = S_{21}^* S_{22} + S_{11}^* S_{21}$$

$$(\text{RT})^{1/2} \left[\exp\left(i\left(\theta_{11} - \theta_{21}\right)\right) + \exp\left(i\left(\theta_{21} - \theta_{22}\right)\right) \right] = 0$$

$$\exp\left(i\left(\theta_{11} + \theta_{22} - 2\theta_{21}\right)\right) = -1$$

$$\theta_{21} = \frac{\theta_{11} + \theta_{22}}{2} + \left(n + \frac{1}{2}\right)\pi$$

Example – π-circuit
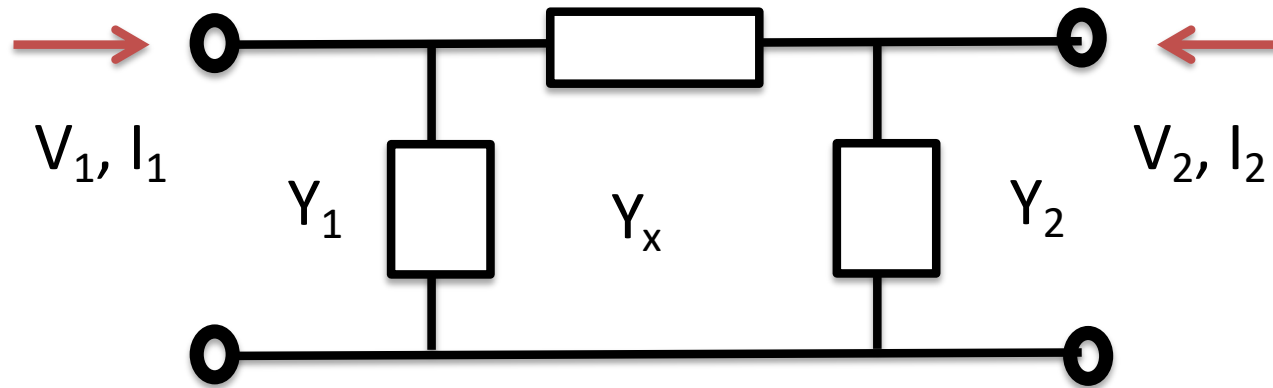
Admittance Matrix

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$
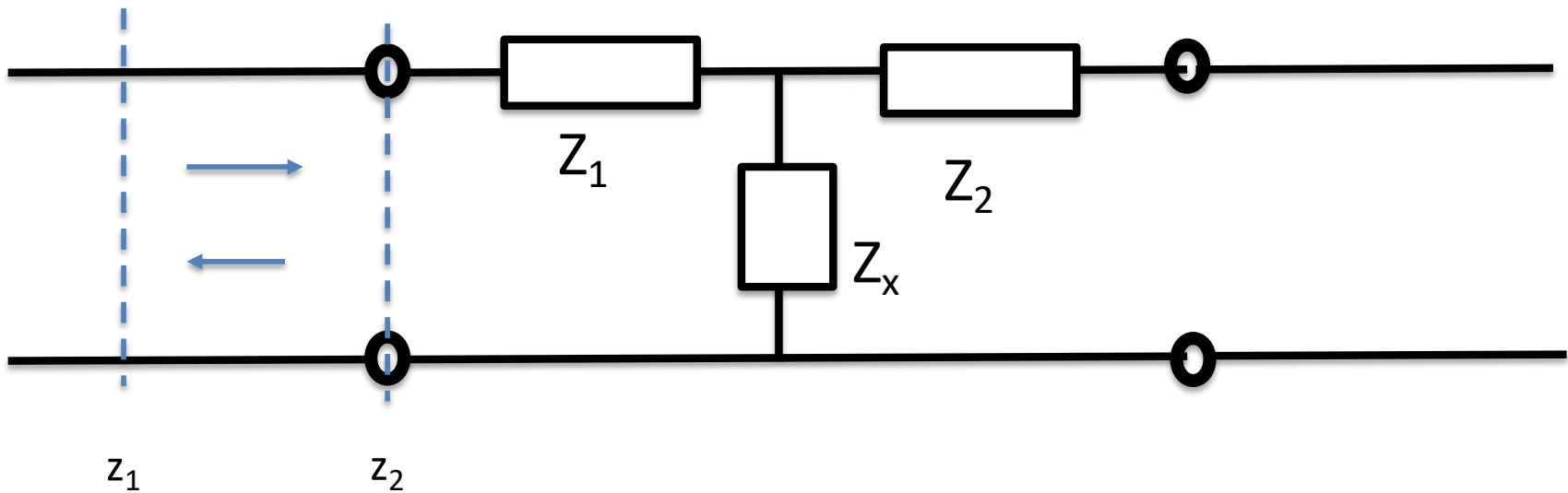
$$Y = Z^{-1}$$

$$Y_{11} = Y_1 + Y_x$$
$$Y_{22} = Y_2 + Y_x$$
$$Y_{12} = Y_{21} = -Y_x$$

Calibration plane

$$A_{1i}(z_2) = A_{1i}(z_1)e^{-jk(z_2-z_1)}$$

$$A_{1r}(z_2) = A_{12}(z_1)e^{jk(z_2-z_1)}$$

How to measure S-parameters

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How does a Vector Network Analyzer (VNA) work?

A Vector Network Analyzer contains both a source, used to generate a known stimulus signal, and a set of receivers, used to determine changes to this stimulus caused by the device-under-test or DUT.

The stimulus signal is injected into the DUT and the Vector Network Analyzer measures both the signal that's reflected from the input side, as well as the signal that passes through to the output side of the DUT. The Vector Network Analyzer receivers measure the resulting signals and compare them to the known stimulus signal. The measured results are then processed by either an internal or external PC and sent to a display.
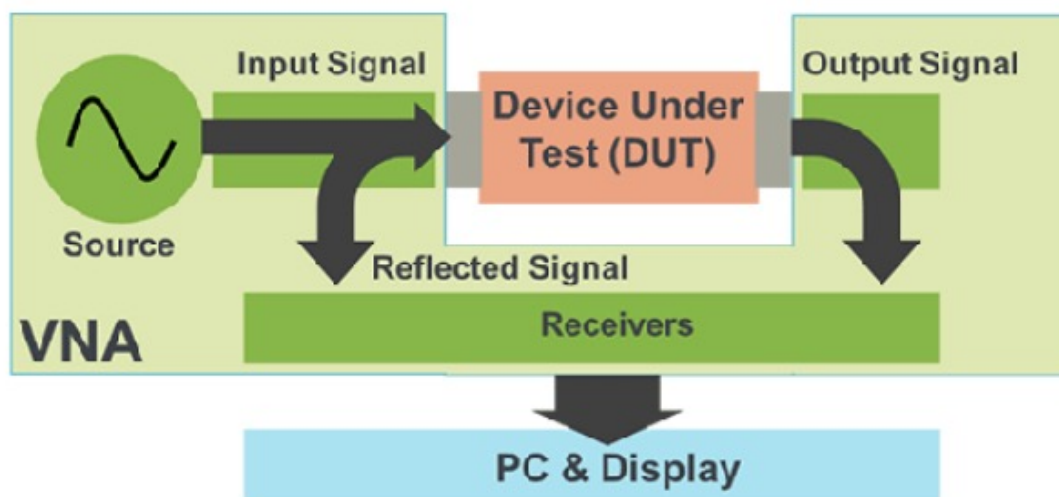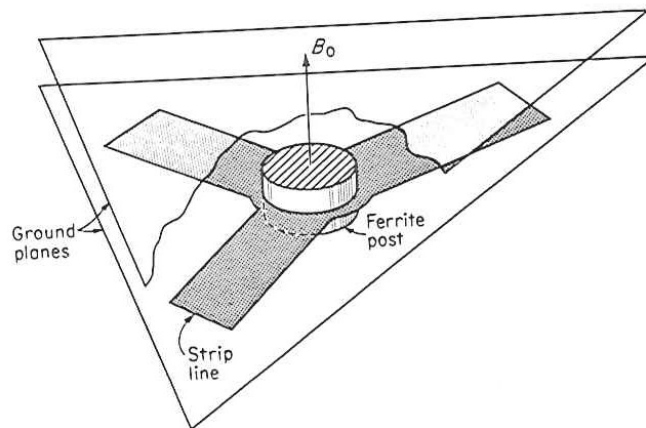
Non Reciprocal - Circulator

3

1

Bo

Ferrite post

(a)

2

Bo

Ground planes

Ferrite post

Strip line
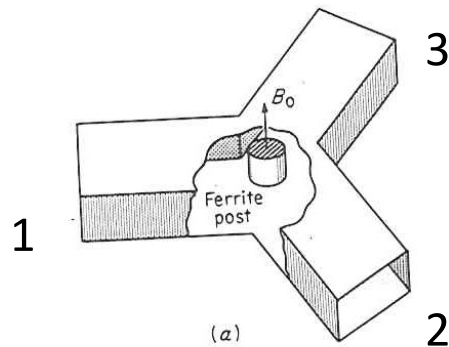
R.E. Collin: Foundations of microwave engineering

If lossless and calibration planes are properly defined

$$\underline{\underline{S}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What does it do?