Dielectric Waveguides

ENEE 681

Basic principles
Slab model
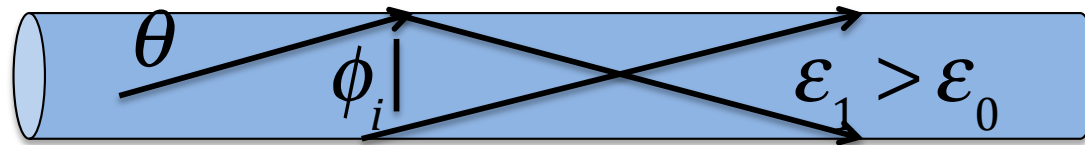   two regions
    boundary conditions
    dispersion relation – odd and even modes
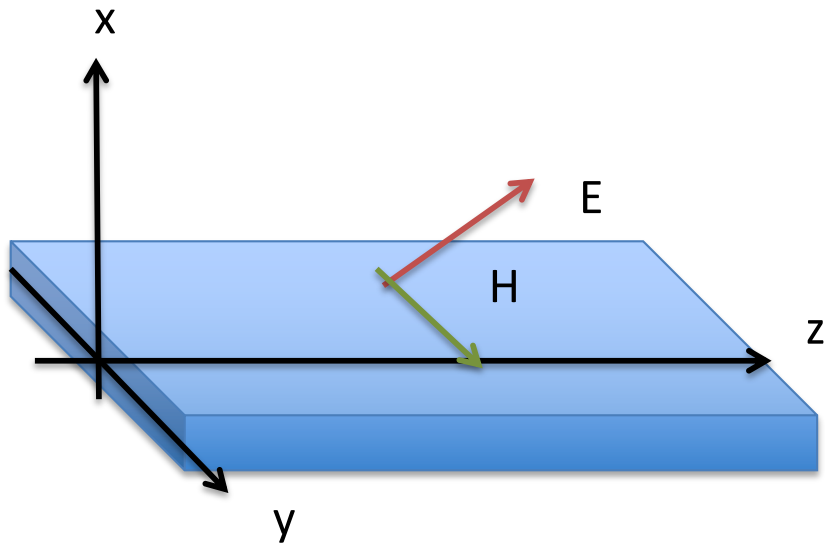Cylindrical geometry

Guiding by total internal reflection

Optical fiber

θ ϕ_i $\varepsilon_1 > \varepsilon_0$

Total reflection if

$$\sin\phi_i = \cos\theta > \sqrt{\varepsilon_0 / \varepsilon_1}$$

Slab Model
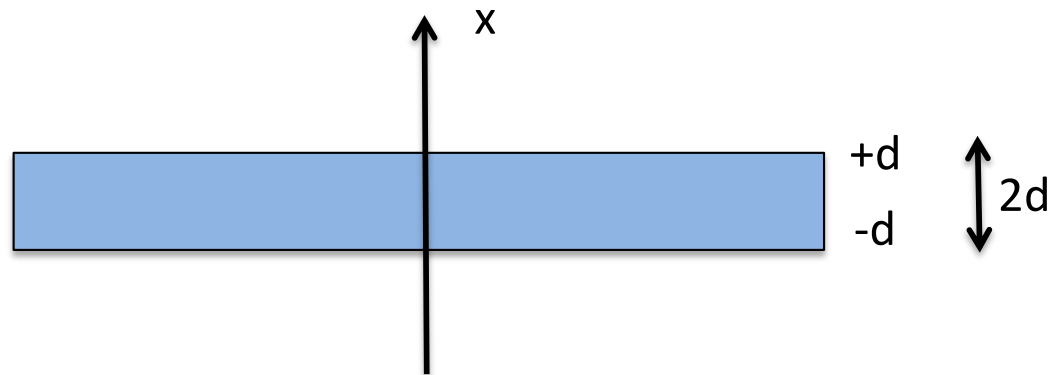

$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d \\ \varepsilon_0 & |x| > d \end{cases}$$

Look for solutions

$$\mathbf{E} = \text{Re}\left\{ (\hat{E}_x, 0, \hat{E}_z) \exp\left[i\left(k_z z - \omega t \right) \right] \right\}$$

$$\mathbf{H} = \text{Re}\left\{ (0, \hat{H}_y, 0) \exp\left[i\left(k_z z - \omega t \right) \right] \right\}$$

Maxwell's Equations

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t}$$

$$\frac{\partial \hat{E}_z}{\partial y} - ik_z \hat{E}_y = i\omega\mu \hat{H}_x$$

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu \hat{H}_y$$

$$\frac{\partial \hat{E}_y}{\partial x} - \frac{\partial \hat{E}_x}{\partial y} = i\omega\mu \hat{H}_z$$

$$\nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\frac{\partial \hat{H}_z}{\partial y} - ik_z \hat{H}_y = -i\omega\varepsilon \hat{E}_x$$

$$ik_z \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} = -i\omega\varepsilon \hat{E}_y$$

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Look for solutions

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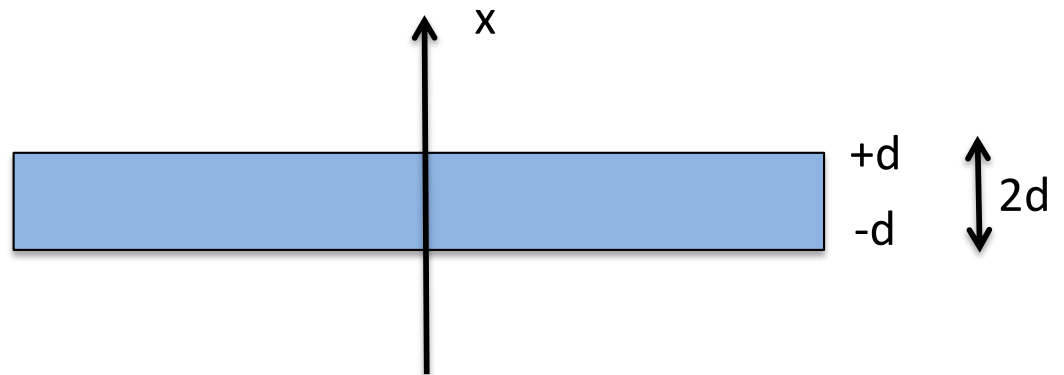
Maxwell's Equations

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y$$

$$-ik_z\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

What are the boundary conditions at x = +/- d?

$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d \\ \varepsilon_0 & |x| > d \end{cases}$$

x

+d
-d
2d

Maxwell's Equations

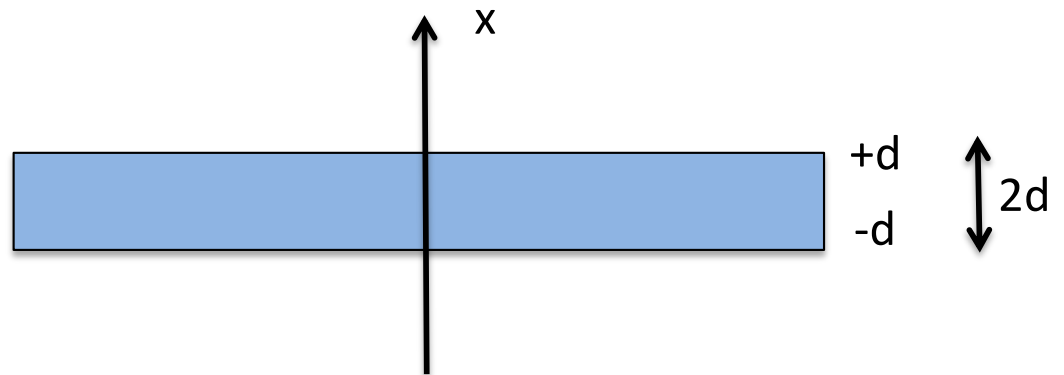$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y$$

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$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

What are the boundary conditions at x = +/- d?

$$\varepsilon = \begin{cases} \varepsilon_1 & |x| < d \\ \varepsilon_0 & |x| > d \end{cases}$$

$$\left.\begin{array}{c} \hat{E}_z \\ \\ \hat{H}_y \end{array}\right\} \text{ continuous at } x = \pm d$$

x

+d
-d
2d

Combine to a single equation

$$ik_z \hat{E}_x - \frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\hat{H}_y$$

$$-ik_z\hat{H}_y = -i\omega\varepsilon\hat{E}_x$$

$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

$$k_z^2\hat{E}_x + ik_z\frac{\partial \hat{E}_z}{\partial x} = i\omega\mu\left(-ik_z\hat{H}_y\right) = \omega^2\varepsilon\mu\hat{E}_x$$

$$\hat{E}_x = \frac{ik_z}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

$$\hat{H}_y = \frac{i\omega\varepsilon}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x}$$

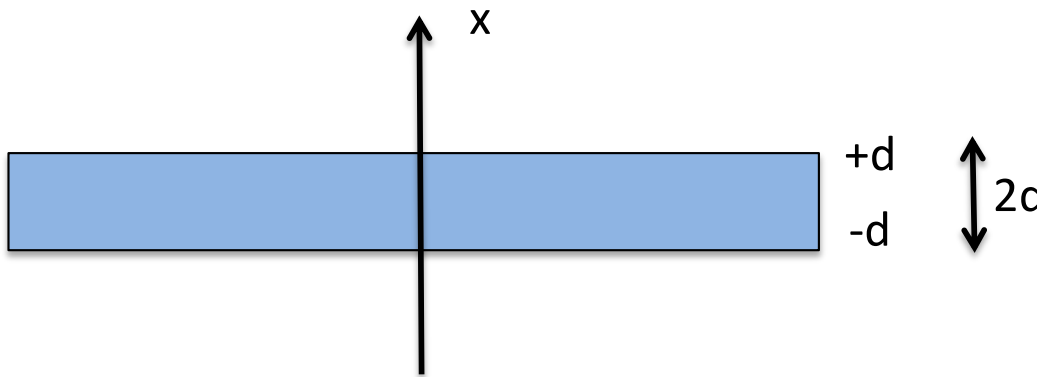$$\frac{\partial \hat{H}_y}{\partial x} = -i\omega\varepsilon\hat{E}_z$$

$$\frac{\partial}{\partial x}\frac{\varepsilon}{\omega^2\varepsilon\mu - k_z^2}\frac{\partial \hat{E}_z}{\partial x} = -\varepsilon\hat{E}_z$$

If ε is constant $\qquad \dfrac{\partial}{\partial x}\dfrac{\partial \hat{E}_z}{\partial x} + \left(\omega^2\varepsilon\mu - k_z^2\right)\hat{E}_z = 0$

Solve in two regions

$$\frac{\partial}{\partial x}\frac{\partial \hat{E}_z}{\partial x}+\left(\omega^2\varepsilon\mu - k_z^2\right)\hat{E}_z = 0$$

If ε is constant

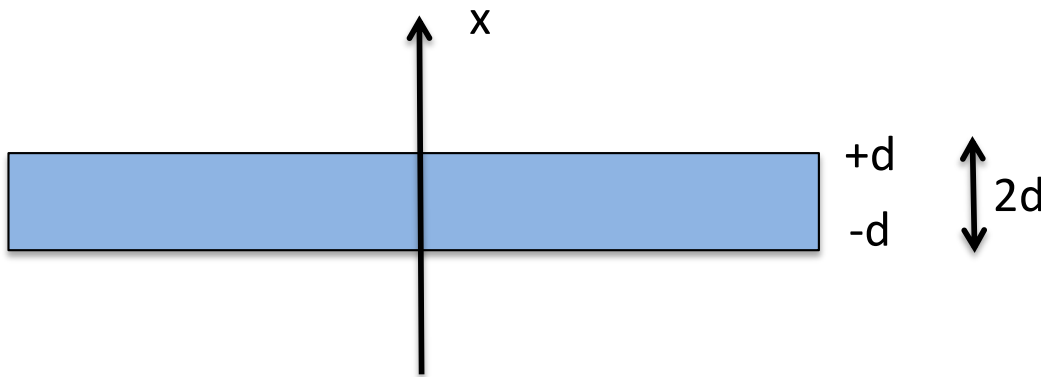$$\varepsilon = \begin{cases} \varepsilon_1 & |x|<d, \quad \omega^2\varepsilon_1\mu - k_z^2 > 0 \\ \varepsilon_0 & |x|>d, \quad \omega^2\varepsilon_0\mu - k_z^2 < 0 \end{cases}$$

x

+d

2d

-d

For x>d, $\quad \varepsilon = \varepsilon_0 \qquad \hat{E}_z = A\exp\left(-\kappa x\right) \quad \kappa = \sqrt{k_z^2 - \omega^2\varepsilon_0\mu}$

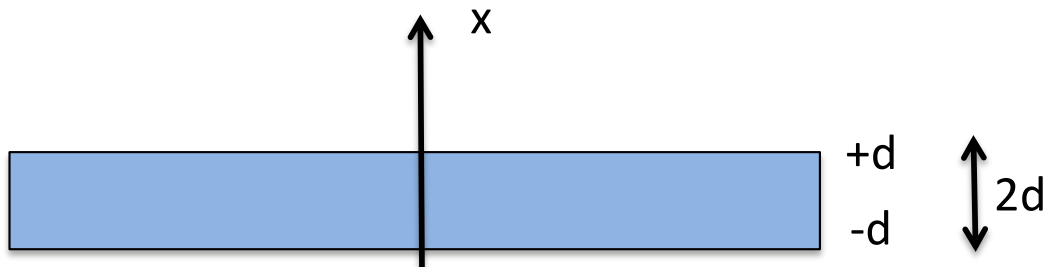Evanescent (Spatially decaying) field

Solve in two regions

$$\frac{\partial}{\partial x} \frac{\partial \hat{E}_z}{\partial x} + \left(\omega^2 \varepsilon \mu - k_z^2\right)\hat{E}_z = 0$$

If ε is constant

x

+d

2d

-d

For $|x| < d,$ $\varepsilon = \varepsilon_1$ $\hat{E}_z = B\sin\left(k_x x\right) + C\cos\left(k_x x\right)$ $k_x = \sqrt{\omega^2 \varepsilon_1 \mu - k_z^2}$

Spatially oscillating field
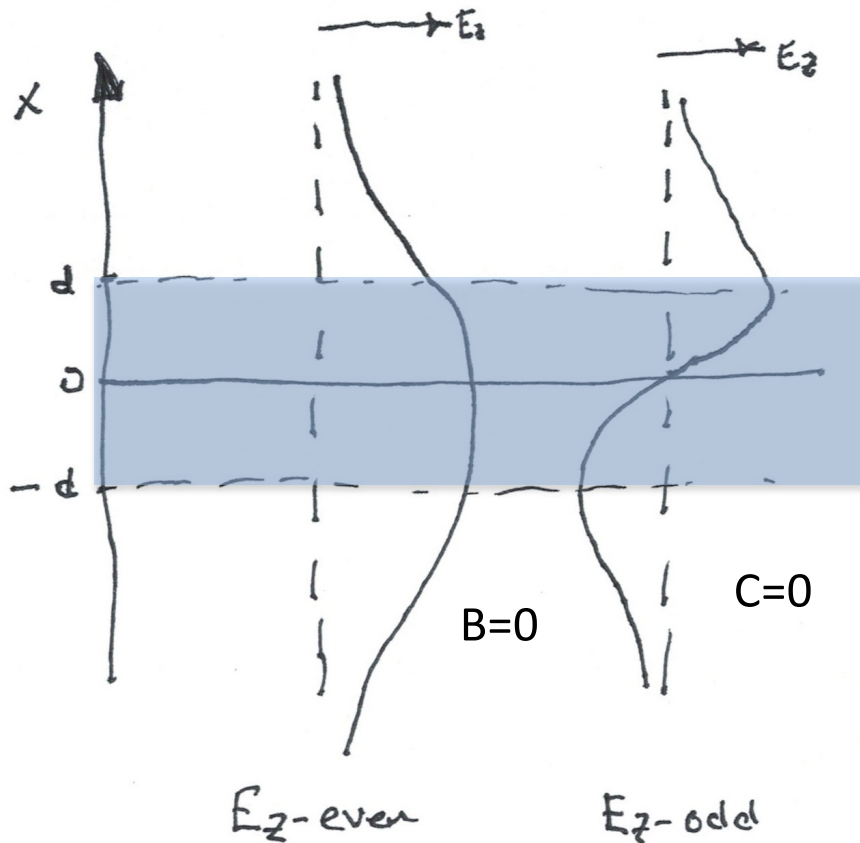
Odd and Even solutions

For $|x| < d$, $\quad \varepsilon = \varepsilon_1$

$$\hat{E}_z = B\sin(k_x x) + C\cos(k_x x)$$

$$k_x = \sqrt{\omega^2 \varepsilon_1 \mu - k_z^2}$$

Either: $\qquad\qquad$ B=0, $\;$ $C \neq 0$

$\qquad$ or $\qquad$ $B \neq 0$, $\;$ C = 0

$$\hat{E}_x = \frac{ik_z}{\omega^2 \varepsilon \mu - k_z^2} \frac{\partial \hat{E}_z}{\partial x}$$

$$\hat{H}_y = \frac{i\omega\varepsilon}{\omega^2 \varepsilon \mu - k_z^2} \frac{\partial \hat{E}_z}{\partial x}$$

E_z –even (E_x – odd) solution

For $|x| < d$, $\hat{E}_z = C\cos(k_x x)$ $k_x = \sqrt{\omega^2 \varepsilon_1 \mu - k_z^2}$ $\hat{E}_x = \dfrac{ik_z}{\omega^2 \varepsilon \mu - k_z^2}\dfrac{\partial \hat{E}_z}{\partial x}$

For $x > d$, $\hat{E}_z = A\exp(-\kappa x)$ $\kappa = \sqrt{k_z^2 - \omega^2 \varepsilon_0 \mu}$ $\hat{H}_y = \dfrac{i\omega\varepsilon}{\omega^2 \varepsilon \mu - k_z^2}\dfrac{\partial \hat{E}_z}{\partial x}$

At x=d $\hat{E}_z = C\cos(k_x d) = A\exp(-\kappa d)$

$$\hat{H}_y = \frac{i\omega\varepsilon_1}{k_x^2}\frac{\partial}{\partial x}C\cos(k_x x)\bigg|_{x=d} = -\frac{i\omega\varepsilon_1}{k_x}C\sin(k_x d)$$

$$\hat{H}_y = \frac{i\omega\varepsilon_0}{-\kappa^2}\frac{\partial}{\partial x}A\exp(-\kappa x)\bigg|_{x=d} = \frac{i\omega\varepsilon_0}{\kappa}A\exp(-\kappa d)$$

Dispersion Relation

$$C\cos(k_x d) = A\exp(-\kappa d) \qquad -\frac{\varepsilon_1}{k_x}C\sin(k_x d) = \frac{\varepsilon_0}{\kappa}A\exp(-\kappa d)$$

Even E$_z$

$$-\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d\cot(k_x d)$$

Odd E$_z$

$$\frac{\varepsilon_1}{\varepsilon_0}\kappa d = k_x d\tan(k_x d)$$

$$\kappa d = d\sqrt{k_z^2 - \omega^2\varepsilon_0\mu} = \sqrt{\omega^2(\varepsilon_1 - \varepsilon_0)\mu d^2 - k_x^2 d^2}$$

$$\kappa d = \sqrt{\omega^2(\varepsilon_1 - \varepsilon_0)\mu d^2 - k_x^2 d^2}$$
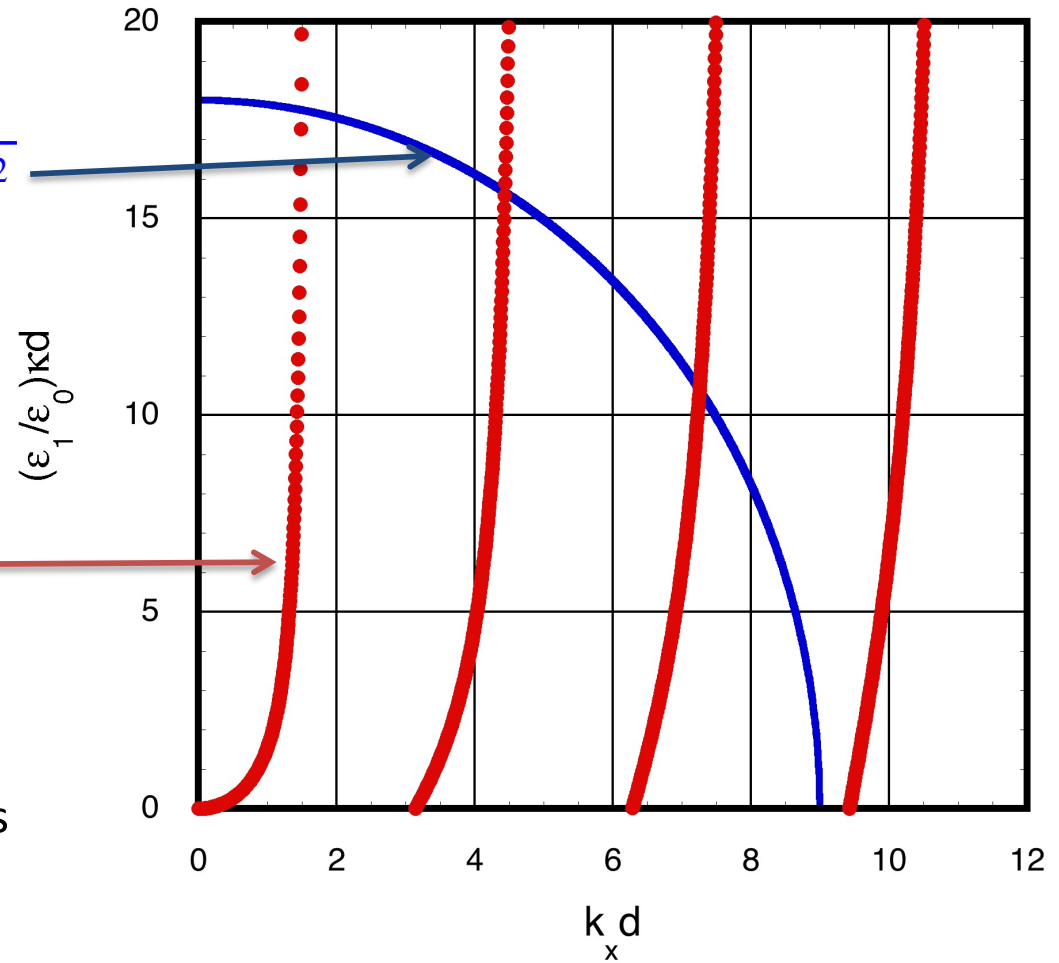
Plot both sides versus $k_x d$

Odd E_z

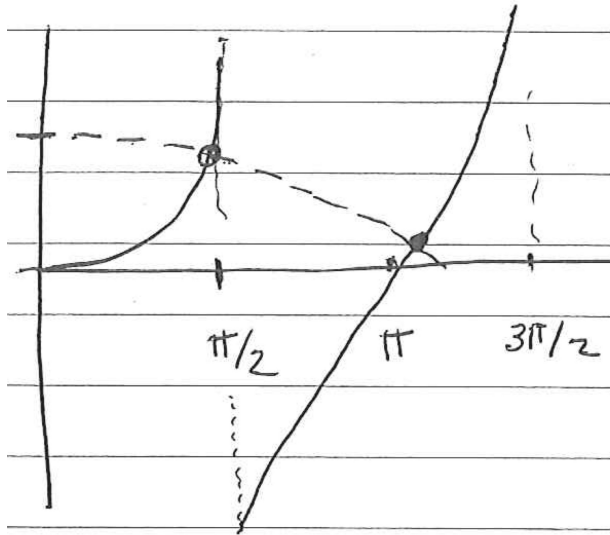$$\frac{\varepsilon_1}{\varepsilon_0} \kappa d = k_x d \tan(k_x d)$$

$$\frac{\varepsilon_1}{\varepsilon_0} \kappa d = \frac{\varepsilon_1}{\varepsilon_0} \sqrt{\omega^2 (\varepsilon_1 - \varepsilon_0) \mu d^2 - k_x^2 d^2}$$

$$\frac{\varepsilon_1}{\varepsilon_0} \kappa d = k_x d \tan(k_x d)$$

In this case three solutions – modes
There will always be at least one.

Comments
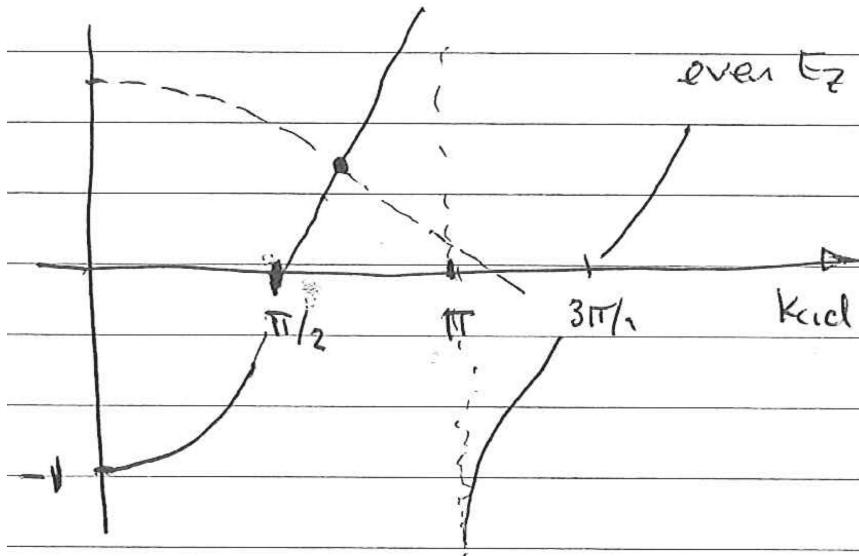

Odd Ez –Even Ex

Always at least one solution
For small diameter d fields outside dielectric
Requires cladding

Even Ez –Odd Ex

Possibly no solutions

Cylindrical Dielectrics

Region II

ϵ_0

Region I

ϵ_i

$\uparrow a \to z$
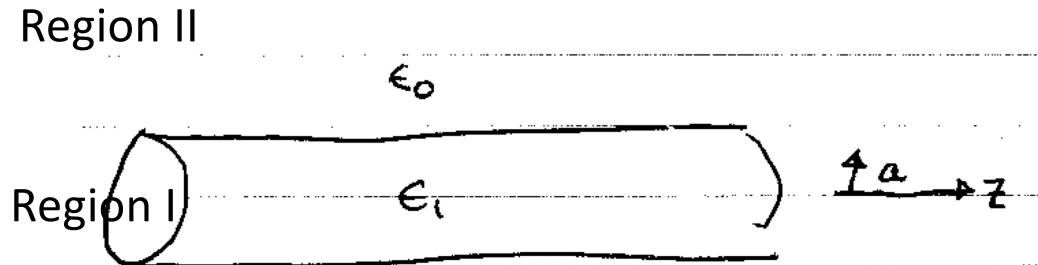
$$\text{Field} \sim \{\hat{E}, \hat{H}(r,\theta)\} \exp[ikz - i\omega t]$$

Express transverse components in terms of axial components

$$\hat{E}_\perp = \frac{1}{k_\perp^2}\left[ik_z \nabla_\perp \hat{E}_z - i\omega\mu\, \underline{e}_z \times \nabla_\perp \hat{H}_z\right\}$$

$$\hat{H}_\perp = \frac{1}{k_\perp^2}\left[ik_z \nabla_\perp \hat{H}_z + i\omega\epsilon\, \underline{e}_z \times \nabla_\perp \hat{E}_z\right]$$

$k_\perp^2 = \omega^2\epsilon\mu - k_z^2$ — different in each region

Equation for Axial Components

$$\nabla_\perp^2 \hat{E}_z + k_\perp^2 \hat{E}_z = 0 \qquad \nabla_\perp^2 \hat{H}_z + k_\perp^2 \hat{H}_z = 0$$

$$k_\perp^2 = \omega^2 \epsilon \mu - k_z^2 \quad - \quad \text{different in each region}$$

Region I: $\quad k_{\perp 1}^2 = \omega^2 \epsilon_1 \mu - k_z^2 \quad > 0$

Region II $\quad k_{\perp 2}^2 = \omega^2 \epsilon_0 \mu - k_z^2 = -\gamma_2^2 < 0$

$$\hat{E}_z = A_1 \exp(i\ell\theta) J_\ell(k_{\perp 1} r)$$

$\qquad\qquad\qquad\qquad\qquad \ell = \text{integer}$

$$\hat{E}_z(r,\theta) = A_2 e^{i\ell\theta} K_\ell(\gamma_2 r)$$

$$\hat{H}_z = B_1 \exp(i\ell\theta) J_\ell(k_{\perp 1} r)$$

$$\hat{H}_z(r,\theta) = B_2 e^{i\ell\theta} K_\ell(\gamma_2 r)$$

Ordinary Bessel functions
Regular at origin

Modified Bessel functions
Goes to zero at infinity

Unknowns and Boundary Conditions

$$\hat{E}_z = A_1 \exp(i\ell\theta) J_\ell(k_{11}r)$$

$$\hat{E}_z(r,\theta) = A_2 e^{i\ell\theta} K_\ell(\gamma_2 r)$$

$$\hat{H}_z = B_1 \exp(i\ell\theta) J_\ell(k_{11}r)$$

$$\hat{H}_z(r,\theta) = B_2 e^{i\ell\theta} K_\ell(\gamma_2 r)$$

Unknowns $\quad A_1, A_2, B_1, B_2 \quad$ & ω for given k_z

Boundary Conditions: $\quad$ at $r=a$

Tangential components of E and H continuous

$$E_z, E_\theta, H_z, H_\theta - \text{continuous at } r=a$$

$$H_z, E_z - \text{continuous}$$

$$A_1 J_\ell(k_{11}a) = A_2 K_\ell(\gamma_2 a)$$

$$B_1 J_\ell(k_{11}a) = B_2 K_\ell(\gamma_2 a)$$

Continuity of Transverse Components

Transverse components in terms
of axial components

$$\hat{E}_{\perp} = \frac{1}{k_{\perp}^2} \left[ik_z \nabla_{\perp} \hat{E}_z - i\omega\mu \, \underline{e}_z \times \nabla_{\perp} \hat{H}_z \right\}$$

$$\hat{H}_{\perp} = \frac{1}{k_{\perp}^2} \left[ik_z \nabla_{\perp} \hat{H}_z + i\omega\epsilon \, \underline{e}_z \times \nabla_{\perp} \hat{E}_z \right]$$

Region I

$$\hat{E}_{\theta}(a) = \frac{1}{k_{\perp 1}^2} \left\{ ik_z \left(i\frac{\ell}{a}\right) A_1 J_{\ell}(k_{1}a) - i\omega\mu \, B_1 k_{1} J_{\ell}'(k_{1}a) \right\}$$

$$H_{\theta}(r=a) = \frac{1}{k_{\perp 1}^2} \left\{ ik_z \left(\frac{i\ell}{a}\right) B_1 J_{\ell}(k_{1}a) + i\omega\epsilon \, k_{1} A_1 J_{\ell}'(k_{1}a) \right\}$$

Region II

$$\hat{E}_{\theta}(a) = -\frac{1}{\gamma_2^2} \left\{ ik_z \left(\frac{i\ell}{a}\right) A_2 K_{\ell}(\gamma_2 a) - i\omega\mu \, \gamma_2 B_2 K_{\ell}'(\gamma_2 a) \right\}$$

$$k_z^2 = -\gamma_z^2$$

$$H_{\theta}(a) = -\frac{1}{\gamma_2^2} \left\{ ik_z \left(\frac{i\ell}{a}\right) B_2 K_{\ell}(\gamma_2 a) + i\omega\epsilon \, \gamma_2 A_2 K_{\ell}'(\gamma_2 a) \right\}$$

Solution

$$A_1 J_\ell (k_{11} a) = A_2 K_\ell (\gamma_2 a)$$

$$B_1 J_\ell (k_{11} a) = B_2 K_\ell (\gamma_2 a)$$

$$\hat{E}_\theta (a) = \frac{1}{k_{11}^2} \left\{ ik_z \left(i\frac{\ell}{a} \right) A_1 J_\ell (k_{11} a) - i\omega\mu B_1 k_{11} J_\ell' (k_{11} a) \right\}$$

$$H_\theta (r=a) = \frac{1}{k_{11}^2} \left\{ ik_z \left(\frac{i\ell}{a} \right) B_1 J_\ell (k_{11} a) + i\omega\epsilon k_{11} A_1 J_\ell' (k_{11} a) \right\}$$

$$\hat{E}_\theta (a) = -\frac{1}{\gamma_2^2} \left\{ ik_z \left(\frac{i\ell}{a} \right) A_2 K_\ell (\gamma_2 a) - i\omega\mu \gamma_2 B_2 K_\ell' (\gamma_2 a) \right\}$$

$$H_\theta (a) = -\frac{1}{\gamma_2^2} \left\{ ik_z \left(\frac{i\ell}{a} \right) B_2 K_\ell (\gamma_2 a) + i\omega\epsilon \gamma_2 A_2 K_\ell' (\gamma_2 a) \right\}$$

Replace $A_2 = A_1 J_\ell / K_\ell$ $\qquad B_2 = B_1 J_\ell / K_\ell$

Solution $\qquad \det (\underline{\underline{M}}) = 0 \qquad$ determines $\omega(k)$

$$\begin{bmatrix} & & \\ & 2 \times 2 & \\ & \underline{\underline{M}} & \end{bmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = 0$$