Cavities

ENEE 681

Role of Cavities

Cavities are resonant structures: Support EM modes at specific frequencies.

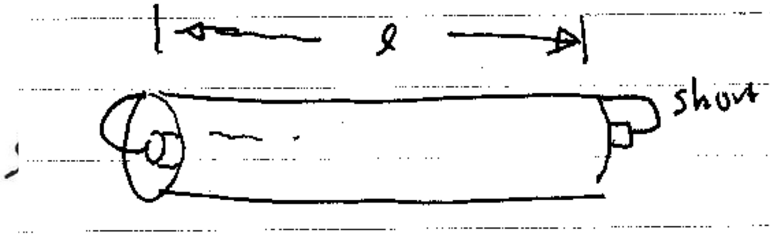
Used in:

- Filters
- Oscillators
- Amplifiers
- Measurement of material properties

Resonance

natural frequency of oscillation of fields

Example:



Transmission line

Terminated in short circuits

$kl = n\pi$ ~~$\omega = k$~~ $k = \frac{\omega}{v}$

$$\frac{d^2}{dx^2}V(x) + \frac{\omega^2}{v^2}V(x) = 0$$

$$\boxed{\omega_n = \frac{n\pi}{l} v}$$

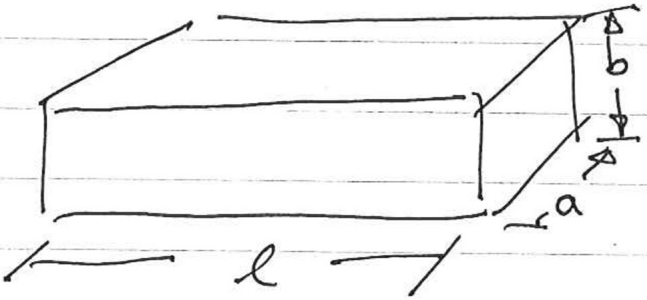
$n = 1, 2, \ldots$

$$V(x) = V_+ e^{i\omega x/v} + V_- e^{-i\omega x/v}$$

$$V(x=0) = 0$$
$$V(x=l) = 0$$

Enclosed Rectangular Prism

enclosed box

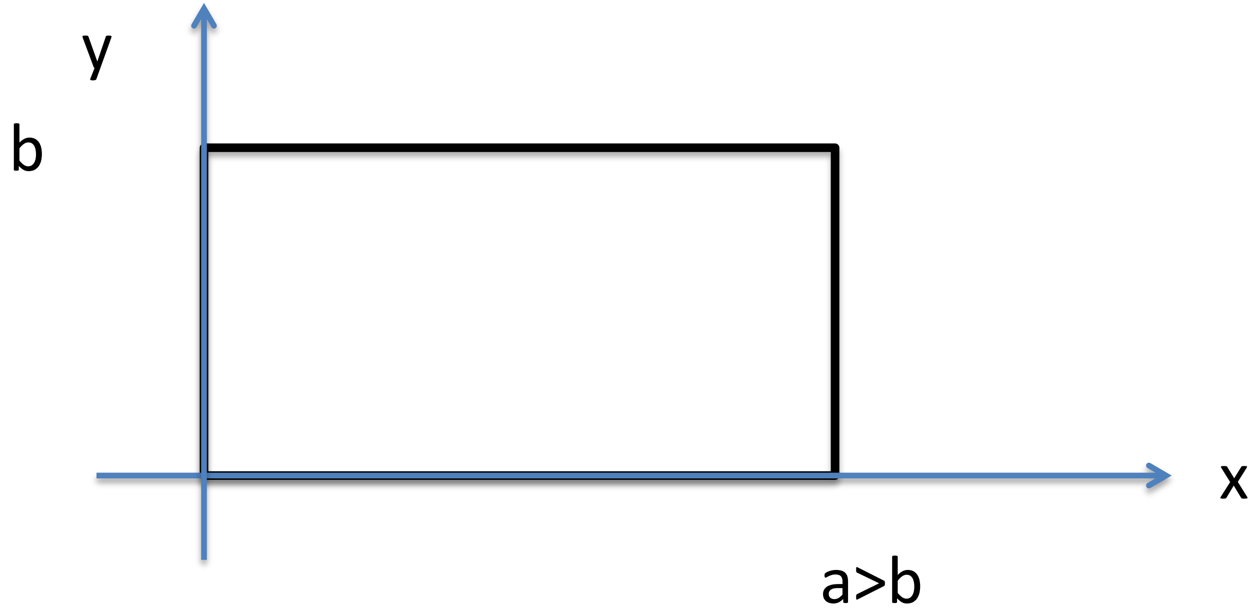


TE$_{nm}$ or TM$_{nm}$

$$\omega_{nmp} = V\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2}$$

n, m, p
are
integers

Modes of a Rectangular WG



$$\mathrm{TM}_{\mathrm{nm}}:\quad \hat{E}_z = E_0 \sin\left(k_x x\right)\sin\left(k_y y\right) \qquad k_x = \frac{n\pi}{a},\quad k_y = \frac{m\pi}{b}:\quad n,m = 1,2,3,....$$

$$\mathrm{TE}_{\mathrm{nm}}:\quad \hat{H}_z = H_0 \cos\left(k_x x\right)\cos\left(k_y y\right) \qquad k_x = \frac{n\pi}{a},\quad k_y = \frac{m\pi}{b}:\quad n,m = 0^*,1,2,3,...$$

* one or the other, but not both

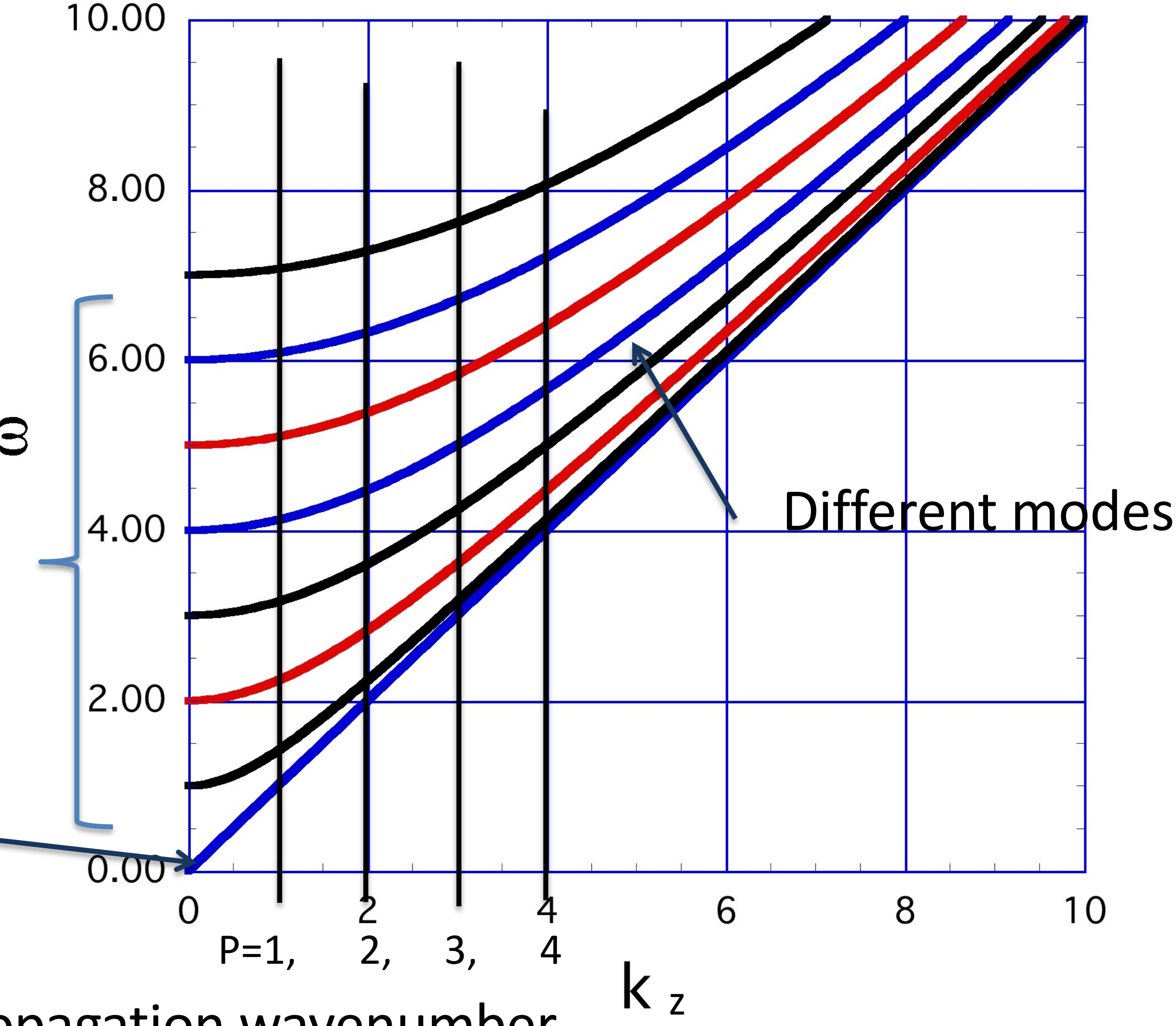
Cut-Off frequencies

$$\omega_{c,n,m} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

WG Dispersion Relations

$$\frac{\omega^2}{v^2} = k_\perp^2 + k_z^2$$

$$k_z = p\frac{\pi}{L}$$



Different modes

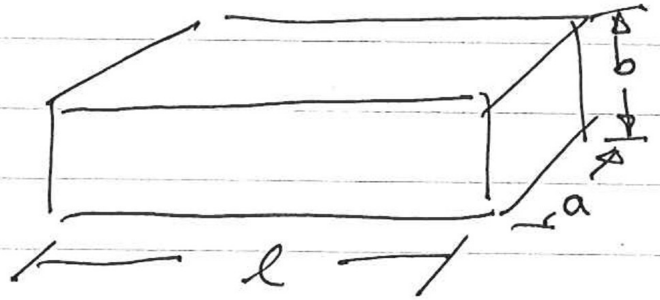
Cut off frequencies

Transmission Lines Only

Propagation wavenumber

Enclosed Rectangular Prism

enclosed box



TE$_{nm}$ or TM$_{nm}$

$$\omega_{nmp} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2}$$

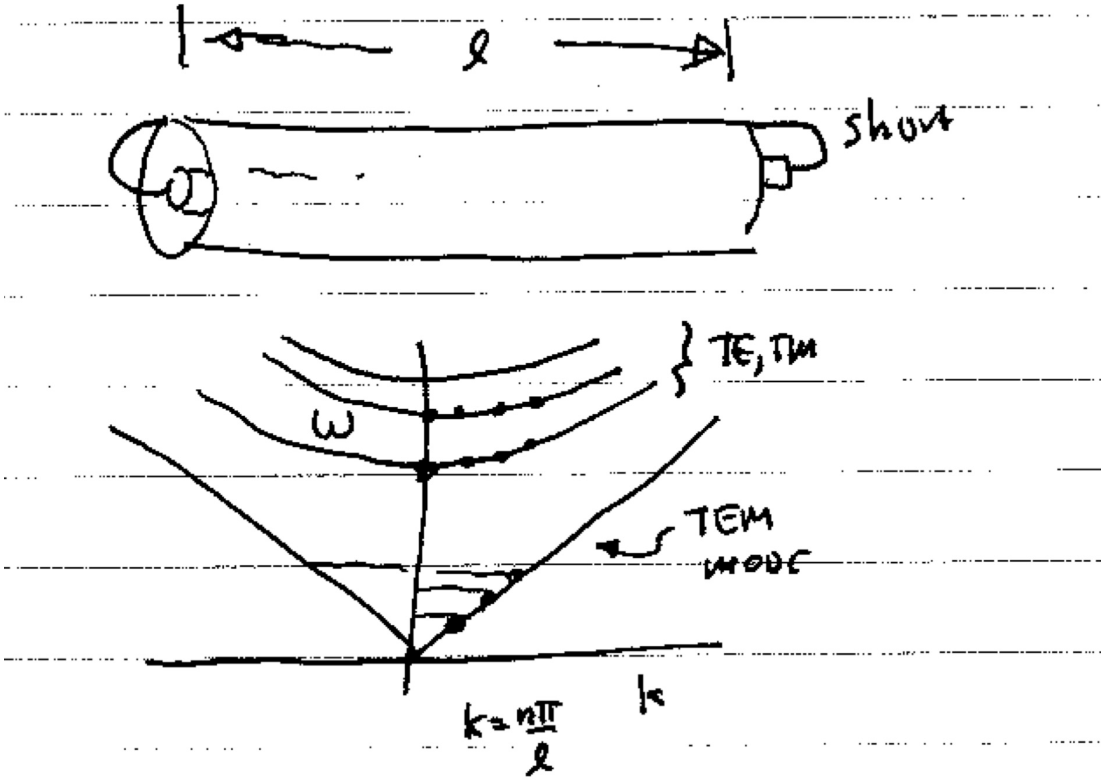
$\boxed{n, m, p \text{ are integers}}$

$$\omega^2 = k_z^2 v^2 + \omega_{c,nm}^2$$

$$\omega_{c,n,m} = v\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

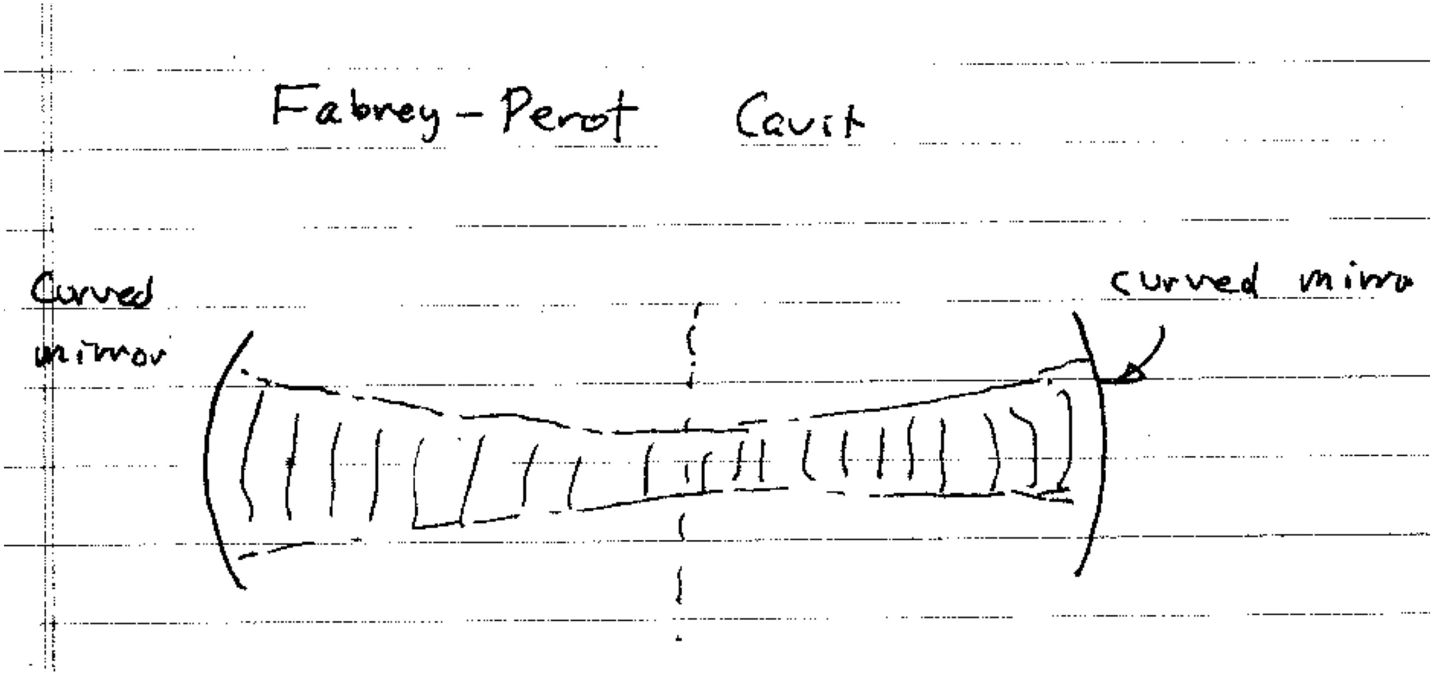
$$k_z = p\frac{\pi}{L}$$

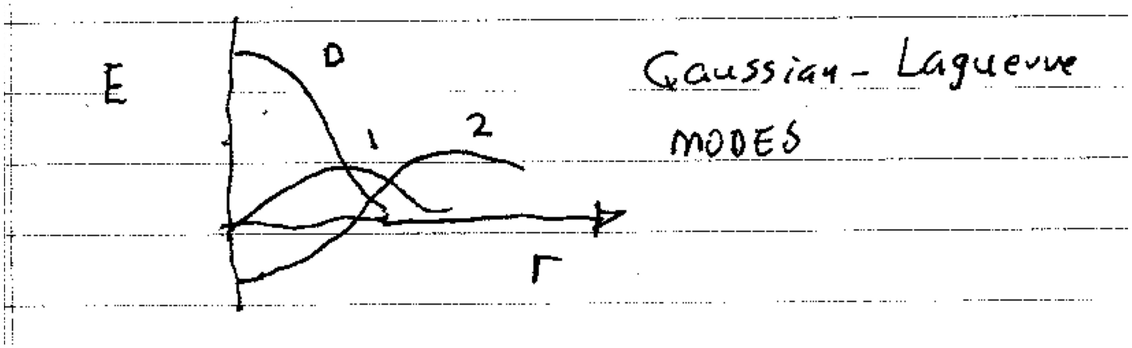
Transmission Line – TEM mode



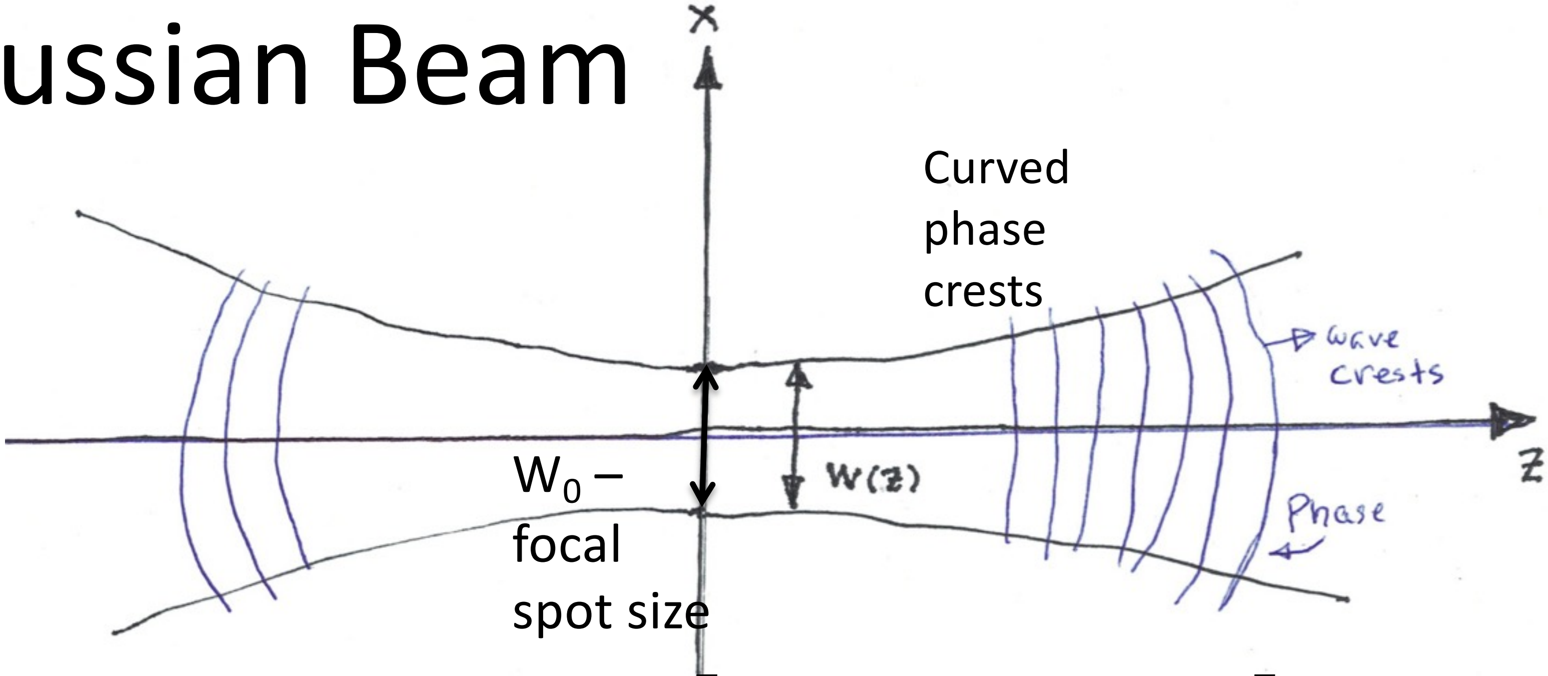
Operate at frequencies well below TE and TM cut-off

Fabry-Perot Cavity





Gaussian Beam



Curved phase crests

W_0 – focal spot size

$$E_{x,y}(x,y,z)=\frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$W(z)=W_0\sqrt{1+z^2/Z_R^2} \qquad Z_R=\frac{1}{2}kW_0^2$$

Rayleigh Length

$$E_{x,y}(0,0,z)=\frac{E_0}{1+iz/Z_R}\exp\left[+ikz\right]=\frac{E_0}{\sqrt{1+\left(z/Z_R\right)^2}}\exp\left[+ikz+i\phi(z)\right]$$

Guoy Phase $\tan\phi=-z/Z_R$

Gaussian Beam

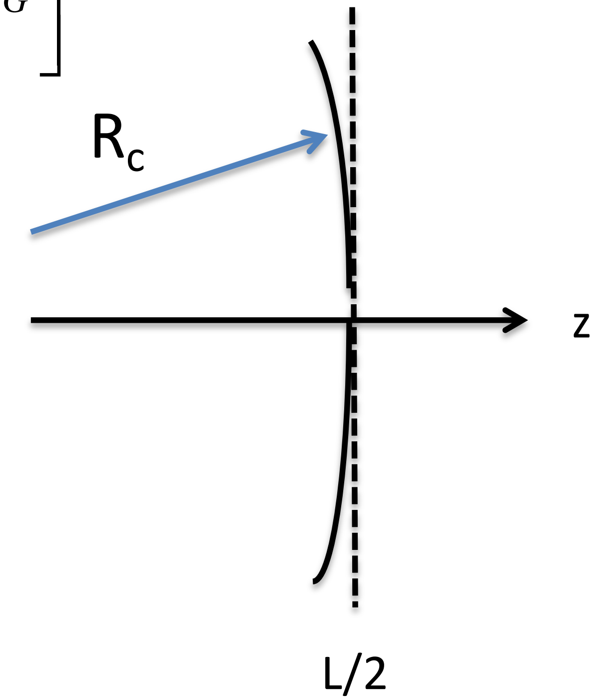
$$E_{x,y}(x,y,z)=\frac{E_0}{1+iz/Z_R}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+iz/Z_R\right)}+ikz\right]$$

$$=\underbrace{\frac{E_0}{\sqrt{1+z^2/Z_R^2}}\exp\left[-\frac{\left(x^2+y^2\right)}{W_0^2\left(1+z^2/Z_R^2\right)}\right]}_{\text{Amplitude}}\underbrace{\exp\left\{i\left[kz+\frac{z\left(x^2+y^2\right)}{Z_R W_0^2\left(1+z^2/Z_R^2\right)}+\phi_G\right]\right\}}_{\text{Phase}}$$

Pick parameters such that phase is constant on surface of mirror.
The pick k such that the phase changes by $p\pi$ in going from one mirror to the next

Wavebeam phase $\left[kz + \frac{z\left(x^2+y^2\right)}{Z_R W_0^2\left(1+z^2/Z_R^2\right)} + \phi_G \right]$

Surface of mirror $z \simeq \frac{L}{2} - \frac{\left(x^2+y^2\right)}{2R_c}$



Will match if $\frac{L}{2R_c} = \frac{(L/2)^2}{Z_R^2+(L/2)^2} < 1$

Phase change mirror to mirror $2\left(k_p \frac{L}{2} + \phi_G\right) = p\pi$

For a given L and R_c Z_R is determined above.
Hence W_0 focal spot determined.

$$Z_R = \frac{1}{2} k W_0^2$$

Design a Fabrey-Perot resonator

Requirements:

Wavelength 1 micron = 10^{-6} m.

Focal spot size = 100 microns= – 10^{-4} m.

Spot size on mirrors = 300 microns = $3x10^{--4}$ m

Find L and R_c

Super bonus: How big must the mirrors be to keep “spill over” below 10%

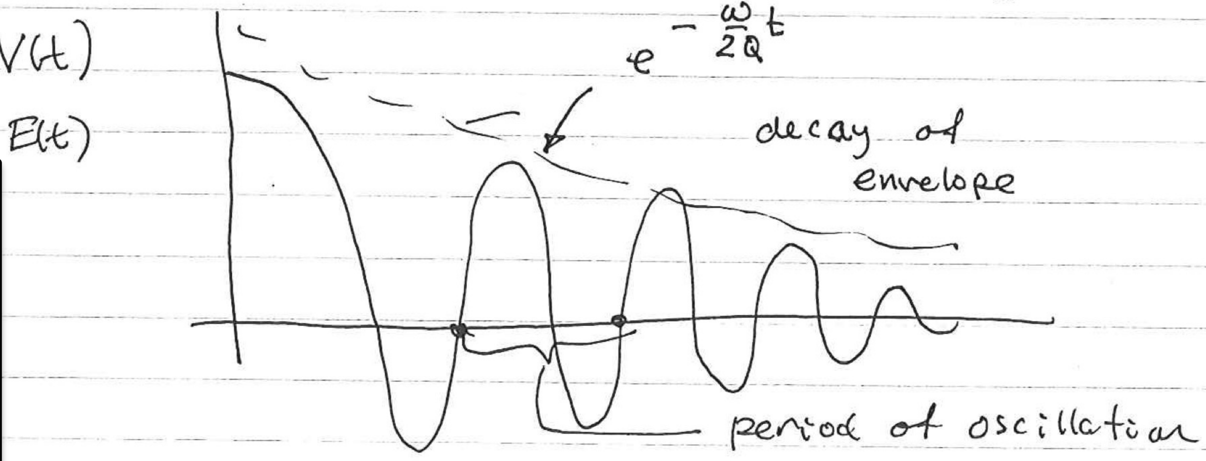
Quality Factor

Quality factor — measures losses Q

big Q low loss low Q high loss

Time domain

Defn $\left(\frac{\omega U}{P_d}\right) = Q$



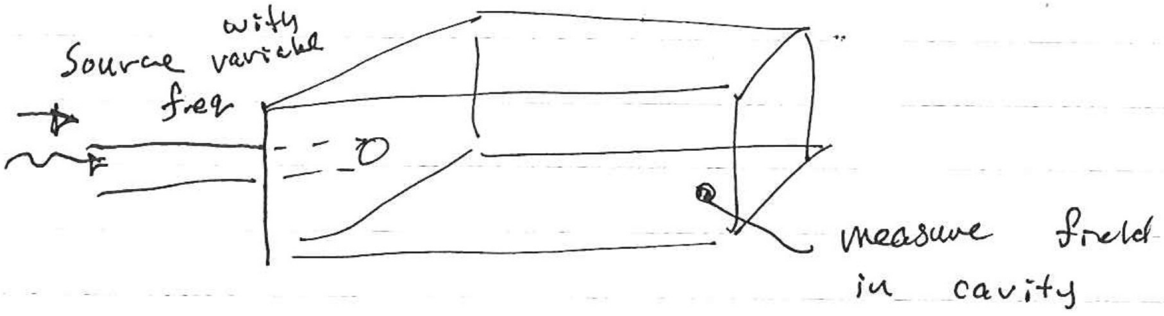
$$\frac{1}{Q} \equiv \frac{\text{Power Dissipated}}{\omega\,\text{Energy Stored}}$$

Field decay rate

$$\text{E, H} \sim \exp\left(-\frac{\omega}{2Q}t\right)$$

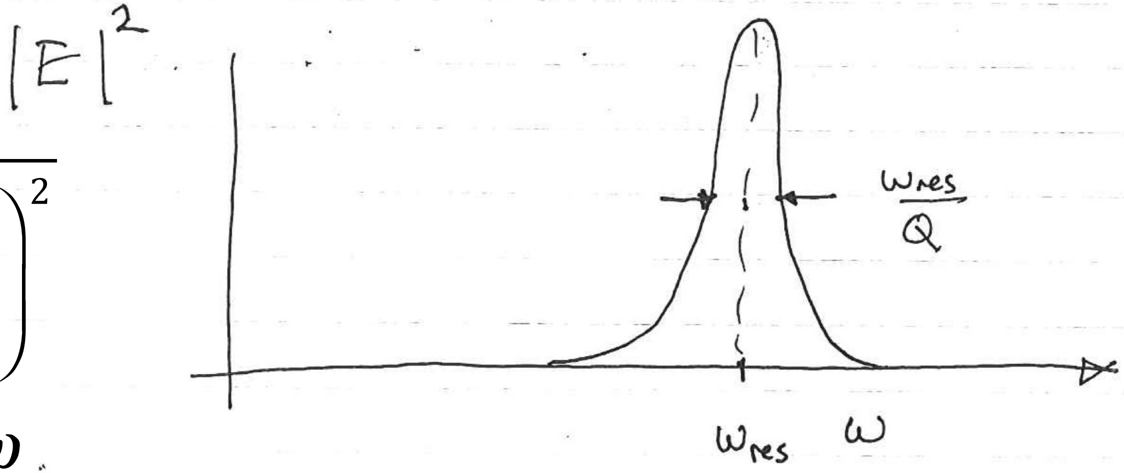
Frequency Domain

Steady state field response



$$\mathrm{E, H} \sim \frac{\text{Source}}{\omega - \omega_{res} + i\dfrac{\omega_{res}}{2Q}}$$

$$|\mathrm{E}|^2, |\mathrm{H}|^2 \sim \frac{1}{\left(\omega - \omega_{res}\right)^2 + \left(\dfrac{\omega_{res}}{2Q}\right)^2}$$

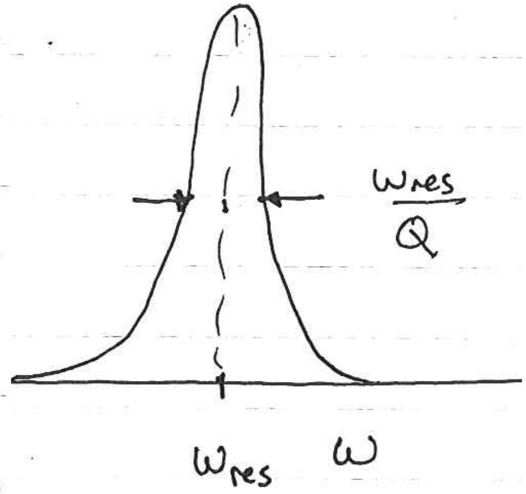


Half maximum $\omega - \omega_{res} = \pm \dfrac{\omega_{res}}{2Q}$

Full Width at Half Maximum (FWHM) = $\dfrac{\omega_{res}}{Q}$

Response Function

$$|E|^2 \propto \frac{1}{(\omega-\omega_{res})^2 + \left(\frac{\omega_{res}}{2Q}\right)^2}$$



when $\omega = \omega_{res} \pm \frac{\omega_{res}}{2Q}$

$|E|^2$ is $\frac{1}{2}$ of peak value

Full width at half max of Power (FWHM) $\longrightarrow \frac{\omega_{res}}{Q}$

Multiple Contributions to Loss

if. losses are small ($Q \gg 1$)

losses are additive

different loss mechan

$$\frac{1}{Q} = \frac{P_d}{\omega U} = \frac{P_{d1}}{\omega U} + \frac{P_{d2}}{\omega U} + \cdots$$

$$= \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots\cdots$$

Reciprocals of Q add.

Dielectric and Conductor Loss

Q due to lossy dielectric

$$\epsilon = \epsilon' - j\epsilon''$$

$$\boxed{Q = \frac{\epsilon'}{\epsilon''}}$$

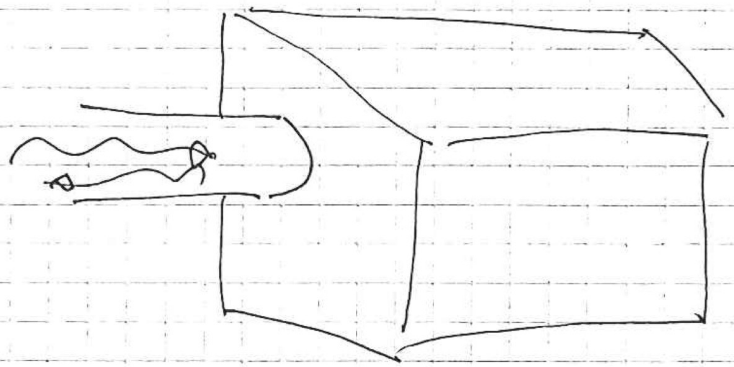
ϵ must completely fill cavity

Losses due to conductors

$$Q = \frac{\omega}{c}\left(\frac{\eta}{R_s}\right)\frac{\int d^3x\,|\underset{\sim}{\hat{H}}|^2 \quad \leftarrow \text{energy stored}}{\int da\,|\underset{\sim}{\hat{H}}_t|^2 \quad \leftarrow \text{losses}}$$

Coupling to Cavities

A closed box is useles



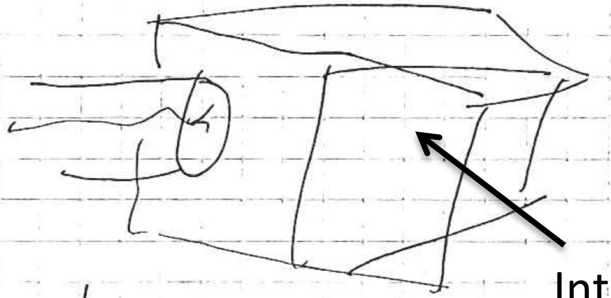
need to be able to get power in it

Adding a hole (or coupling port does two things) power can come in and power can go out.

Coupling Also Characterized by Q

Modifies Q

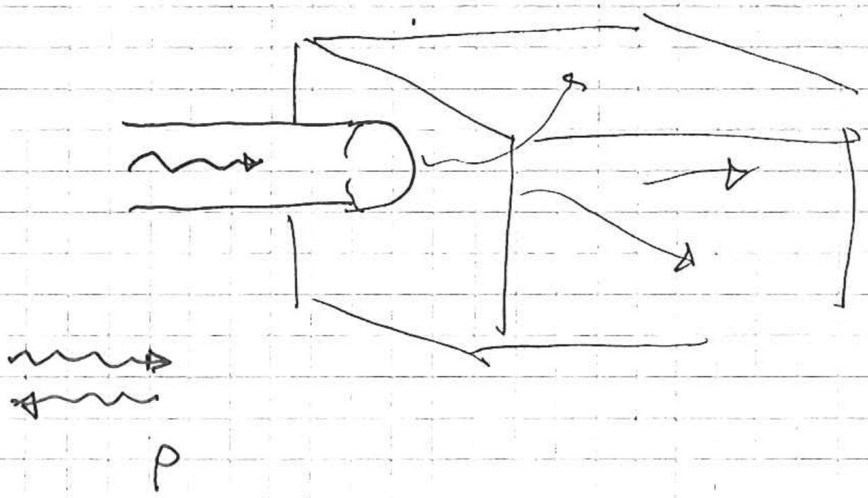
$$\frac{1}{Q_T} = \frac{1}{Q_{\text{internal}}} + \frac{1}{Q_{\text{coupling}}}$$


Internal losses

Q_{internal}

Critically Coupled

$$Q_{coupling} = Q_{internal}$$



At resonance all power from source is absorbed in cavity!

$V_{incident}$

$V_{reflected}$

Voltage reflection coefficient

$$\rho = - \frac{i\left(\frac{\omega}{\omega_{res}} - 1\right) - \left(\frac{1}{2Q_i} - \frac{1}{2Q_e}\right)}{i\left(\frac{\omega}{\omega_{res}} - 1\right) - \left(\frac{1}{2Q_i} + \frac{1}{2Q_e}\right)}$$

$$\rho = \frac{V_{reflected}}{V_{incident}}$$

Universal Response

$$\rho = -\frac{i\left(\frac{\omega}{\omega_{res}}-1\right)-\left(\frac{1}{2Q_i}-\frac{1}{2Q_e}\right)}{i\left(\frac{\omega}{\omega_{res}}-1\right)-\left(\frac{1}{2Q_i}+\frac{1}{2Q_e}\right)}$$

$$\rho = \frac{V_{reflected}}{V_{incident}}$$

Knowing Q_i, Q_e and ω_{res} determines ρ

far from resonance $\rho = -1$

Reflectivity at resonance $|\rho|^2_{Res} = \left|\frac{Q_i^{-1}-Q_e^{-1}}{Q_i^{-1}+Q_e^{-1}}\right|^2$

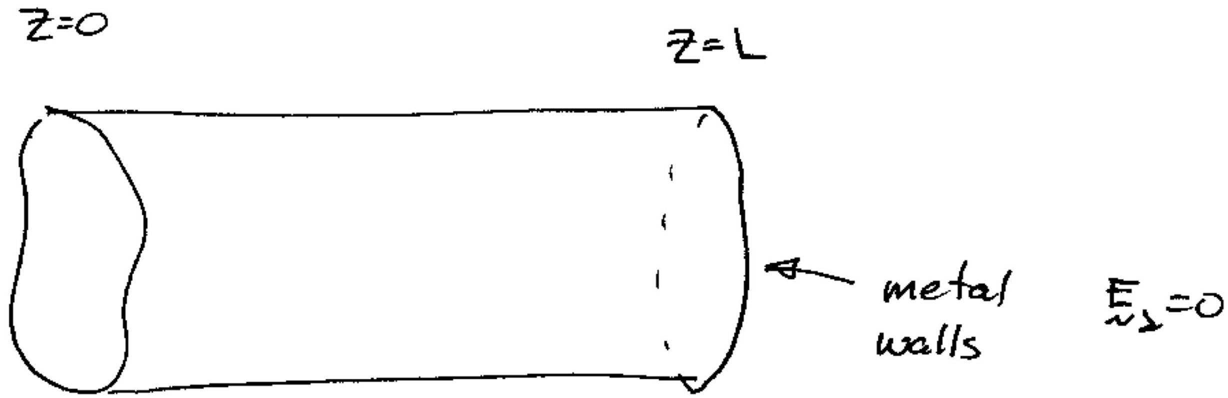
0 - if $Q_i = Q_e$

damping rate for fields $\rho \to \infty$

$$\frac{\omega}{\omega_{res}} = 1 - i\frac{1}{2Q_T} \qquad \frac{1}{Q_T} = \frac{1}{Q_i} + \frac{1}{Q_e}$$

Waveguide Cavities

CAVITIES constructed from cylindrical waveguides will have normal modes



Waveguide fields

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\mathbf{H} = \mathrm{Re}\left\{\hat{\mathbf{H}}(x,y)\exp\left[i\left(k_z z - \omega t\right)\right]\right\}$$

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -\left(\left(\omega / v\right)^2 - k_z^2\right)\hat{H}_z$$

$$\frac{\partial^2 \hat{E}_z}{\partial x^2} + \frac{\partial^2 \hat{E}_z}{\partial y^2} = -\left(\left(\omega / v\right)^2 - k_z^2\right)\hat{E}_z$$

$$\hat{\mathbf{E}}_\perp = \frac{i}{\left(\omega / v\right)^2 - k_z^2}\left[k_z \nabla_\perp \hat{E}_z - \omega\mu\hat{\mathbf{z}} \times \nabla_\perp \hat{H}_z\right]$$

$$\hat{\mathbf{H}}_\perp = \frac{i}{\left(\omega / v\right)^2 - k_z^2}\left[k_z \nabla_\perp \hat{H}_z + \omega\varepsilon\hat{\mathbf{z}} \times \nabla_\perp \hat{E}_z\right]$$

$$\hat{E}_z\Big|_{wall} = 0, \quad \mathbf{n}\cdot\nabla_\perp \hat{H}_z\Big|_{wall} = 0$$

Forward and Backward Waves

TM modes

BC: $\hat{\mathbf{E}}_\perp(z=0,L)=0$

$$\hat{E}_z=\hat{E}_{z,nm}(x,y)\Big(A_+\exp(ik_z z)+A_-\exp(-ik_z z)\Big)$$

$A_+ = A_-$ $A_+ = A_- e^{-2ik_zL}$

$$\hat{\mathbf{E}}_\perp=\frac{ik_z\nabla_\perp\hat{E}_{z,nm}}{\left(\omega/v\right)^2-k_z^2}\Big(A_+\exp(ik_z z)-A_-\exp(-ik_z z)\Big)$$

$k_z L = p\pi,\quad p=0,1,2,\ldots$

TE modes

$$\hat{H}_z=H_{z,nm}(x,y)\Big(A_+\exp(ik_z z)+A_-\exp(-ik_z z)\Big)$$

$A_+ = -A_-$ $A_+ = -A_- e^{-2ik_zL}$

$$\hat{\mathbf{E}}_\perp=\frac{-i\omega\mu\hat{\mathbf{z}}\times\nabla_\perp\hat{H}_{z,nm}}{\left(\omega/v\right)^2-k_z^2}\Big(A_+\exp(ik_z z)+A_-\exp(-ik_z z)\Big)$$

$k_z L = p\pi,\quad p=1,2,\ldots$

Resonant Frequencies

$k_{\parallel} = \frac{\pi p}{L}$

$p \neq 0$ not allowed

all fields zero

determined by cross sectin

$$\frac{\omega^2}{c^2}\epsilon\mu = k_c^2 + \left(\frac{\pi p}{L}\right)^2$$

$$\omega_{Res} = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{k_c^2 + \left(\frac{\pi p}{L}\right)^2}$$

TE_{nmp}

cavity mode

Rectangular cross section

$$\omega_{Res} = \omega_{nmp} = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{L}\right)^2}$$

TM Modes

$$E_z = \frac{1}{2}\left\{ \hat{E}_{\parallel}(\underset{\sim}{x}_\perp)\left[A_+ e^{i(k_\parallel z-\omega t)} + A_- e^{-ik_\parallel z - i\omega t}\right] + c.c.\right\}$$

$$\underset{\sim}{E}_\perp = \frac{1}{2}\left\{ \frac{i\, k_\parallel \nabla_\perp \hat{E}_\parallel}{k_c^2}\left[A_+ e^{i(k_\parallel z-\omega t)} - A_- e^{-ik_\parallel z - i\omega t}\right]\right\}$$

at $z=0$ $\qquad A_+ - A_- = 0 \qquad A_+ = A_-$

at $z=L$ $\qquad A_+ e^{ik_\parallel L} = A_- e^{-ik_\parallel L}$

again $\quad 2k_\parallel L = 2\pi p \qquad p=0 \quad$ o.k.

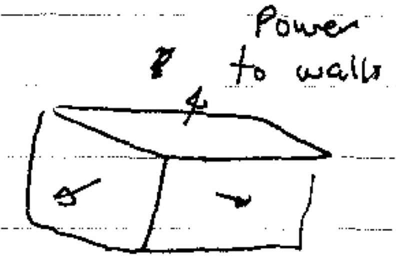
Cavity Losses

$$Q = \frac{\omega U}{P_d}$$ ← energy store ← power dissipated

Quality Factor

Poyntings Theorem

$$\underbrace{\frac{\partial}{\partial t}\int d^3x\, \frac{1}{4}\left(\epsilon|\hat{\underline{E}}|^2 + \mu|\hat{\underline{H}}|^2\right)}_{U} + \underbrace{\int_S dA\, \underline{n}\cdot\frac{1}{2}\mathrm{Re}\{\hat{\underline{E}}\times\hat{\underline{H}}^*\}}_{} = 0$$



For cavity modes

$$\int d^3x\, \frac{\epsilon|\hat{\underline{E}}|^2}{4} = \int d^3x\, \frac{\mu|\hat{\underline{H}}|^2}{4}$$ average energy stored in $\underline{E}$ & H equal

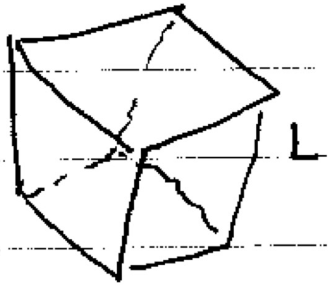
$$P_d = \int dA\, \frac{1}{2} R_s |\hat{\underline{H}}_t|^2$$

$$Q = \frac{\omega}{c}\sqrt{\frac{\mu}{\epsilon}}\frac{1}{R_s}\frac{\int d^3x\, |\hat{\underline{H}}|^2}{\int dA\, |\hat{\underline{H}}_t|^2}$$

Weyl's Formula

How many modes in a cavity of volume V have $\omega_{res} < \omega$?

Consider a cubic cavity of side L $V = L^3$



Resonant Frequencies

$$\omega_{\underline{n}} = \frac{\pi c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

$$\underline{n} = (n_x, n_y, n_z)$$

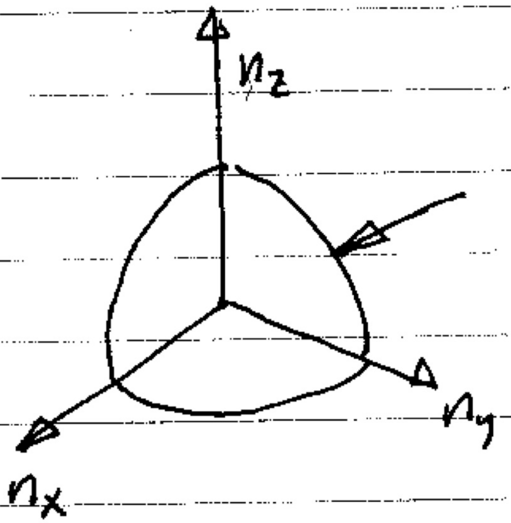
Estimate of Number of Modes

How many combinations of integers (n_x, n_y, n_z) have

$$n_x^2+n_y^2+n_z^2 < \left(\frac{\omega L}{\pi c}\right)^2 = \left(\frac{kL}{\pi}\right)^2$$

$$\omega_{\underline{n}} = \frac{\pi c}{L}\sqrt{n_x^2+n_y^2+n_z^2}$$

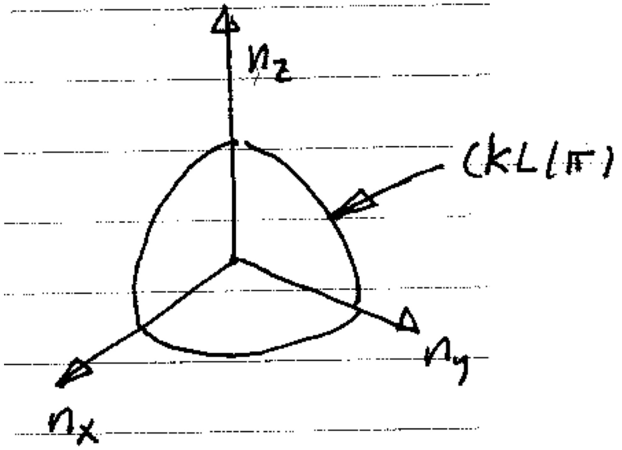
$$\underline{n} = (n_x, n_y, n_z)$$



Spherical surface of radius (kL/π)

Each combination occupies a cube of unit volume

Volume Inside Spherical Surface



$$N = \frac{1}{8}\,\frac{4}{3}\pi\left(\frac{kL}{\pi}\right)^3$$

fraction of sphere

$$N(k) = \frac{(kL)^3}{6\pi^2} = \frac{k^3V}{6\pi^2}$$

But wait! For each set of integers there are 2 polarizations

For EM modes $N(k) = \frac{1}{3}\,\frac{k^3V}{\pi^2}$

Example

Volume $= 1\ \mathrm{m}^3$

$f = 1\ \mathrm{GHz}$

$$k = \frac{2\pi f}{c} = \frac{2\pi \times 10^9}{3 \times 10^8} = 21\ \mathrm{m}^{-1}$$

$$N(k) \cong 310$$

What is the typical spacing

δk = spacing in $k = \omega / c$

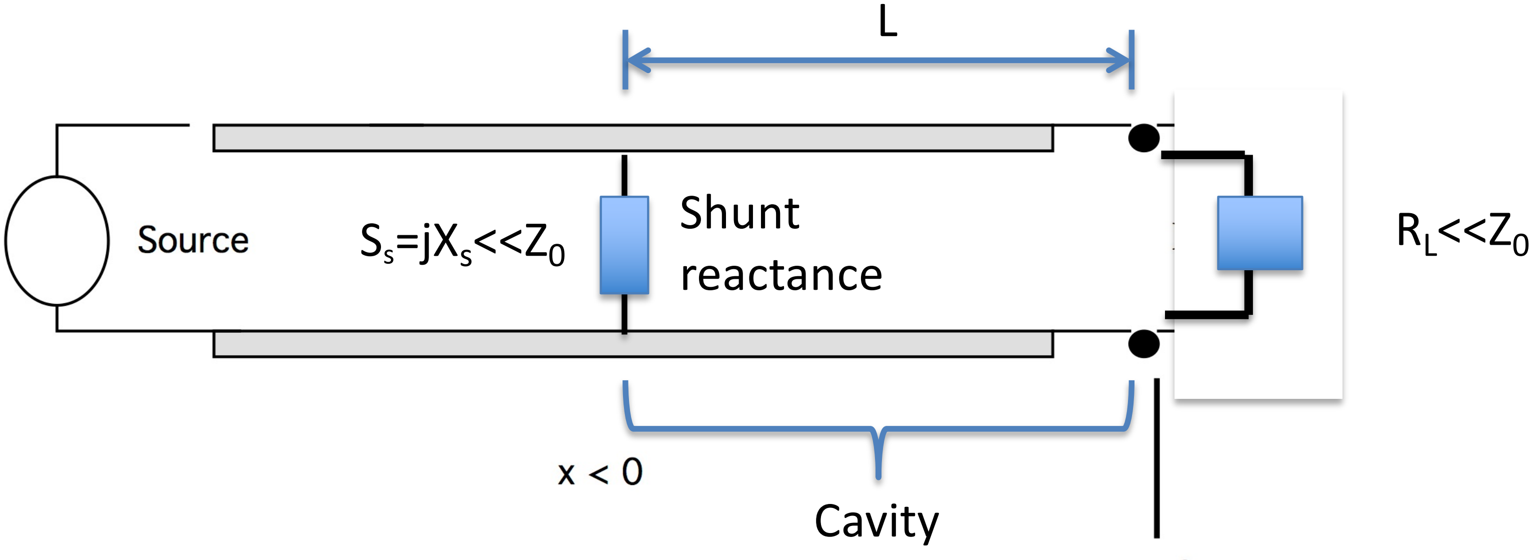
$$N(k+\delta k) = N(k) + 1$$

$$N(k+\delta k) \simeq N(k) + \delta k \frac{dN}{dk} \qquad \frac{dN}{dk} = \frac{k^2 V}{\pi^2}$$

fractional spacing $$\frac{\delta k}{k} = \frac{1}{k\, dN/dk} = \frac{\pi^2}{k^3 V} \cong 1.07 \times 10^{-3}$$

High Q cavity model

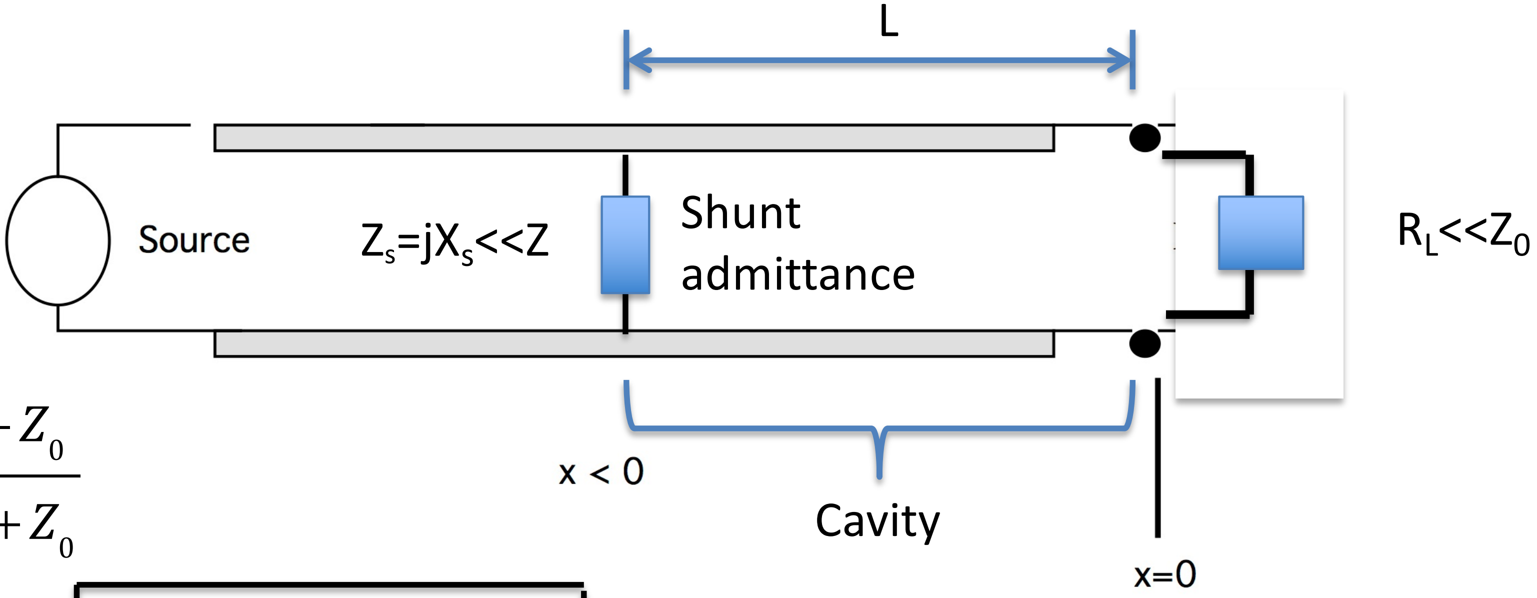
$$e^{-i\omega t} \to e^{j\omega t}$$



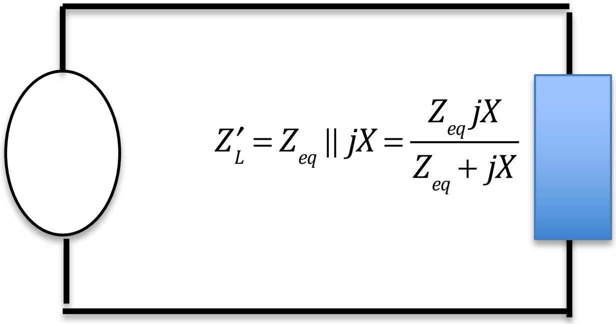
$$Z'_L = Z_{eq} \,||\, jX = \frac{Z_{eq}\, jX}{Z_{eq} + jX}$$

$$\rho == \frac{Z'_{eq} - Z_0}{Z'_{eq} + Z_0}$$

High Q Cavity Model



$$\rho_{cav} = \frac{\left(Z_{eq} \,||\, jX_s\right) - Z_0}{\left(Z_{eq} \,||\, jX_s\right)_s + Z_0}$$



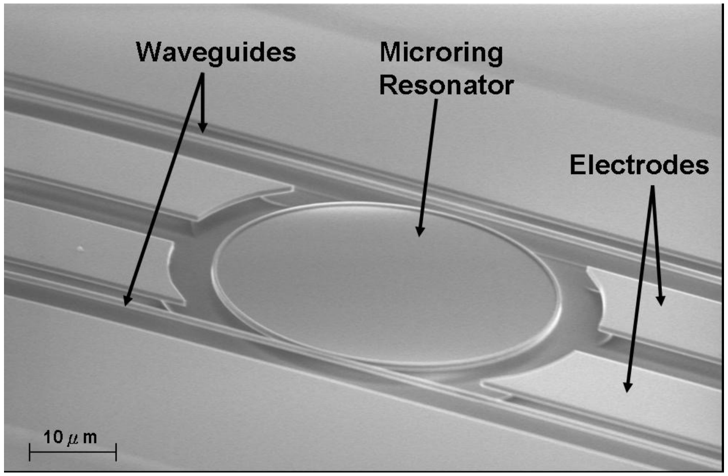
$$Z_L' = Z_{eq} \,||\, jX = \frac{Z_{eq}\, jX}{Z_{eq} + jX}$$

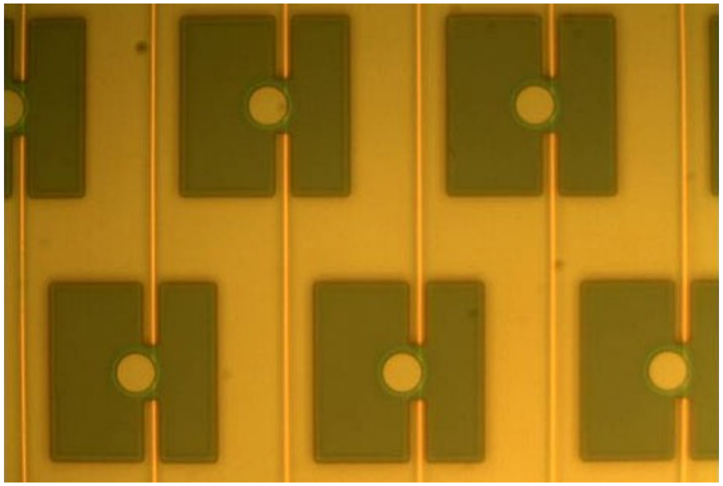
$$Z_{eq} = Z_0 \frac{R_L \cos(kL) + jZ_0 \sin(kL)}{jR_L \sin(kL) + Z_0 \cos(kL)} \simeq jZ_0 \tan(kL) + R_L$$

$$\rho_{cav} \simeq -\frac{j\left(Z_0 \tan(kL) + X_s\right) + R_L - X_s^2 / Z_0}{j\left(Z_0 \tan(kL) + X_s\right) + R_L + X_s^2 / Z_0} = -\frac{j2\left(\omega - \omega_c\right)/\omega_c + Q_{\text{int}}^{-1} - Q_{ext}^{-1}}{j2\left(\omega - \omega_c\right)/\omega_c + Q_{\text{int}}^{-1} + Q_{ext}^{-1}}$$

Integrated photonics

(Courtesy Edo Waks)





nature photonics

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www.nature.com/naturephotonics

PHOTOVOLTAICS
Perovskite shines

TERAHERTZ OPTICS
High-harmonic solution

QUANTUM OPTICS
Harnessing algorithmic cooling

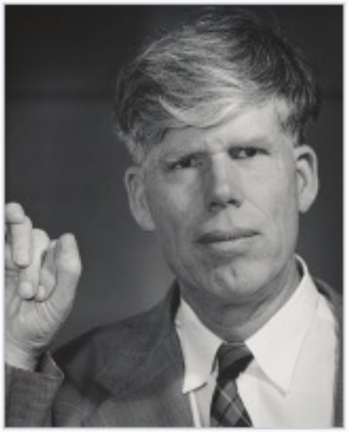
Quantum interference on a chip

Klystron – Beam Driven HPM source

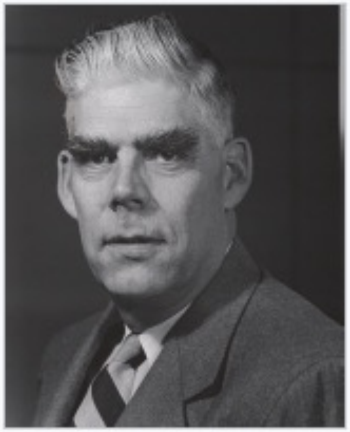
Klystron: invented in 1937 by the Varian brothers. One of the first Palo Alto High Tech. firms.

High Power Source of Microwaves

Radar, Particle Accelerators, (LHC 16 x 300 kW), etc

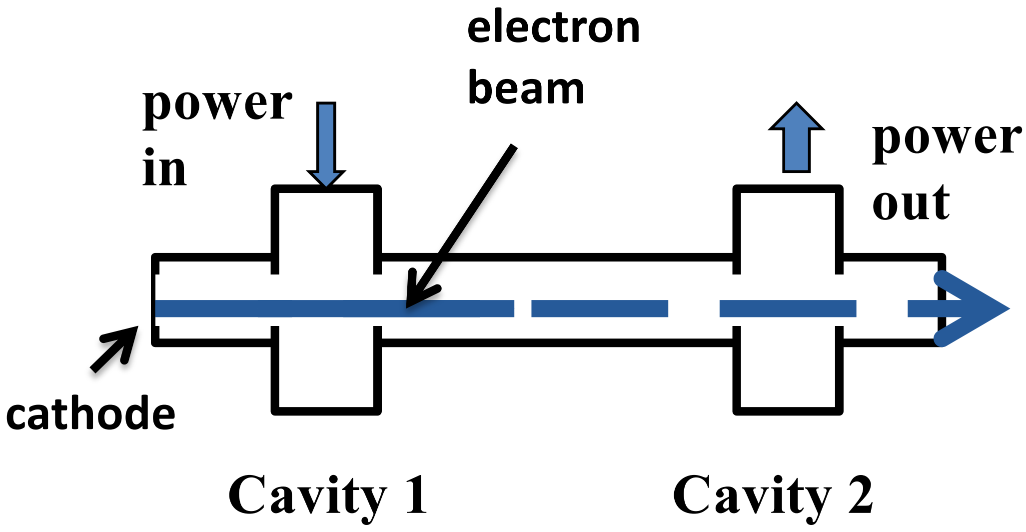


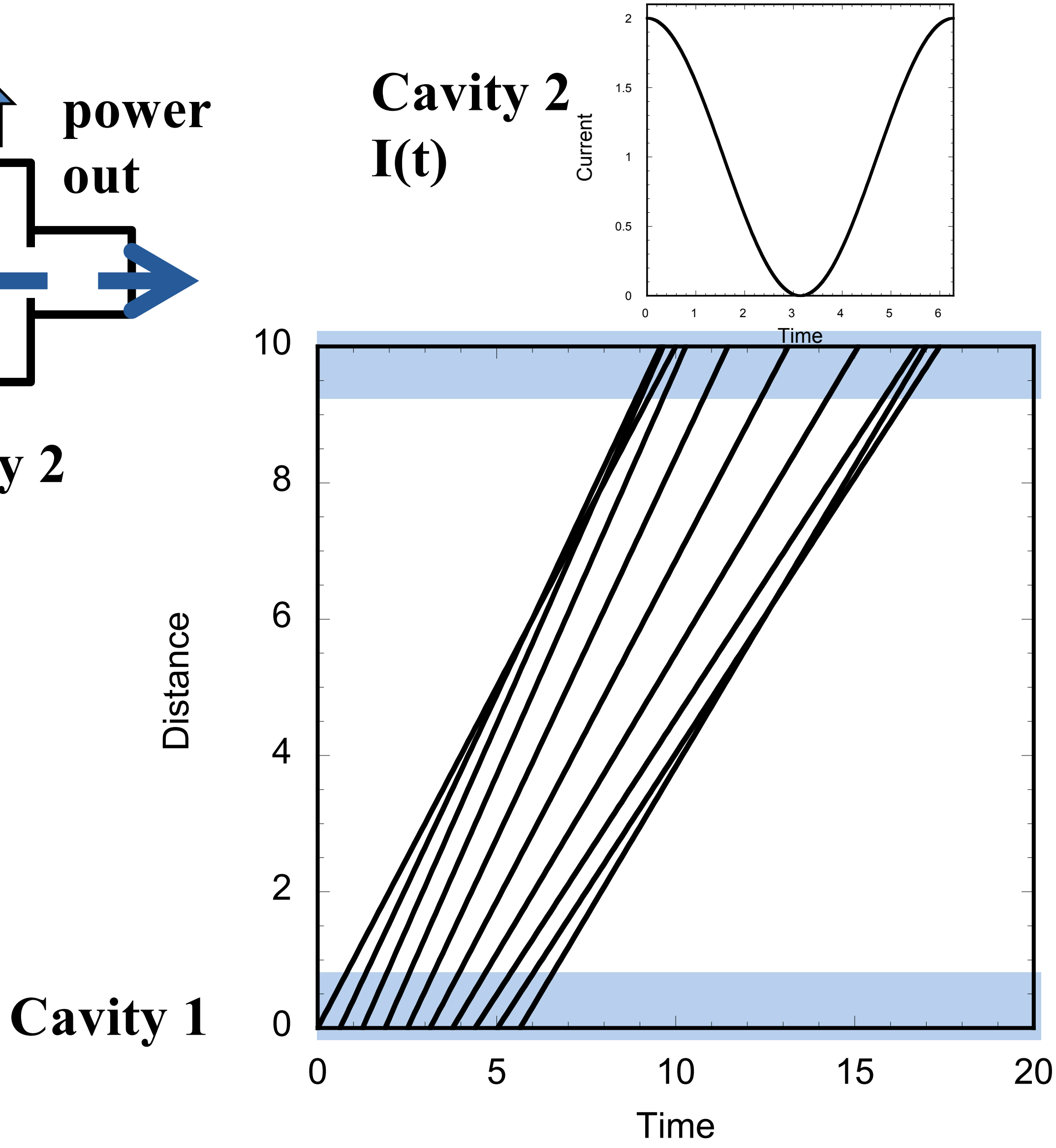
Russell Varian (1898–1959). Photograph by Ansel Adams.



Sigurd Varian (1901–1961) Photograph by Ansel Adams.

Velocity Modulation   Ballistic Bunching





Field in cavity 1 gives small time dependent velocity modulation

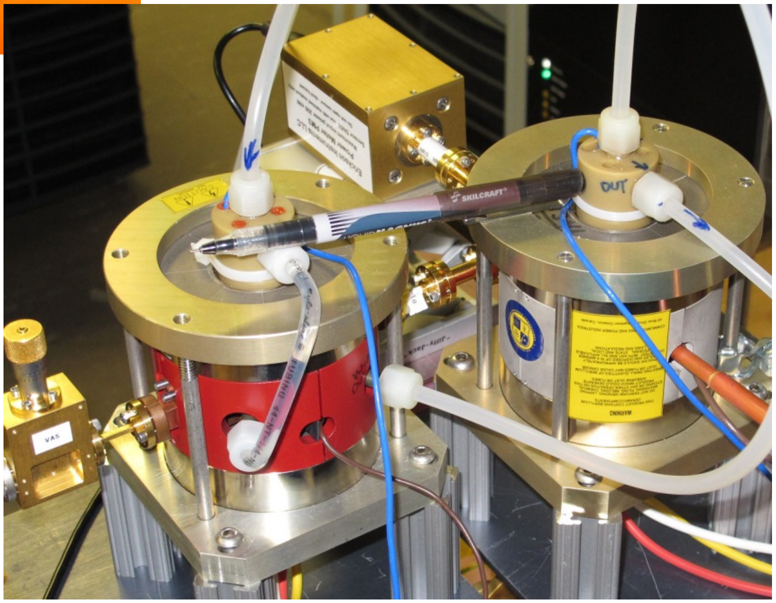
Fast electrons catch up to slow electrons giving large current modulation.

Examples

Monica Blank

170 GHz CPI Gyrotron
IEEE IVEC
http://ieeexplore.ieee.org

L3 Ka Band Power Module http://www.linkmicrotek.com



Experimental high power set-up showing the CPI 218.4 GHz EIK driving the compact NRL Serpentine Waveguide (SWG) TWT.