ENEE681

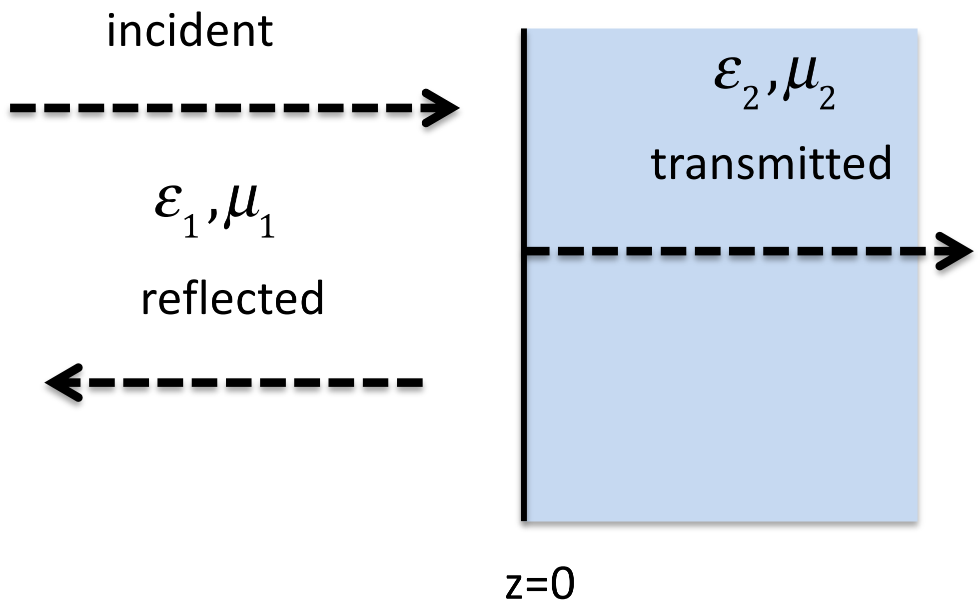
Lecture 9

Reflections at Boundaries

Review: Normal Incidence Linear Polarization

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$k_1 = \omega\sqrt{\varepsilon_1\mu_1}$$



$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$k_2 = \omega\sqrt{\varepsilon_2\mu_2}$$

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc}e^{ik_1z} + \hat{E}_{ref}e^{-ik_1z}\right)e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_1}\left(\hat{E}_{inc}e^{ik_1z} - \hat{E}_{ref}e^{-ik_1z}\right)e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{trans}e^{ik_2z}\right)e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_2}\left(\hat{E}_{trans}e^{ik_2z}\right)e^{-i\omega t}\right\}$$

At z=0 tangential E and tangential H are continuous

At z=0 tangential E and tangential H are continuous

$$\hat{E}_{inc} + \hat{E}_{ref} = \hat{E}_{trans}$$

$$\frac{\hat{E}_{inc} - \hat{E}_{ref}}{\eta_1} = \frac{\hat{E}_{trans}}{\eta_2}$$

solve →

$$\frac{\hat{E}_{ref}}{\hat{E}_{inc}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \equiv \rho$$

$$\frac{\hat{E}_{trans}}{\hat{E}_{inc}} = \frac{2\eta_2}{\eta_2 + \eta_1} \equiv \tau = 1 + \rho$$

$$P_{inc} = \frac{\left|\hat{E}_{inc}\right|^2}{2\eta_1}$$

$$P_{ref} = \frac{\left|\hat{E}_{ref}\right|^2}{2\eta_1} = |\rho|^2 P_{inc}$$

$$P_{trans} = \frac{\left|\hat{E}_{trans}\right|^2}{2\eta_1} = \left(1 - |\rho|^2\right) P_{inc}$$

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

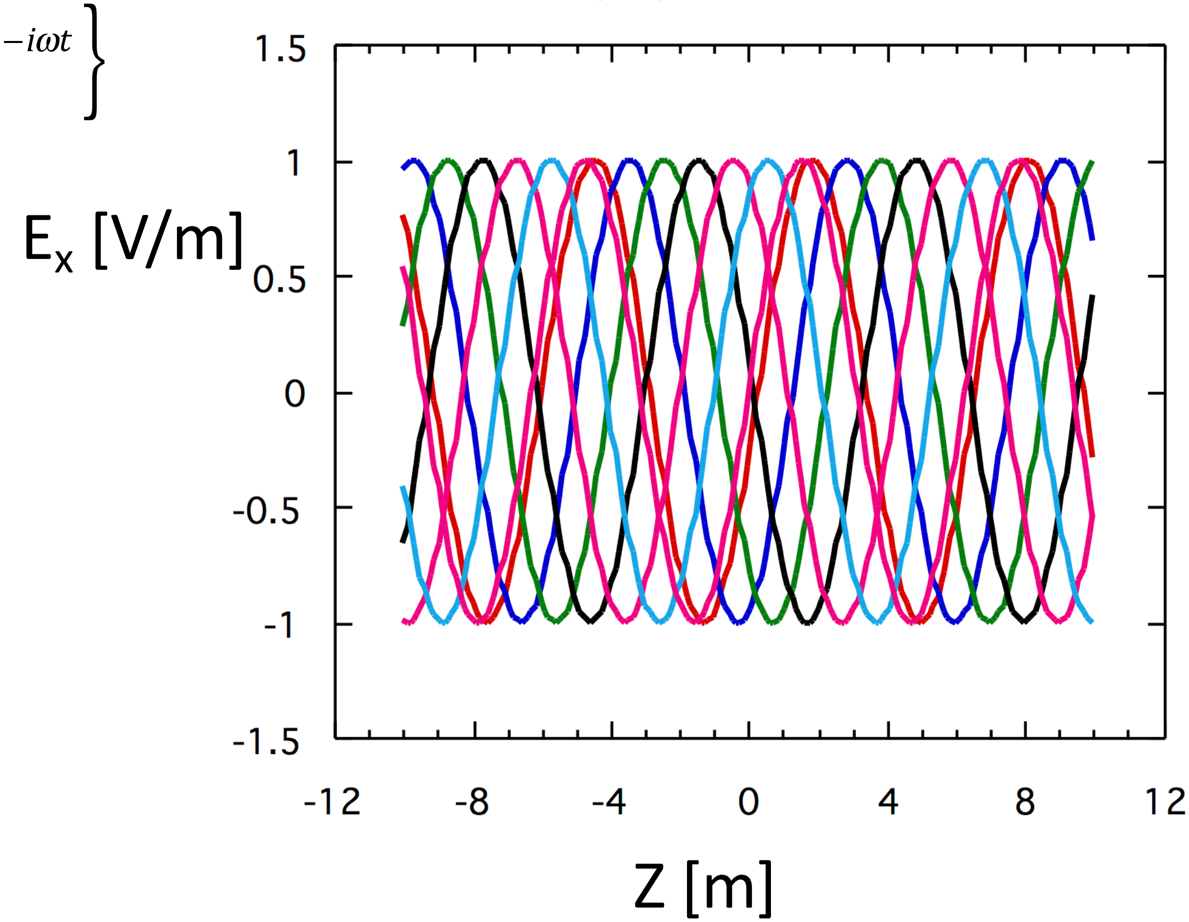
Voltage Standing Wave Ratio (VSWR)

Pronounced "Vizwarr"

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\hat{E}_{inc}\left(e^{ik_1 z} + \rho e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

Plots of Ex(z,t) at different times



Case #1 No reflection

Travelling Wave

Assume vacuum

Find: $k_1, \omega, \left|\hat{E}_x\right|$

Voltage Standing Wave Ratio (VSWR)

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

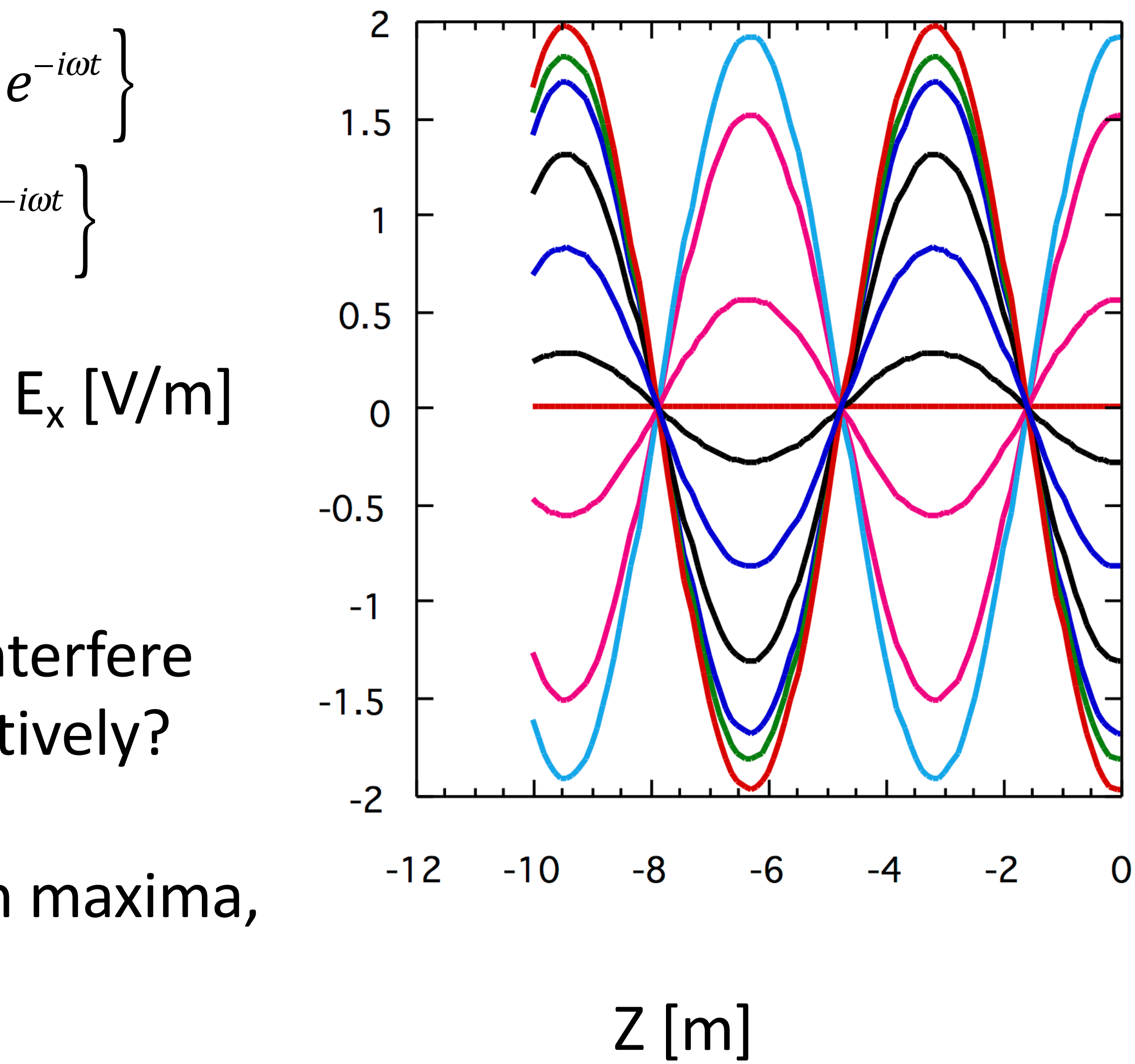
$$E_x = \mathrm{Re}\left\{\hat{E}_{inc} \left(e^{ik_1 z} + \rho e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

Case #2 $\rho = 1$

Where do the waves interfere constructively, destructively?

How far apart between maxima, minima?

Plots of Ex(z,t) at different times



Voltage Standing Wave Ratio (VSWR)

$$E_x = \text{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

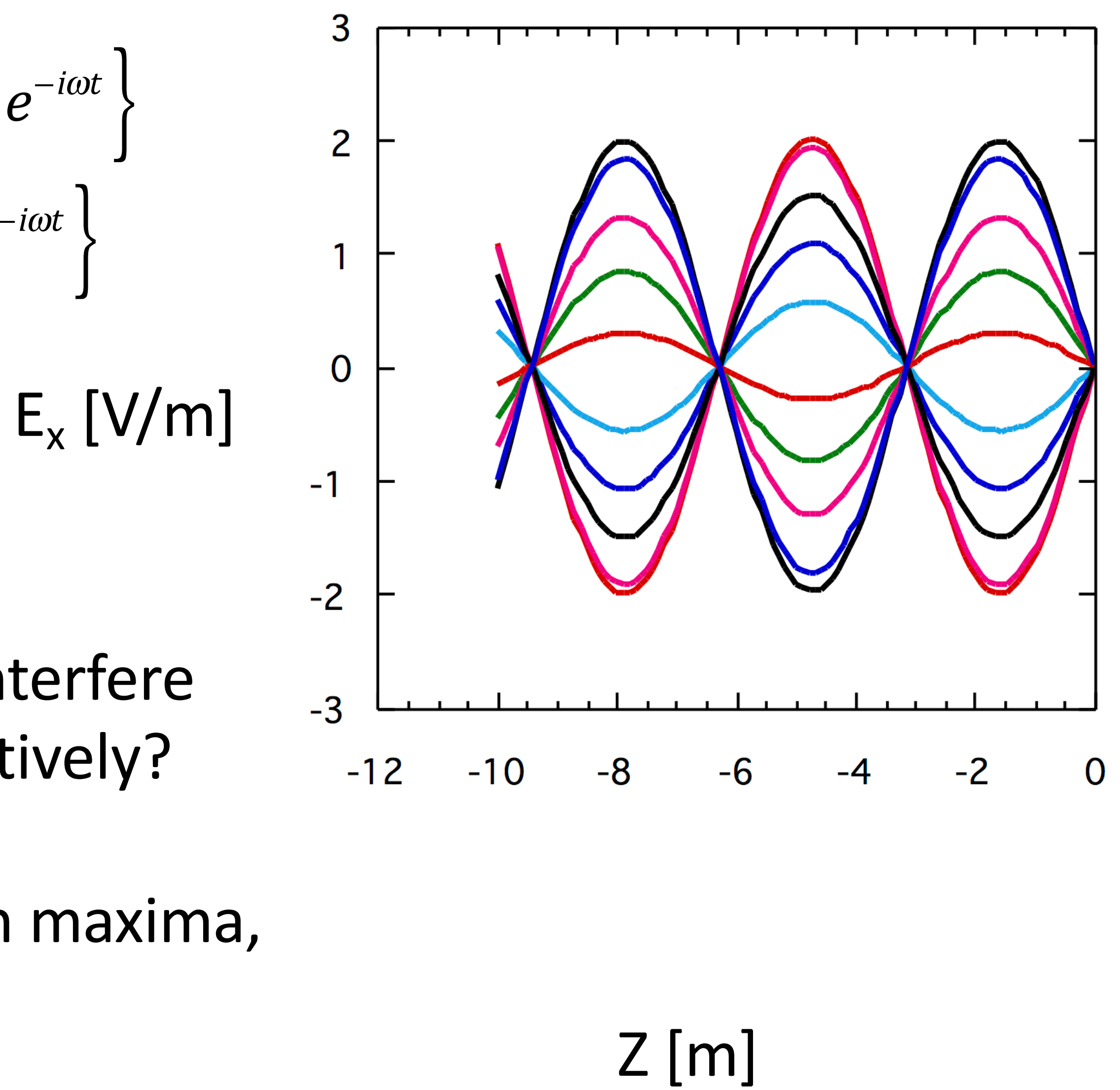
$$E_x = \text{Re}\left\{\hat{E}_{inc}\left(e^{ik_1 z} + \rho e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

Case #3 $\rho = -1$

Where do the waves interfere constructively, destructively?

How far apart between maxima, minima?

Plots of Ex(z,t) at different times



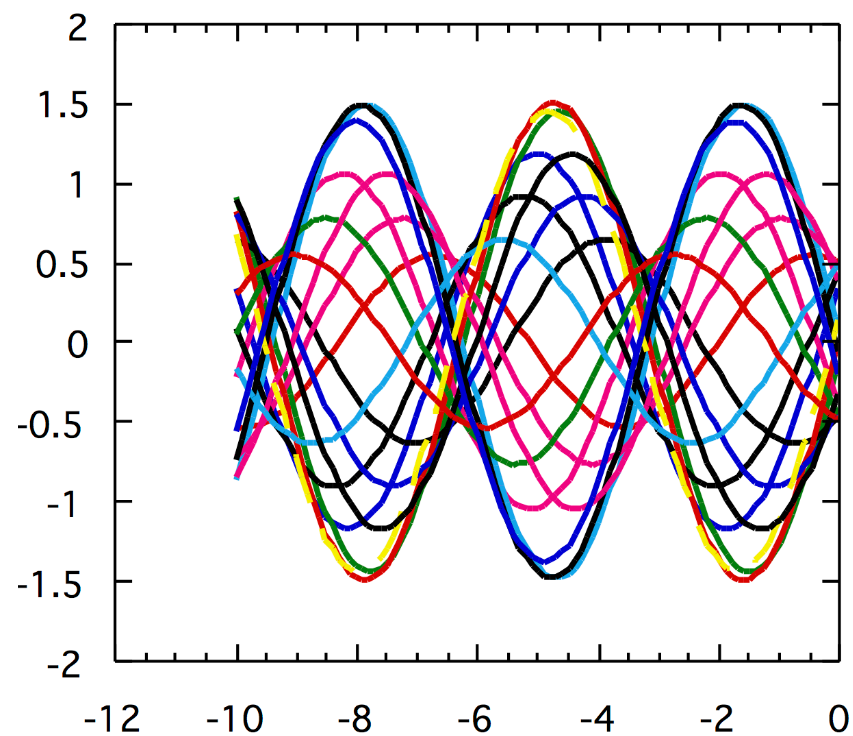
Voltage Standing Wave Ratio (VSWR)

Plots of Ex(z,t) at different times

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc}e^{ik_1 z} + \hat{E}_{ref}e^{-ik_1 z}\right)e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\hat{E}_{inc}\left(e^{ik_1 z} + \rho e^{-ik_1 z}\right)e^{-i\omega t}\right\}$$

E_x [V/m]



Case #3 $\rho = -0.5$

Where do the waves interfere constructively, destructively?

How far apart between maxima, minima?

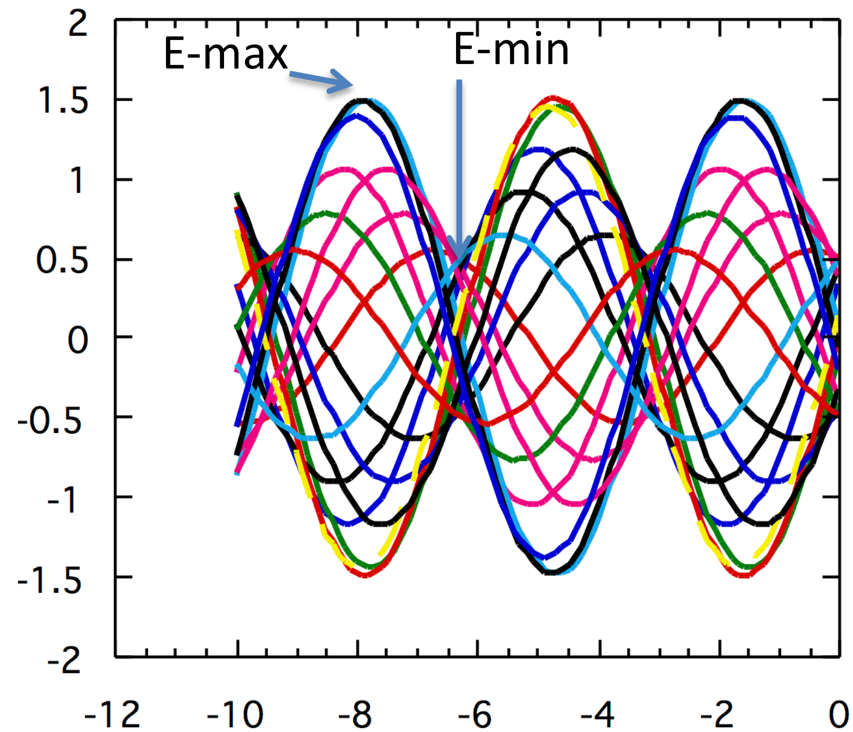
Z [m]

Voltage Standing Wave Ratio (VSWR)

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\hat{E}_{inc}\left(e^{ik_1 z} + \rho e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

Plots of Ex(z,t) at different times



E_x [V/m]

Z [m]

$$\rho = -0.5$$

$$VSWR = \frac{E_{\max}}{E_{\min}} = \frac{1+|\rho|}{1-|\rho|} \geq 1$$

$$VSWR = \frac{1.5}{.5} = 3$$

Problem: where is E_{max} , E_{min}?

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\hat{E}_{inc}\left(e^{ik_1 z} + \rho e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

$$\rho = |\rho| e^{i\theta_\rho}$$

$$E_x = \mathrm{Re}\left\{\hat{E}_{inc} e^{ik_1 z - i\omega t}\left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right)\right\}$$

$$\left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right) = \left|\left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right)\right| e^{i\alpha}$$

Find the set of points where magnitude of Ex is maximum, minimum.

Find VSWR

$$\hat{E}_{inc} e^{ik_1 z - i\omega t}\left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right) = \left|\hat{E}_{inc}\right|\left|\left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right)\right| \exp\left(i\left(\theta_{E_{inc}} + k_1 z - \omega t + \alpha\right)\right)$$

Peak Electric field

VSWR and maxima, minima

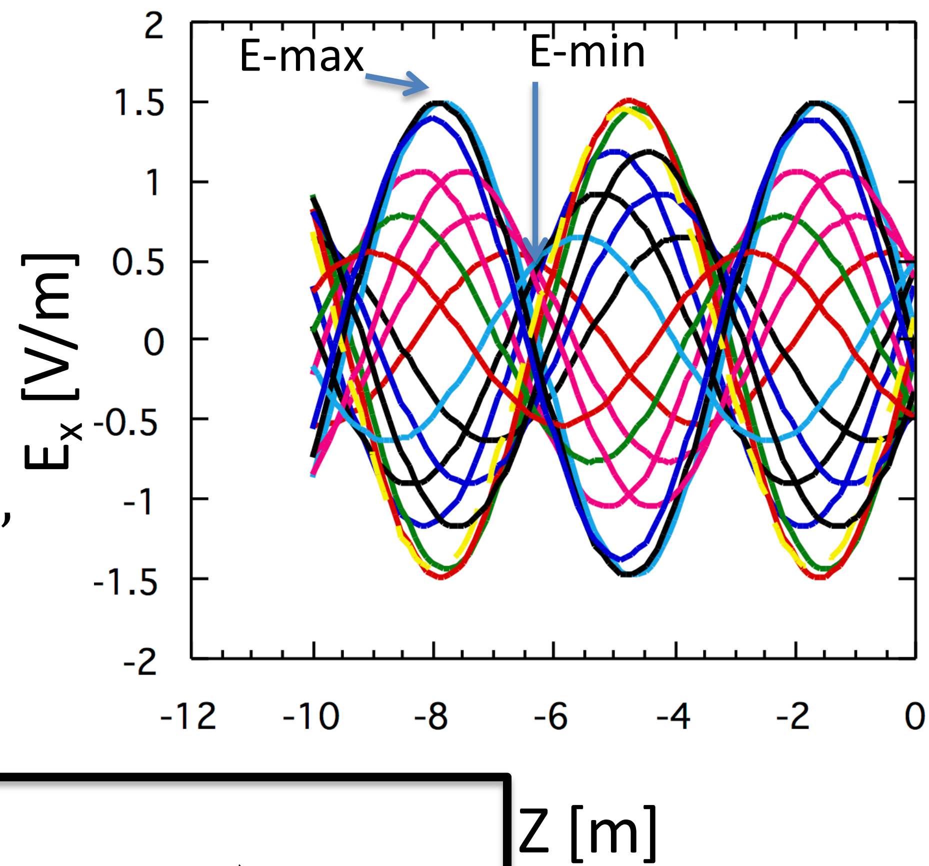
$$\text{Peak-E(z)} = \left|\hat{E}_{inc}\right| \left| \left(1 + |\rho| e^{i\theta_\rho - 2ik_1 z}\right) \right|$$

$$\text{E-max} = \left|\hat{E}_{inc}\right| \left| \left(1 + |\rho|\right) \right|, \qquad e^{i\theta_\rho - 2ik_1 z} = 1,$$

$$\theta_\rho - 2k_1 z_{\max} = 2\pi n$$

$$\text{E-min} = \left|\hat{E}_{inc}\right| \left| \left(1 - |\rho|\right) \right|, \qquad e^{i\theta_\rho - 2ik_1 z} = -1,$$

$$\theta_\rho - 2k_1 z_{\min} = 2\pi n + \pi$$



$$VSWR = \frac{\text{E-max}}{\text{E-min}} = \frac{\left(1 + |\rho|\right)}{\left(1 - |\rho|\right)}$$

$$|z_{\max}| = \left(\pi n - \theta_\rho / 2\right) / k_1$$

$$|z_{\min}| = \left(\pi\left(n + \frac{1}{2}\right) - \theta_\rho / 2\right) / k_1$$

Features of Oblique Incidence

Specularly Reflected wave

Transmitted Wave is refracted (Snell's Law)

Polarization matters

Wave Impedance depends on polarization, angle of incidence

No reflection for special angle (Brewster's angle)

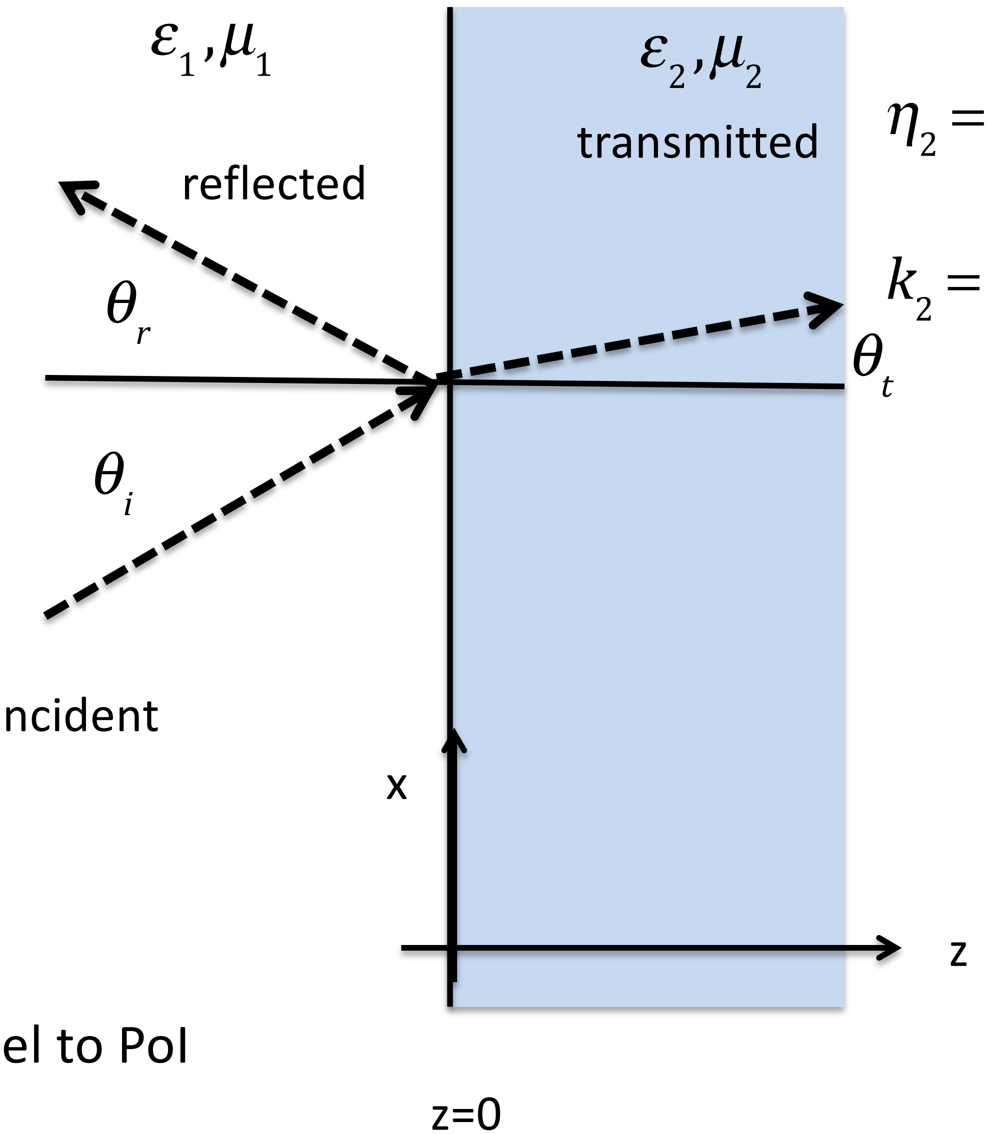
Oblique Incidence

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

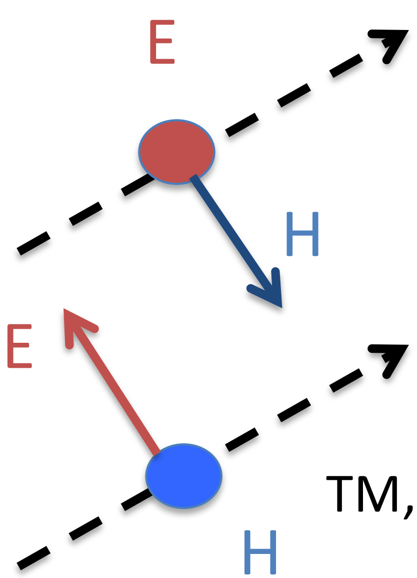
$$k_1 = \omega\sqrt{\varepsilon_1\mu_1}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$k_2 = \omega\sqrt{\varepsilon_2\mu_2}$$



Polarizations
TE, E perpendicular to PoI



TM, E parallel to PoI

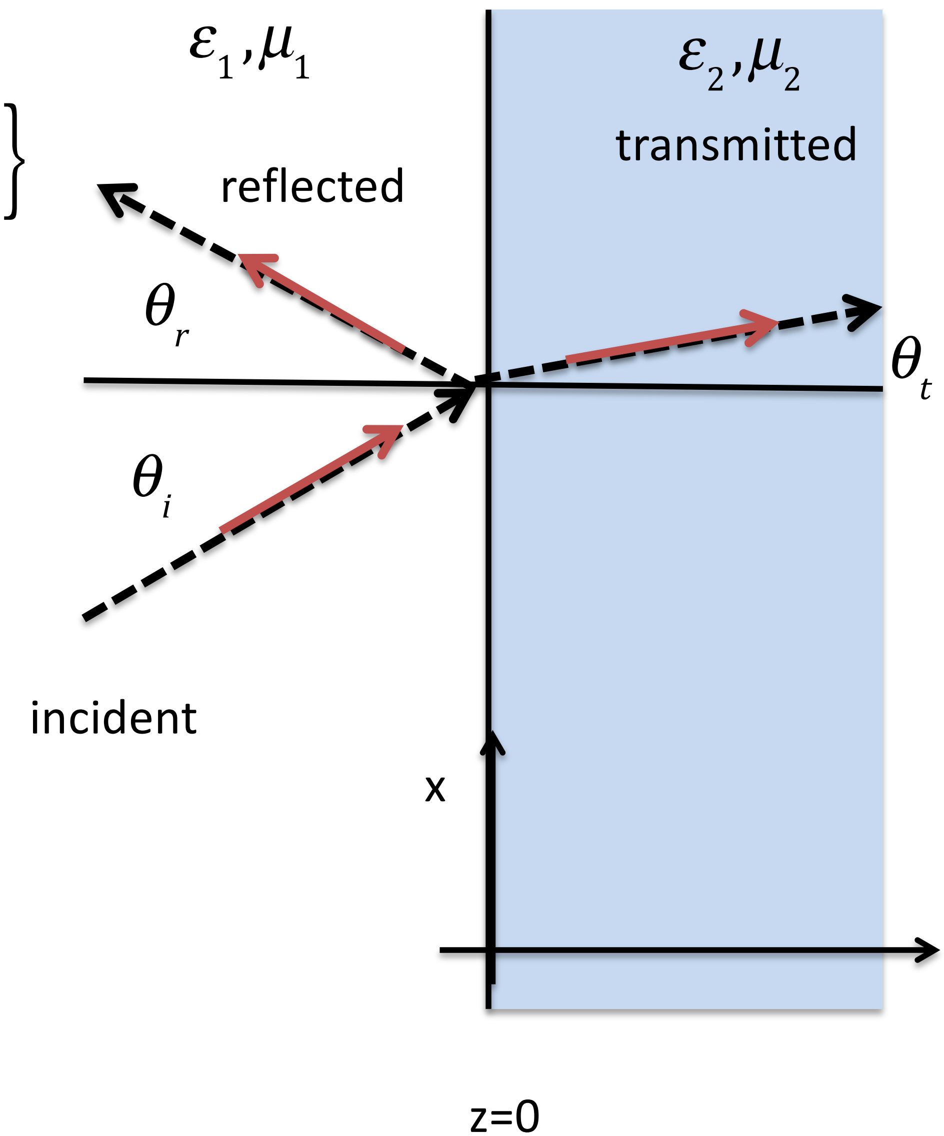
Angles of reflection/transmission

Region 1

$$\mathbf{E} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{inc} e^{i\mathbf{k}_i\cdot\mathbf{x}} + \hat{\mathbf{E}}_{ref} e^{i\mathbf{k}_r\cdot\mathbf{x}}\right) e^{-i\omega t}\right\}$$

Region 2

$$\mathbf{E} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{tran} e^{i\mathbf{k}_t\cdot\mathbf{x}}\right) e^{-i\omega t}\right\}$$



For boundary conditions at z=0 to be satisfied:

$$e^{ik_{ix}x} = e^{ik_{rx}x} = e^{ik_{tx}x}$$

$$k_{ix} = k_{rx} = k_{tx}$$

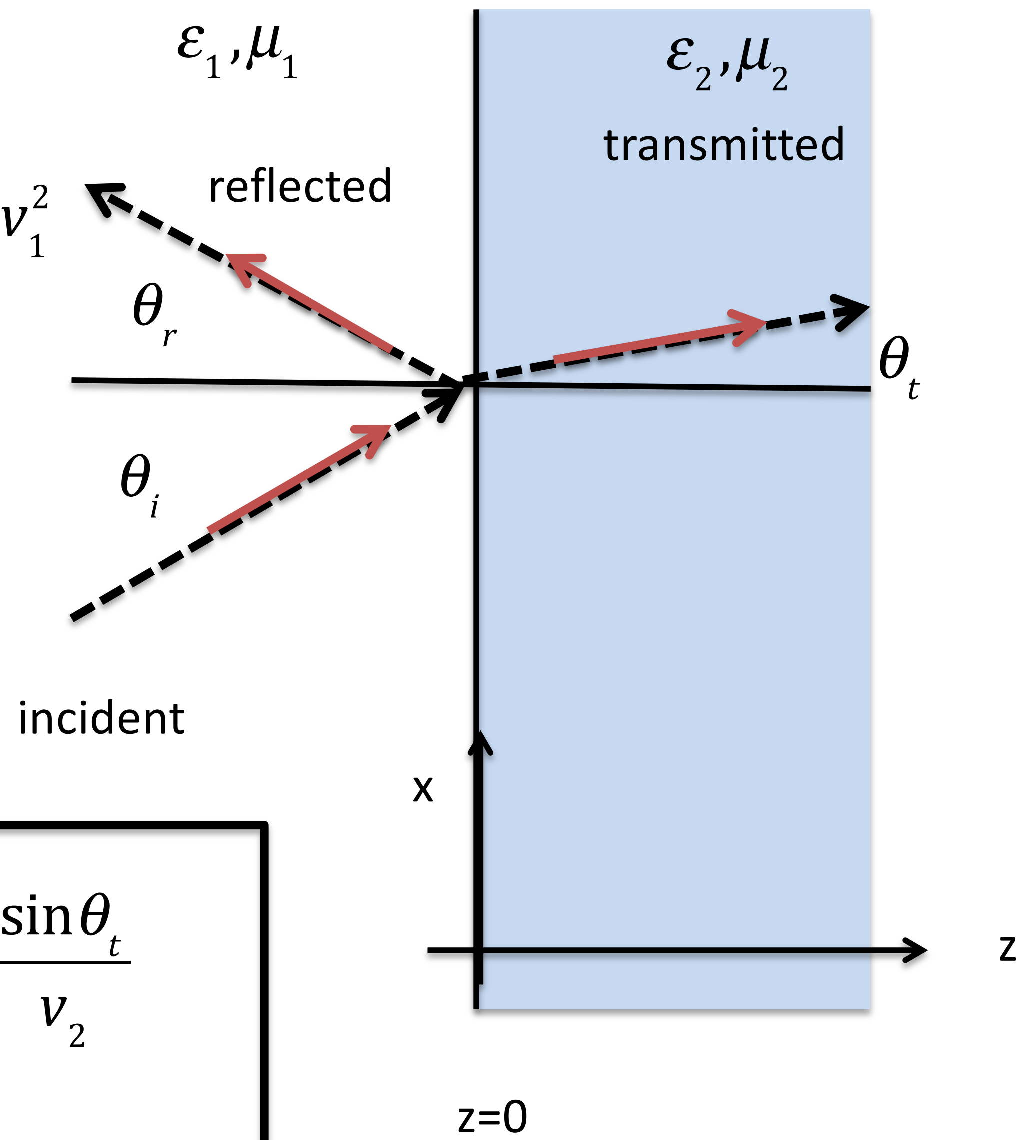
Angles of reflection/transmission

Region 1

$$k_{z,i/r} = \pm\sqrt{k_1^2 - k_x^2} \quad k_1^2 = \omega^2 \varepsilon_1 \mu_1 = \omega^2 / v_1^2$$

Region 2

$$k_{z,t} = \pm\sqrt{k_2^2 - k_x^2} \quad k_2^2 = \omega^2 \varepsilon_2 \mu_2 = \omega^2 / v_2^2$$



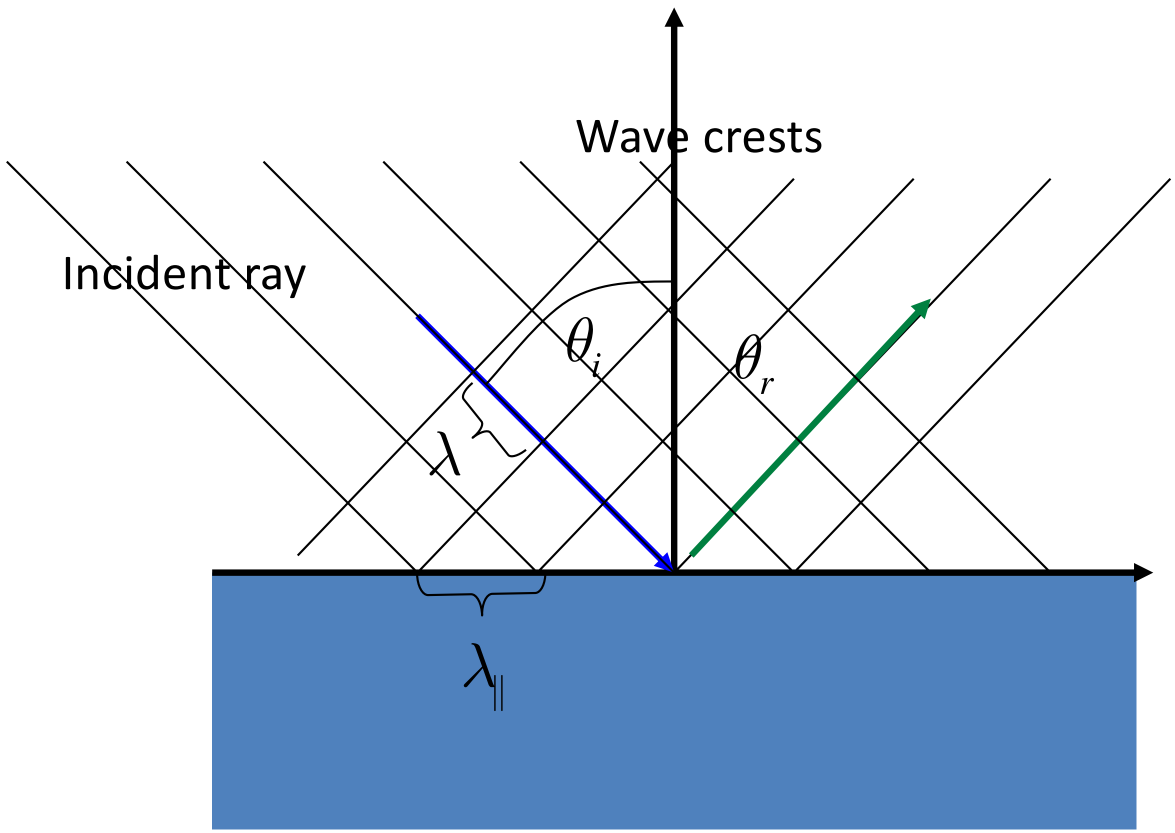
Snell's Law

$$\sin\theta_i = \frac{k_x}{k_1} = \frac{k_x}{\omega} v_1$$

$$\sin\theta_t = \frac{k_x}{k_t} = \frac{k_x}{\omega} v_2$$

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_t}{v_2}$$

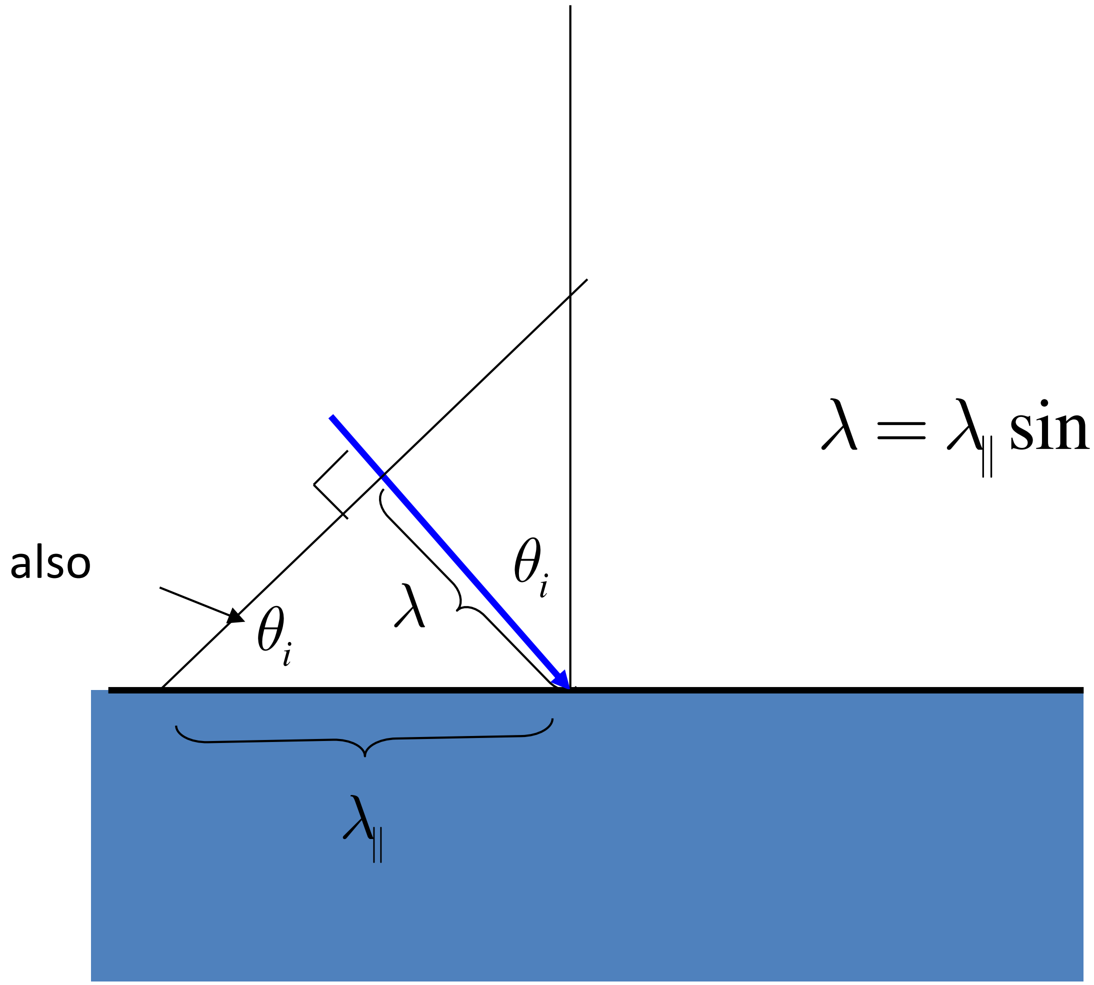
Why does angle of incidence = angle of reflection?



$$\lambda_{\parallel} = \frac{\lambda}{\sin\theta_i} = \frac{\lambda}{\sin\theta_r}$$

$$\theta_i = \theta_r$$

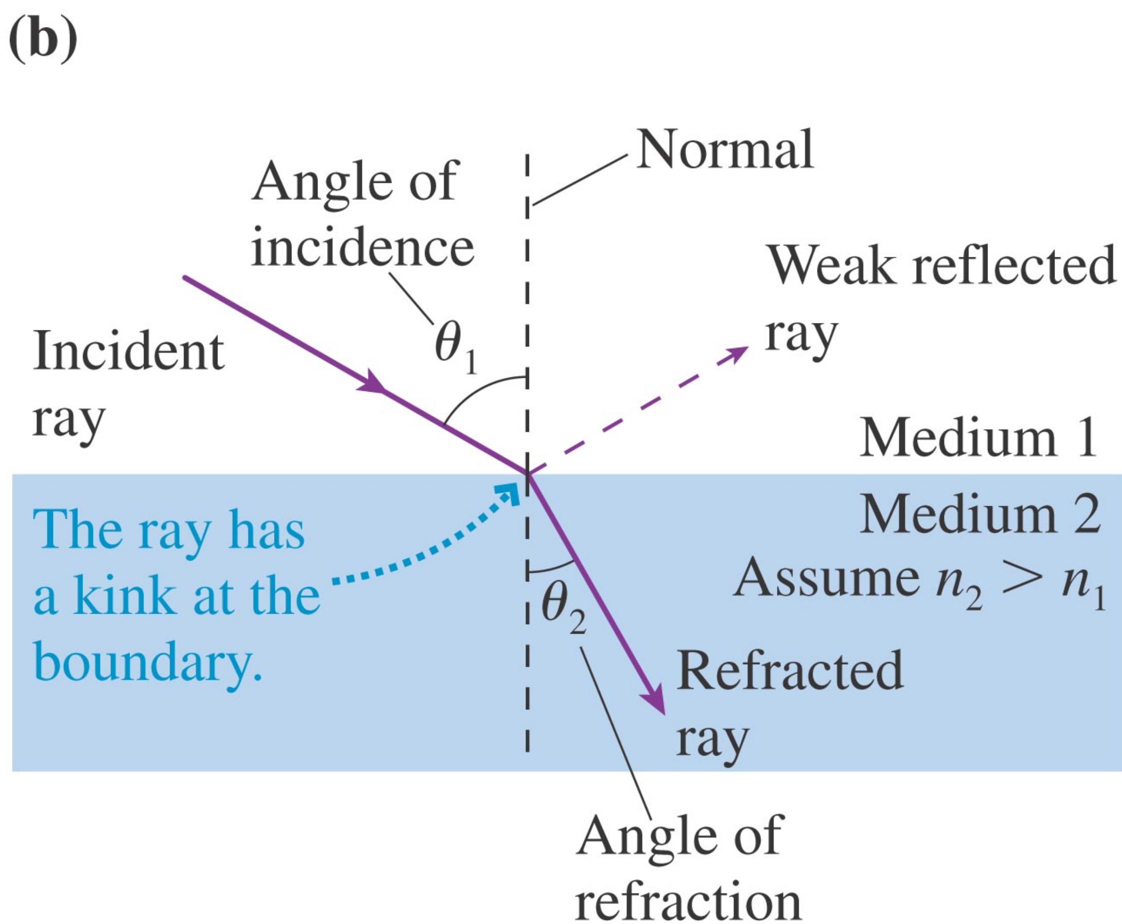
Incident and Reflected wave crests must match up along surface geometry



$$\lambda = \lambda_{\parallel} \sin\theta_i$$

Snell's Law

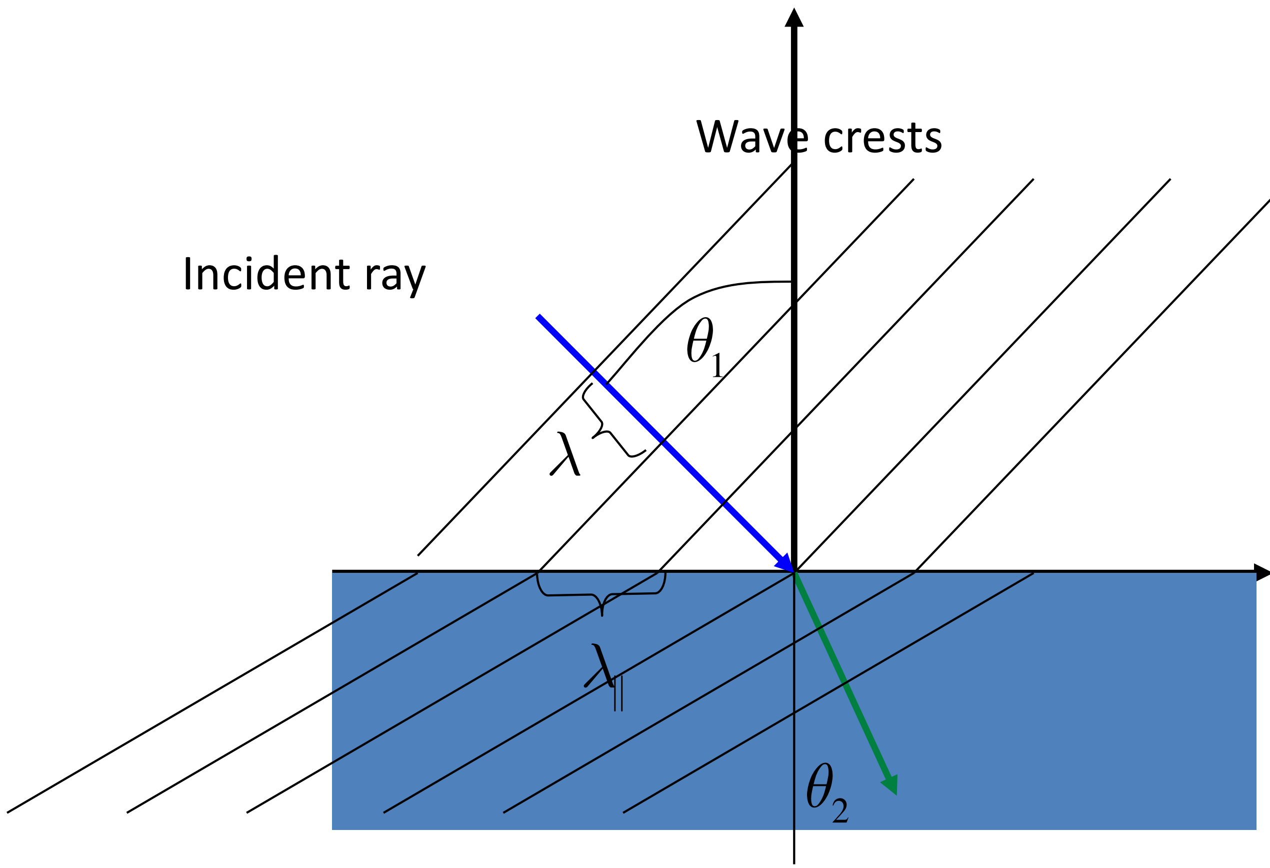
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$



Remember definition of index of refraction

$$n = \frac{c}{v_{em}}$$

Why Snell's Law?



$$\lambda_\parallel = \frac{\lambda_1}{\sin\theta_1} = \frac{\lambda_2}{\sin\theta_2}$$

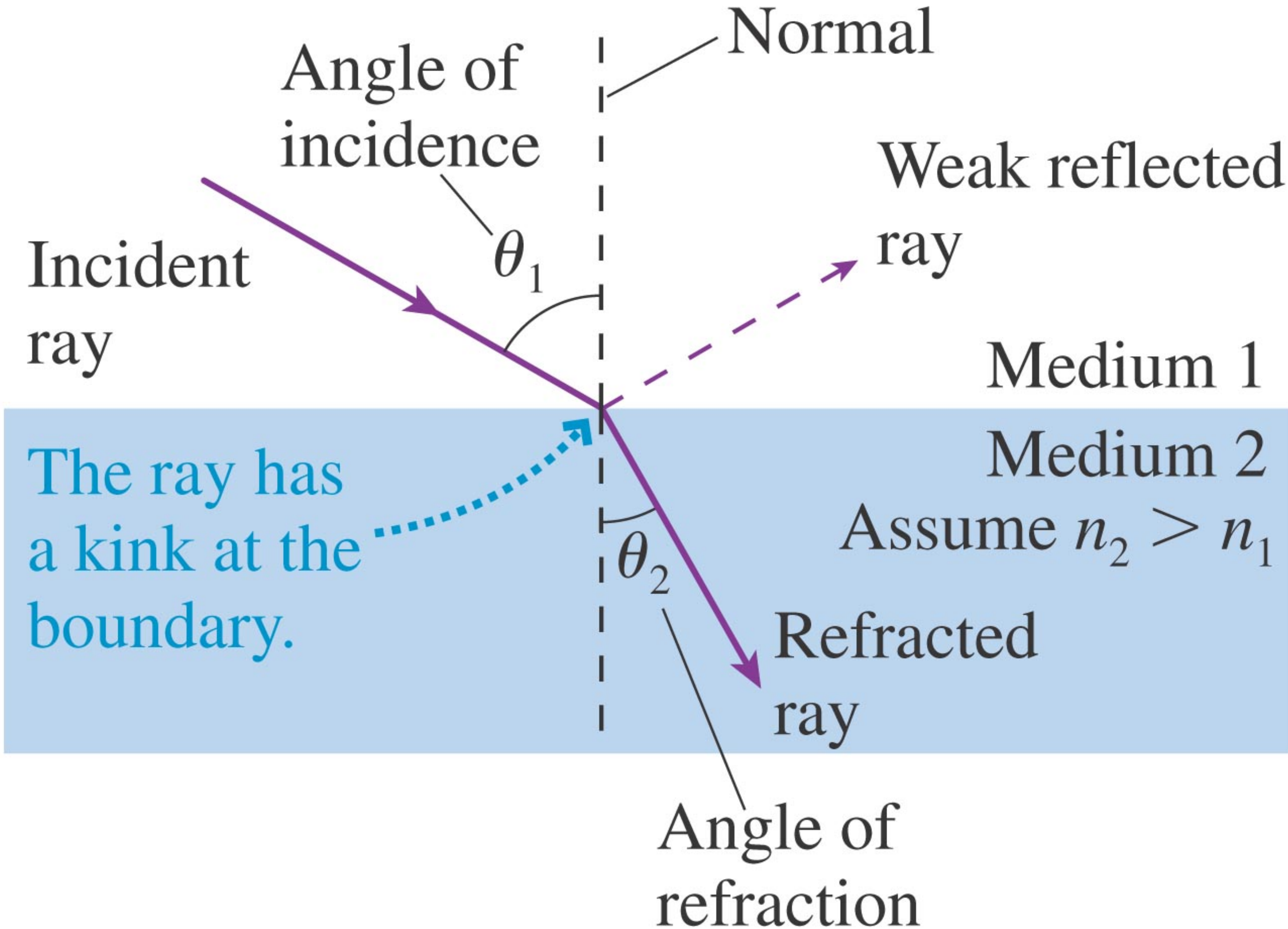
$$\lambda_1 = \frac{\lambda_{vac}}{n_1}$$

$$\lambda_2 = \frac{\lambda_{vac}}{n_2}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Incident and Transmitted wave crests must match up along surface

(b)


Normal
Angle of incidence
θ_1
Incident ray
Weak reflected ray
Medium 1
Medium 2
Assume $n_2 > n_1$
The ray has a kink at the boundary.
θ_2
Refracted ray
Angle of refraction

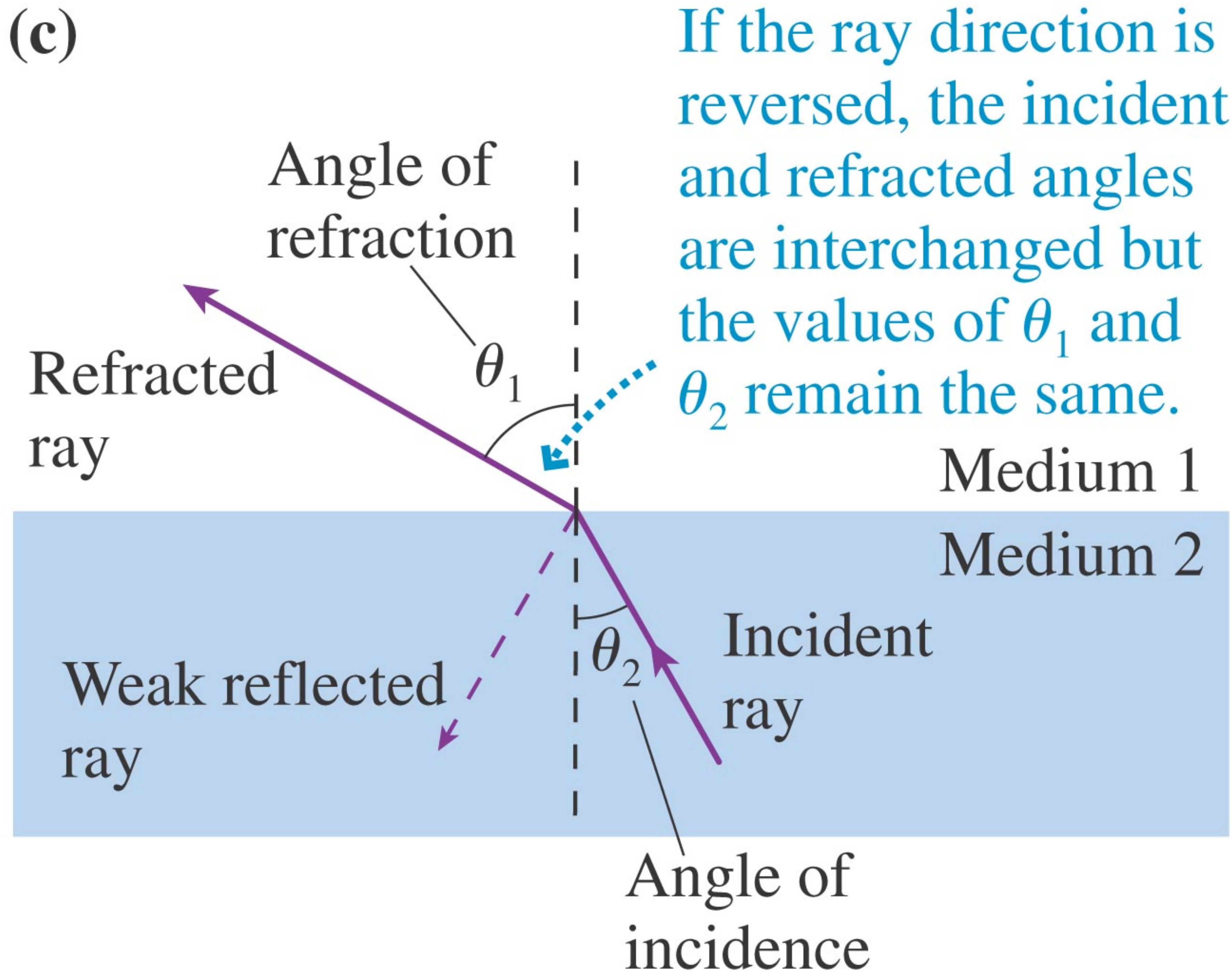

(c)
If the ray direction is reversed, the incident and refracted angles are interchanged but the values of θ_1 and θ_2 remain the same.
Angle of refraction
θ_1
Refracted ray
Medium 1
Medium 2
Weak reflected ray
θ_2
Incident ray
Angle of incidence

TABLE 23.1 Indices of refraction

Medium	n
Vacuum	1.00 exactly
Air (actual)	1.0003
Air (accepted)	1.00
Water	1.33
Ethyl alcohol	1.36
Oil	1.46
Glass (typical)	1.50
Polystyrene plastic	1.59
Cubic zirconia	2.18
Diamond	2.41
Silicon (infrared)	3.50

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

For most material
n>1

Plasma
n<1

Total Internal Reflection

Region 1

$$\mathbf{E} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{inc} e^{i\mathbf{k}_i\cdot\mathbf{x}} + \hat{\mathbf{E}}_{ref} e^{i\mathbf{k}_r\cdot\mathbf{x}}\right) e^{-i\omega t}\right\}$$

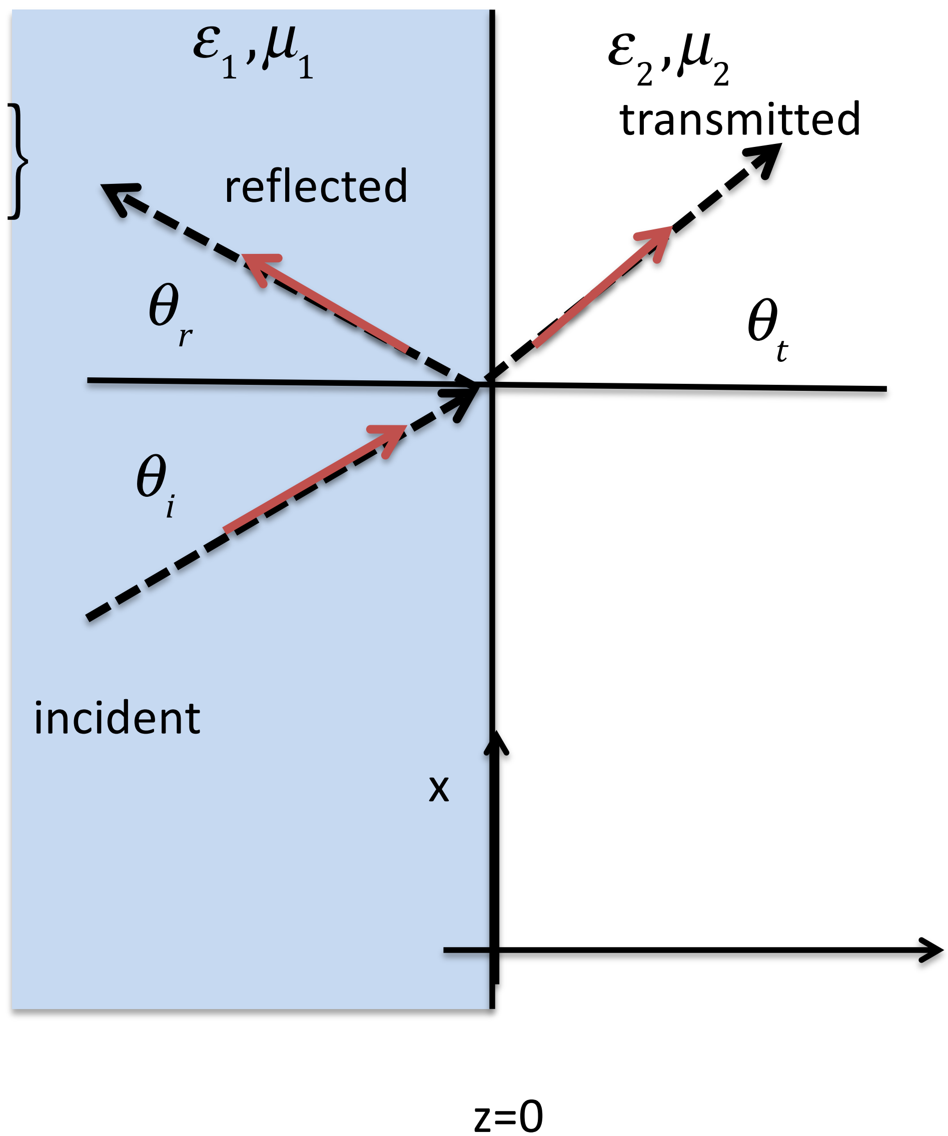
Region 2

$$\mathbf{E} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{tran} e^{i\mathbf{k}_t\cdot\mathbf{x}}\right) e^{-i\omega t}\right\}$$

For boundary conditions at z=0 to be satisfied:

$$e^{ik_{ix}x} = e^{ik_{rx}x} = e^{ik_{tx}x}$$

$$k_{ix} = k_{rx} = k_{tx}$$



Region 1

$$k_1^2 = \omega^2 \varepsilon_1 \mu_1 \qquad k_1^2 = k_x^2 + k_{z1}^2$$

Region 2

$$k_2^2 = \omega^2 \varepsilon_2 \mu_2 \qquad k_2^2 = k_x^2 + k_{z2}^2$$

Snell's Law

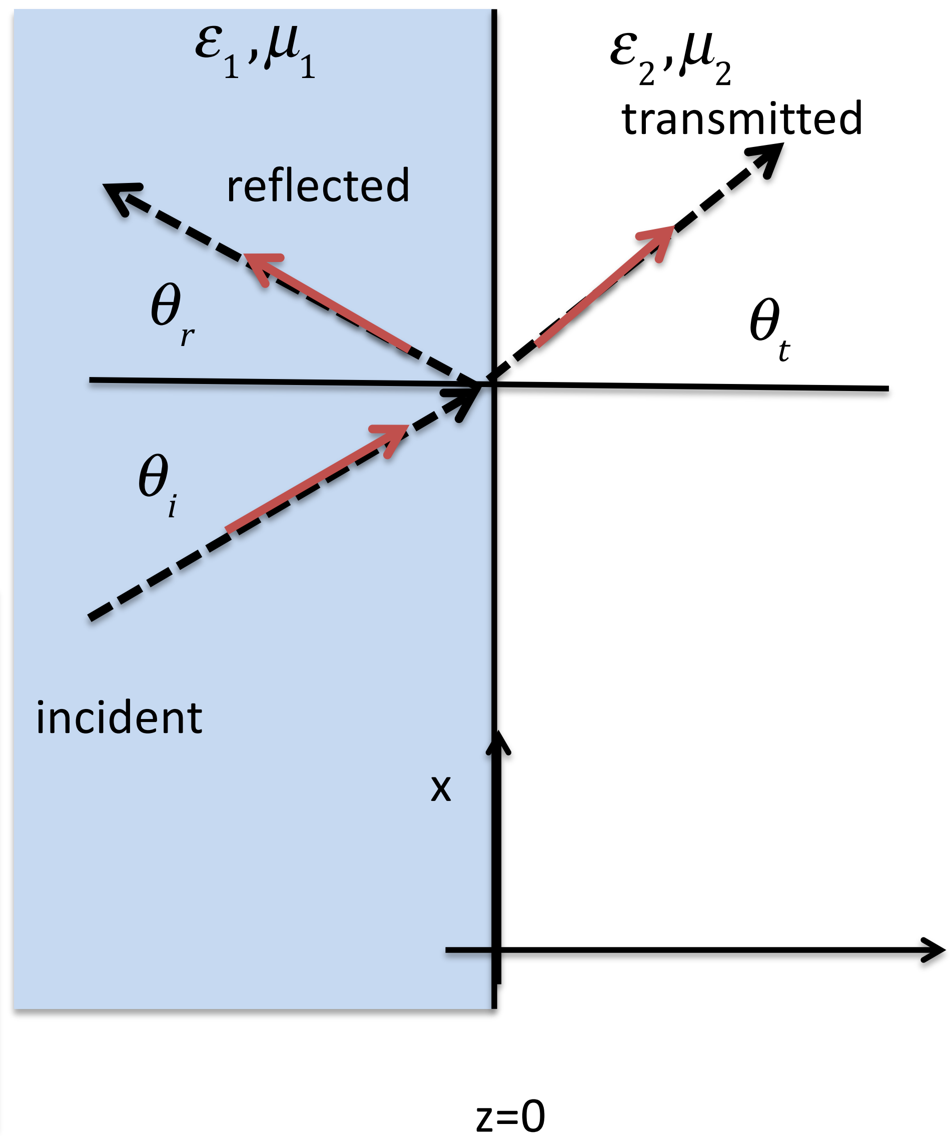
$$\sin\theta_i = \frac{k_x}{k_1} = \frac{k_x}{\omega} v_1$$

$$\sin\theta_t = \frac{k_x}{k_t} = \frac{k_x}{\omega} v_2$$

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_t}{v_2}$$

No solution

$$\sin\theta_i > \frac{v_1}{v_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$



Evanescent Field

Region 2

$$\mathbf{E} = \mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{tran} e^{ik_x x + ik_{z2} z}\right) e^{-i\omega t}\right\}$$

$$k_2^2 = k_{z2}^2 + k_x^2 = \omega^2 \varepsilon_2 \mu_0$$

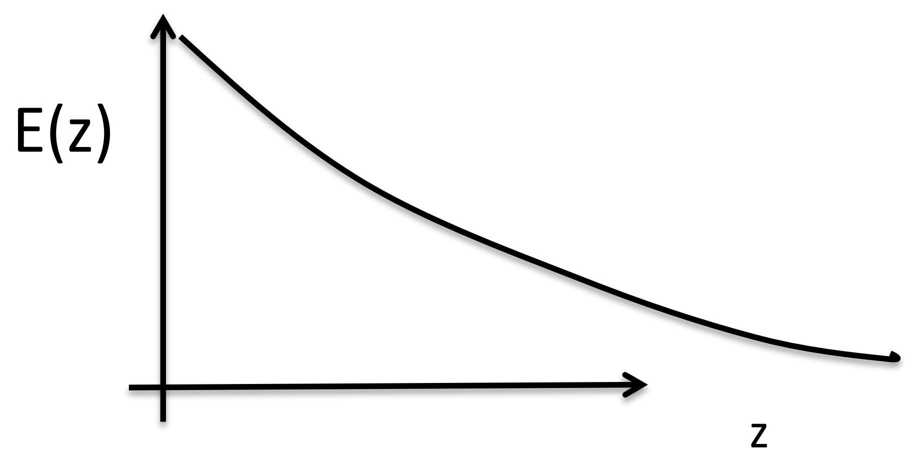
$$k_{z2}^2 = \omega^2 \varepsilon_2 \mu_0 - k_x^2 \quad k_x^2 = \omega^2 \varepsilon_1 \mu_0 \sin^2\theta_i$$

$$k_{z2}^2 = \omega^2 \varepsilon_1 \mu_0 \left[\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i\right]$$

If $\quad 1 > \sin^2\theta_i > \frac{\varepsilon_2}{\varepsilon_1}$ then $k_{z2}^2 < 0$

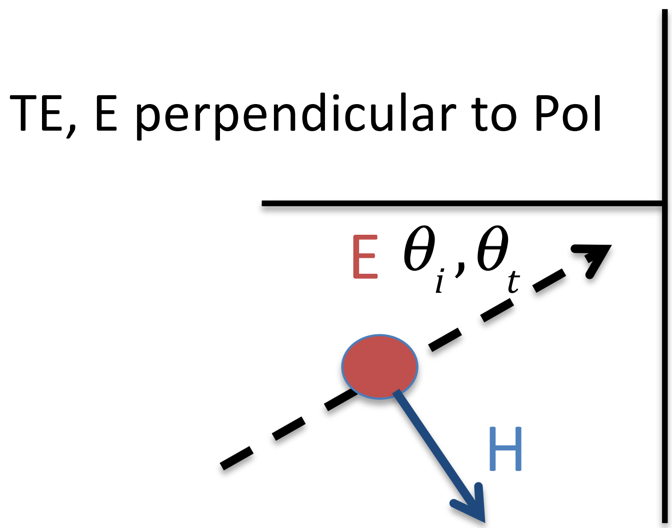
$$k_{z2} = \pm i k_1 \sqrt{\left[\sin^2\theta_i - \frac{\varepsilon_2}{\varepsilon_1}\right]} \equiv \pm i\kappa$$

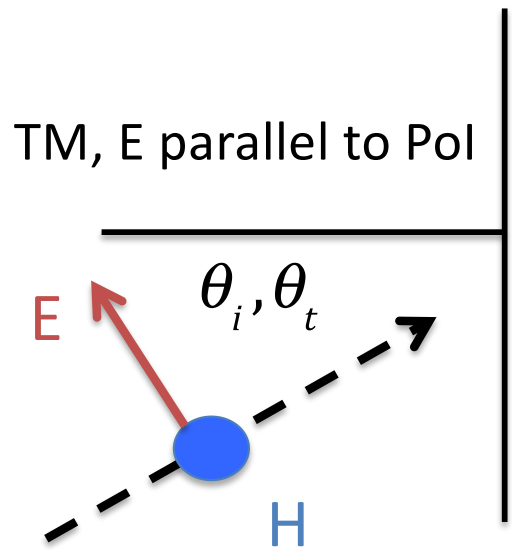
$$\mathbf{E} = e^{\pm\kappa z}\,\mathrm{Re}\left\{\left(\hat{\mathbf{E}}_{tran} e^{ik_x x}\right) e^{-i\omega t}\right\}$$



What is the reflection coefficient?

$$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

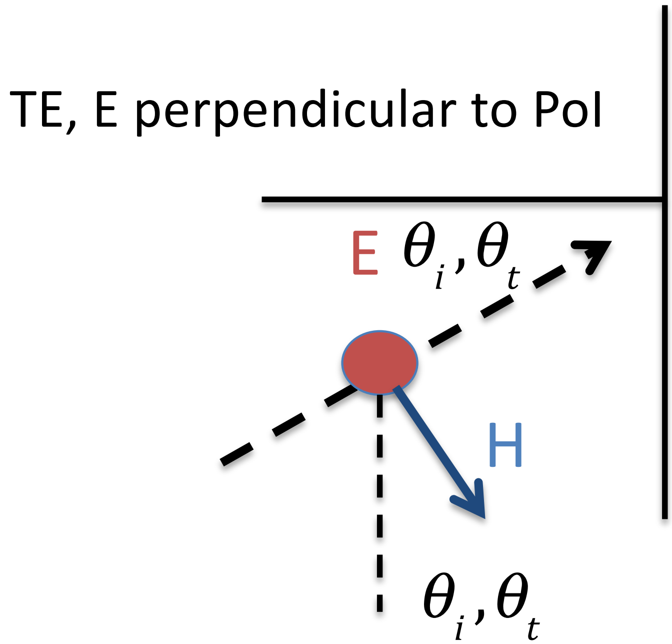


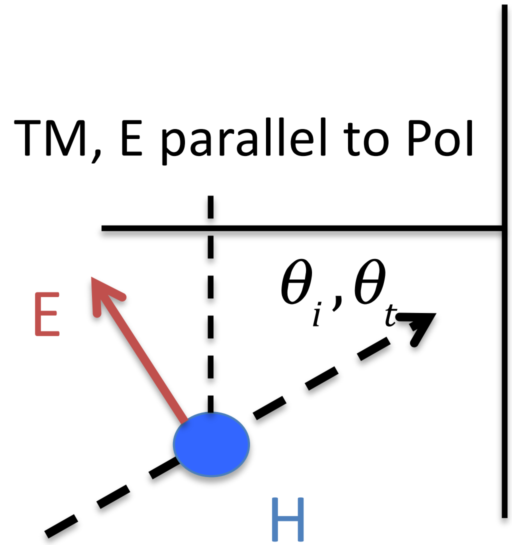


$$Z = \frac{E_{\tan}}{H_{\tan}} = ?$$

$$Z = \frac{E_{\tan}}{H_{\tan}} = ?$$

What is the reflection coefficient?





$$Z = \frac{E_{\tan}}{H_{\tan}} = \frac{|E|}{|H|\cos\theta} = \frac{\eta}{\cos\theta}$$

$$Z = \frac{E_{\tan}}{H_{\tan}} = \frac{|E|\cos\theta}{|H|} = \eta\cos\theta$$

Reflection Coefficient

$$\rho_{TE} = \frac{\eta_2 / \cos\theta_2 - \eta_1 / \cos\theta_1}{\eta_2 / \cos\theta_2 + \eta_1 / \cos\theta_1}$$

$$\rho_{TM} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

Specialize to case $\mu_1 = \mu_2 \quad \eta_2 / \eta_1 = \sqrt{\varepsilon_1 / \varepsilon_2}$

By Snell's Law

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

TE

$$\rho_{TE} = \frac{\eta_2 / \cos\theta_2 - \eta_1 / \cos\theta_1}{\eta_2 / \cos\theta_2 + \eta_1 / \cos\theta_1}$$

By Snell's Law

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\rho_{TE} = \frac{\sqrt{\varepsilon_1/\varepsilon_2}\cos\theta_1 - \sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1}}{\sqrt{\varepsilon_1/\varepsilon_2}\cos\theta_1 + \sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1}}$$

TM

$$\rho_{TM} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}$$

By Snell's Law

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1}$$

$$\rho_{TM} = \frac{\sqrt{\varepsilon_1/\varepsilon_2}\sqrt{1-\left(\varepsilon_1/\varepsilon_2\right)\sin^2\theta_1} - \cos\theta_1}{\sqrt{\varepsilon_1/\varepsilon_2}\sqrt{1-\left(\varepsilon_1/\varepsilon_2\right)\sin^2\theta_1} + \cos\theta_1}$$

TE

$$\rho_{TE} = \frac{\sqrt{\varepsilon_1/\varepsilon_2}\cos\theta_1 - \sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1}}{\sqrt{\varepsilon_1/\varepsilon_2}\cos\theta_1 + \sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1}}$$

TM

$$\rho_{TM} = \frac{\sqrt{\varepsilon_1/\varepsilon_2}\sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1} - \cos\theta_1}{\sqrt{\varepsilon_1/\varepsilon_2}\sqrt{1-(\varepsilon_1/\varepsilon_2)\sin^2\theta_1} + \cos\theta_1}$$

Normal incidence

$$\theta = 0$$

$$\rho_{TE} = \frac{\sqrt{\varepsilon_1/\varepsilon_2} - 1}{\sqrt{\varepsilon_1/\varepsilon_2} + 1}$$

$$\rho_{TM} = \frac{\sqrt{\varepsilon_1/\varepsilon_2} - 1}{\sqrt{\varepsilon_1/\varepsilon_2} + 1}$$

Grazing incidence

$$\theta = \pi/2, \quad \varepsilon_1 < \varepsilon_2$$

$$\rho_{TE} = -1$$

$$\rho_{TM} = 1$$

Critical angle

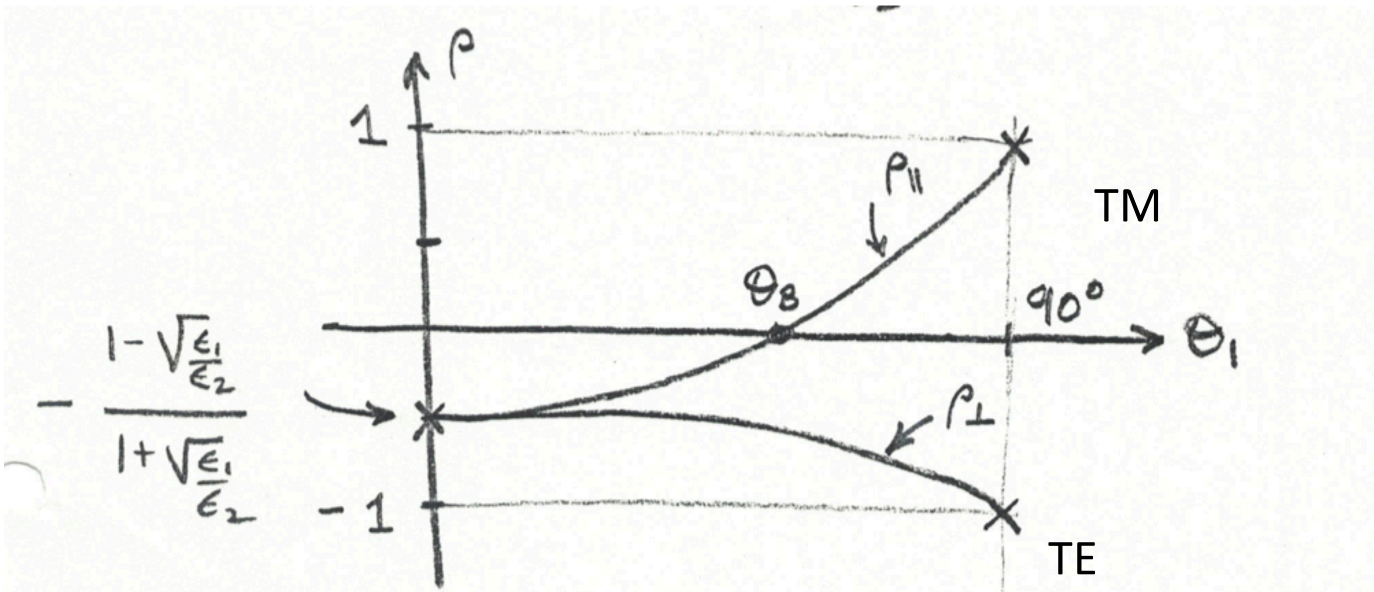
$$\sin\theta = \sqrt{\varepsilon_2/\varepsilon_1},$$

$$\varepsilon_2/\varepsilon_1 < 1$$

$$\rho_{TE} = 1$$

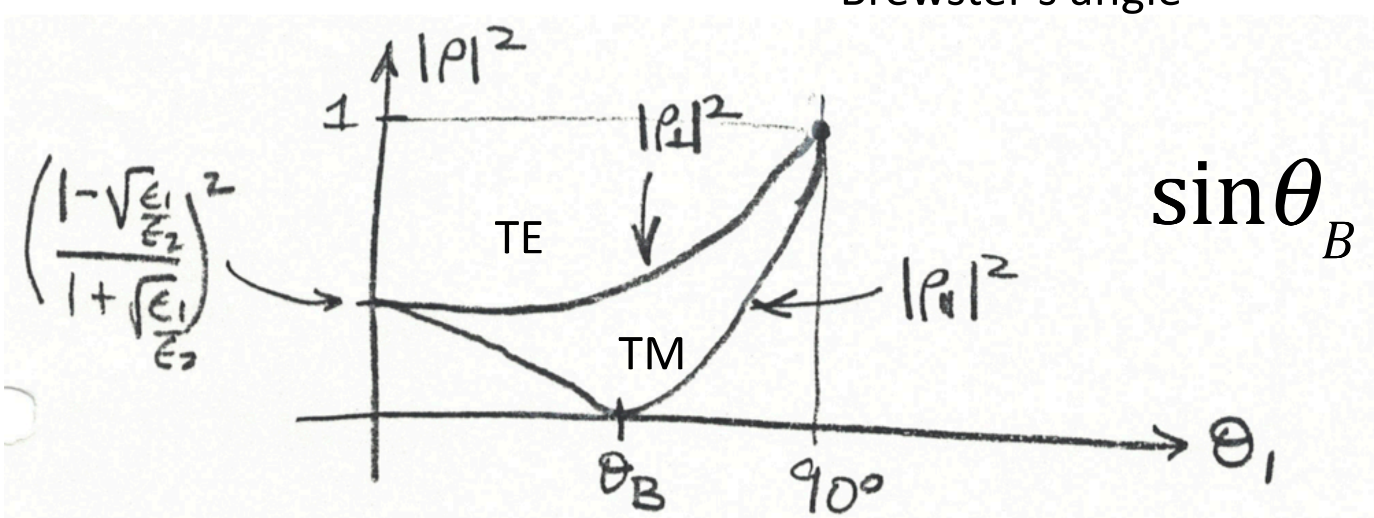
$$\rho_{TM} = -1$$

Case1 $\varepsilon_1 < \varepsilon_2$



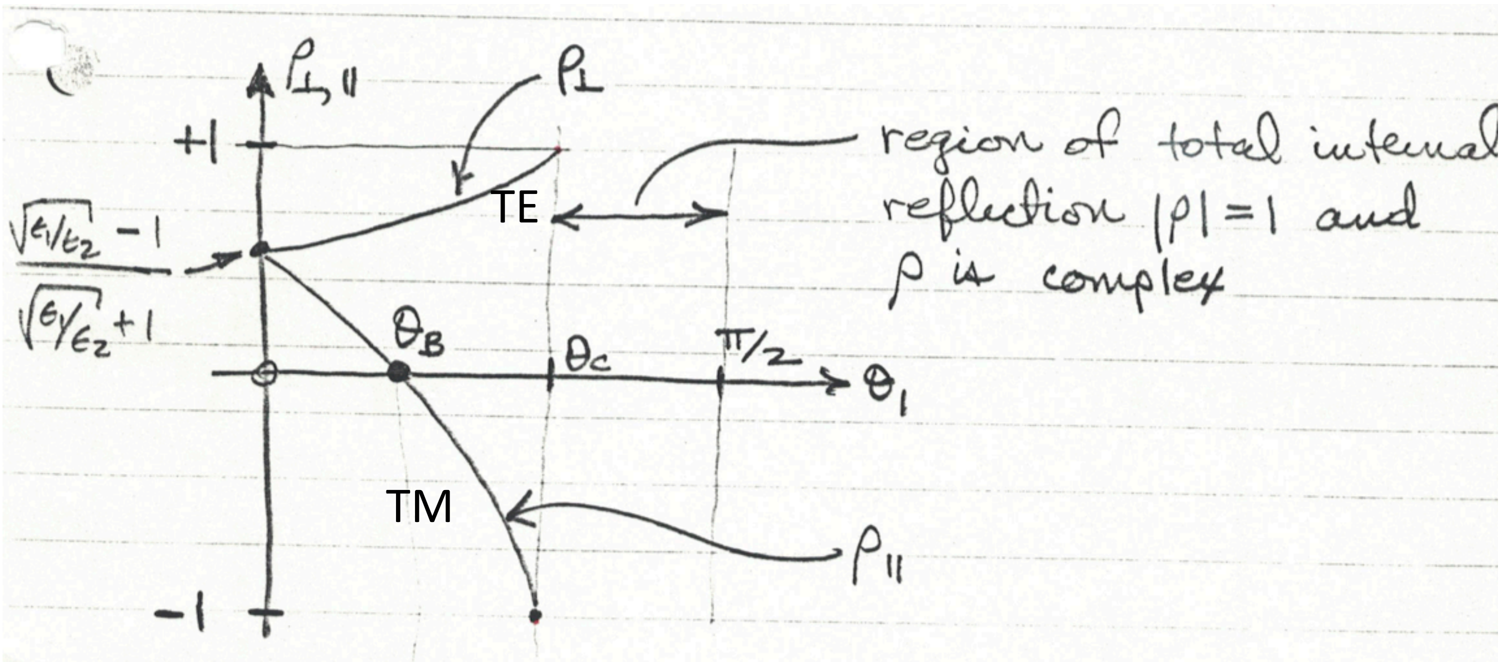
Power reflection

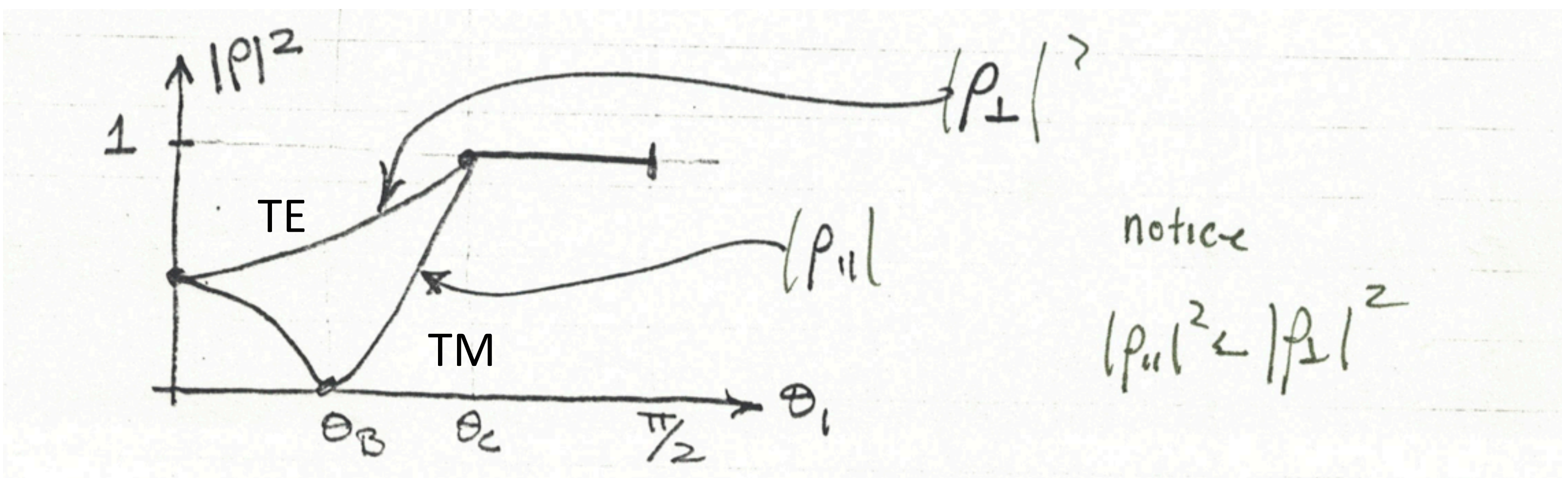
Brewster's angle



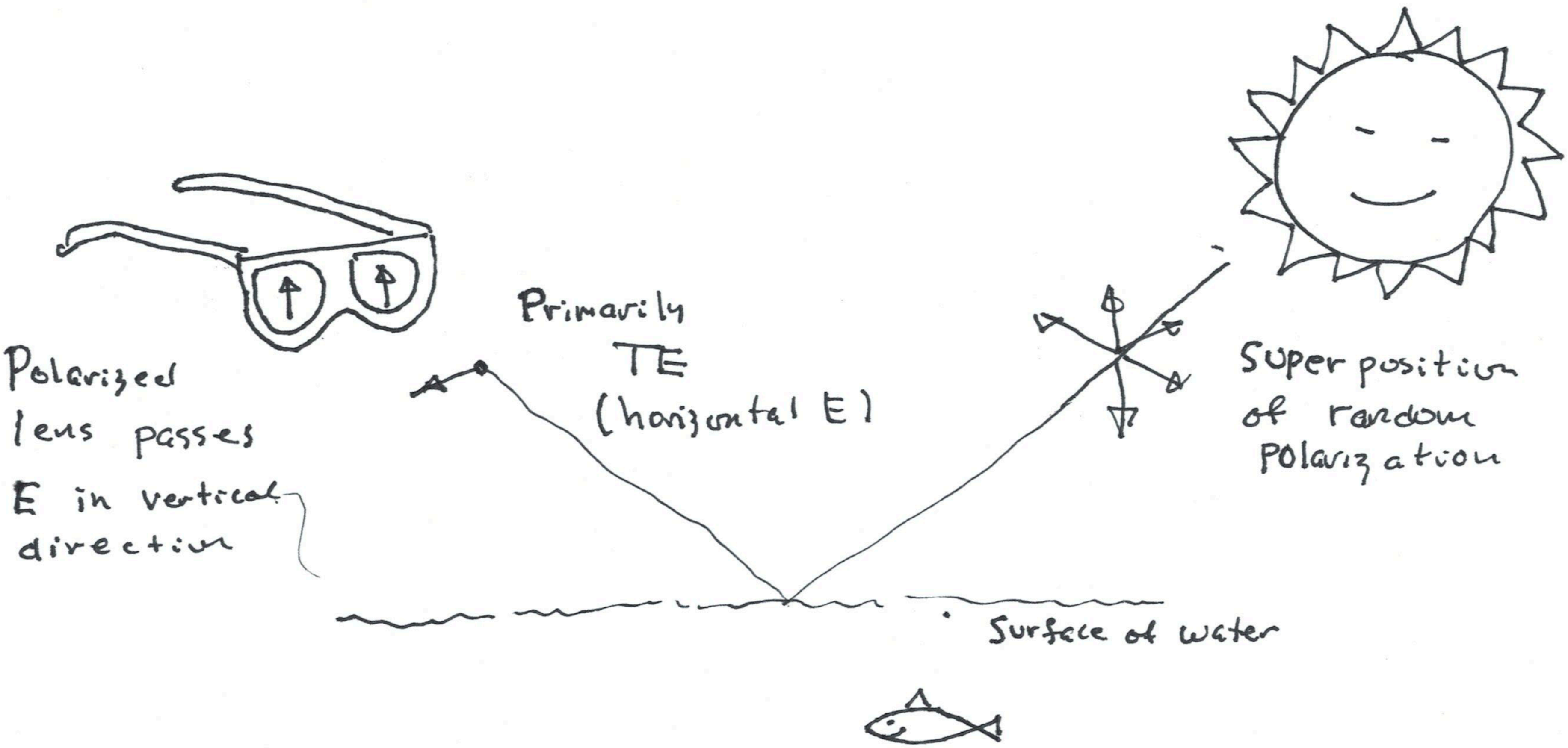
$$\sin\theta_B = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}$$

Case2 $\varepsilon_1 > \varepsilon_2$



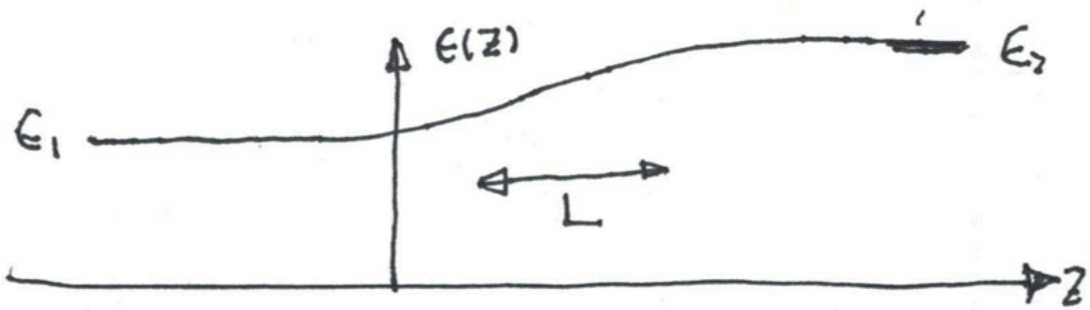


Polarized Lenses



Geometric Optics

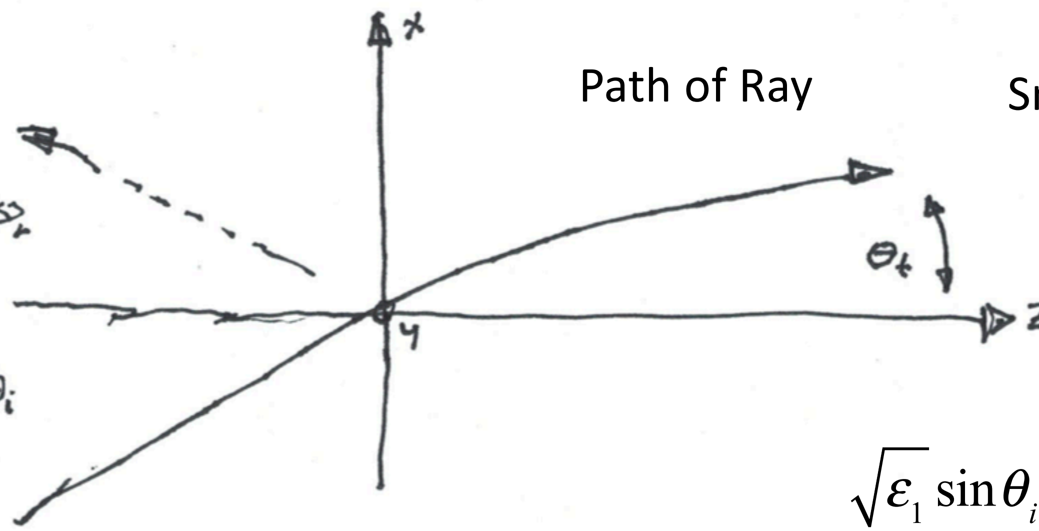
What happens when boundary is diffuse? $L >> \lambda$



Small reflection

$$\left|\rho\right| \sim \exp\left[-L/\lambda\right] << 1$$

Path of Ray

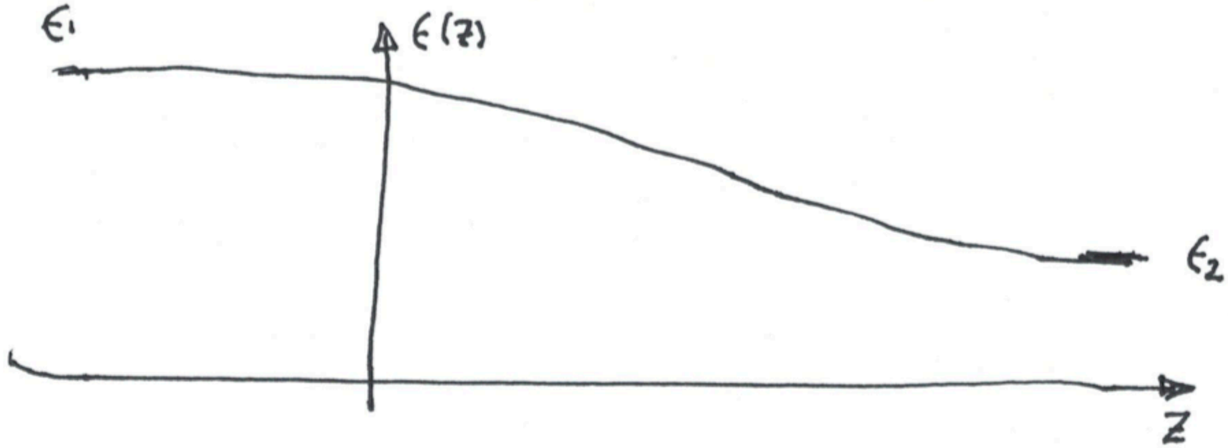


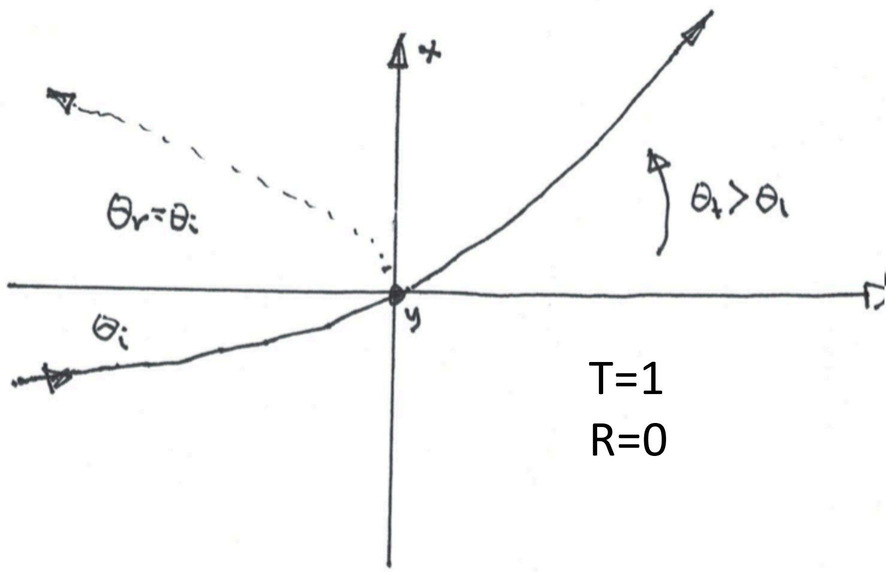
Snell's Law still valid

$$\frac{d}{dx}\varepsilon(x) = 0$$

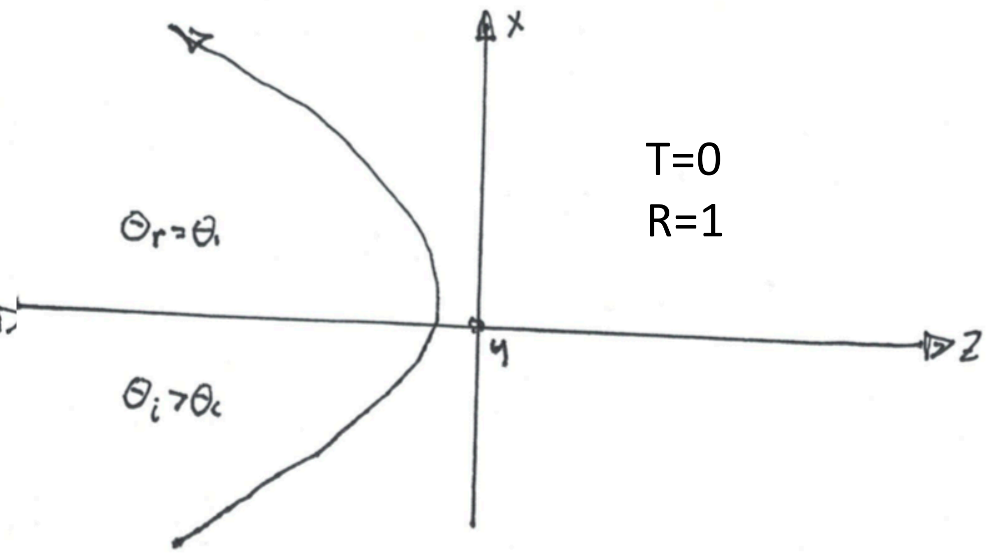
$$\rightarrow k_x = const.$$

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t$$





$\theta_i < \theta_{cr}$



$\theta_i > \theta_{cr}$

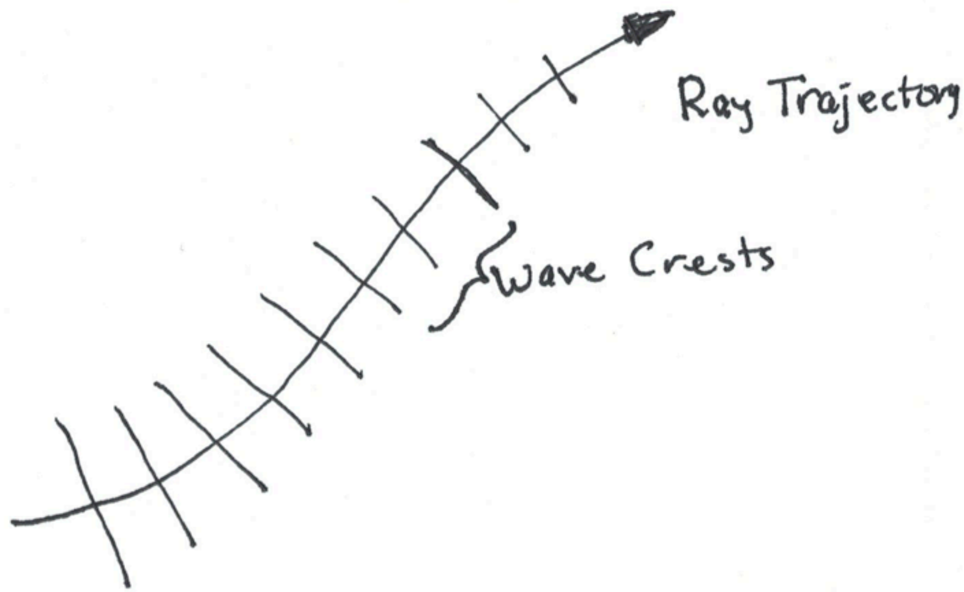
Geometric Optics (Ray Tracing)

Properties of Medium depend weakly on position

$$\varepsilon(\boldsymbol{x},\omega), \quad \mu(\boldsymbol{x},\omega)$$

$$\left|\frac{1}{\varepsilon(\boldsymbol{x},\omega)}\nabla\varepsilon(\boldsymbol{x},\omega)\right| \sim \frac{1}{L} << \lambda$$

Locally medium appears homogeneous



$$\boldsymbol{E} = \mathrm{Re}\left\{\hat{\boldsymbol{E}}(\boldsymbol{x})\exp\left[i\phi(\boldsymbol{x}) - i\omega t\right]\right\}$$

$\phi(\boldsymbol{x})$ the Eikonal

$\boldsymbol{k}(x) = \nabla\phi(\boldsymbol{x})$ is the local wave vector

$\phi(\boldsymbol{x})$ the Eikonal

$\nabla\phi(\boldsymbol{x})$ is the local wave vector

Near a point $\boldsymbol{x}_0$, $\phi(\boldsymbol{x}) \simeq \phi(\boldsymbol{x}_0) + (\boldsymbol{x} - \boldsymbol{x}_0)\cdot\nabla\phi(\boldsymbol{x})\big|_{x_0}$

$$\boldsymbol{E} = \mathrm{Re}\left\{ \hat{\boldsymbol{E}}(\boldsymbol{x}) \exp\left[i\phi(\boldsymbol{x}) - i\omega t \right] \right\}$$

$\nabla\phi(\boldsymbol{x})$ is the local wave vector

Local dispersion relation

$\omega = \omega(\boldsymbol{x}, \boldsymbol{k})$

$\boldsymbol{k} = \nabla\phi(\boldsymbol{x})$ is the local wave vector

As the wave propagates through space

$\boldsymbol{k}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x})$

will vary, but at every point $\omega = \omega(\boldsymbol{x}, \boldsymbol{k})$, constant

$$\frac{d}{dx}\omega(\boldsymbol{x}, \boldsymbol{k}) = \frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial \boldsymbol{k}} \cdot \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}} = 0$$

$$\frac{d}{dy}\omega(\boldsymbol{x}, \boldsymbol{k}) = \frac{\partial\omega}{\partial y} + \frac{\partial\omega}{\partial \boldsymbol{k}} \cdot \frac{\partial \boldsymbol{k}}{\partial y} = 0$$

$$\frac{d}{dz}\omega(\boldsymbol{x}, \boldsymbol{k}) = \frac{\partial\omega}{\partial z} + \frac{\partial\omega}{\partial \boldsymbol{k}} \cdot \frac{\partial \boldsymbol{k}}{\partial z} = 0$$

Consider x derivative

$$\frac{d}{dx}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial \boldsymbol{k}} \cdot \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}}$$

$$\frac{d}{dx}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial k_x}\frac{\partial k_x}{\partial \boldsymbol{x}} + \frac{\partial \omega}{\partial k_y}\frac{\partial k_y}{\partial x} + \frac{\partial \omega}{\partial k_z}\frac{\partial k_z}{\partial x} = 0$$

Replace derivatives above

But, $\boldsymbol{k} = \nabla \phi(\boldsymbol{x}) \quad \rightarrow \quad \nabla \times \boldsymbol{k}(\boldsymbol{x}) = 0$

$$\frac{\partial k_y}{\partial x} = \frac{\partial k_x}{\partial y}, \qquad \frac{\partial k_z}{\partial x} = \frac{\partial k_x}{\partial z}$$

$$\frac{d}{dx}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial k_x}\frac{\partial k_x}{\partial \boldsymbol{x}} + \frac{\partial \omega}{\partial k_y}\frac{\partial k_x}{\partial y} + \frac{\partial \omega}{\partial k_z}\frac{\partial k_x}{\partial z} = 0$$

Introduce group velocity,

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}$$

$$\frac{d}{dx}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial x} + \boldsymbol{v}_g \cdot \nabla k_x = 0$$

$$\frac{d}{dy}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial y} + \boldsymbol{v}_g \cdot \nabla k_y = 0$$

$$\frac{d}{dz}\omega(\boldsymbol{x},\boldsymbol{k}) = \frac{\partial \omega}{\partial z} + \boldsymbol{v}_g \cdot \nabla k_z = 0$$

How to interpret

$$\frac{\partial \omega}{\partial x} + \boldsymbol{v}_g \cdot \nabla k_x = 0$$

$$\frac{\partial \omega}{\partial y} + \boldsymbol{v}_g \cdot \nabla k_y = 0$$

$$\frac{\partial \omega}{\partial z} + \boldsymbol{v}_g \cdot \nabla k_z = 0$$

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}$$

As you follow a trajectory defined by the group velocity

$$\frac{d}{d\tau}\boldsymbol{x}(\tau) = \boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}$$

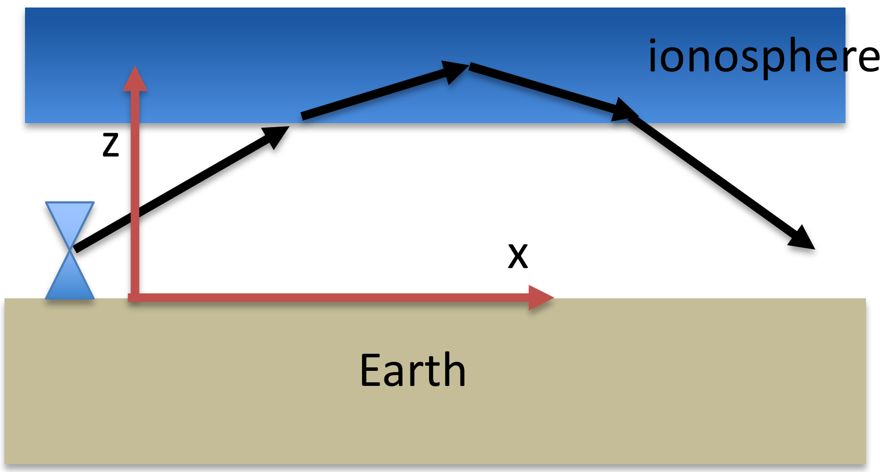
The wave vector changes according to

$$\frac{d}{d\tau}\boldsymbol{k}(\tau) = \boldsymbol{v}_g \cdot \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}} = -\frac{\partial \omega}{\partial \boldsymbol{x}}$$

Hamilton's Equations

$$\frac{d}{d\tau}\boldsymbol{x}(\tau) = \frac{\partial \omega(\boldsymbol{k}, \boldsymbol{x})}{\partial \boldsymbol{k}} \qquad \frac{d}{d\tau}\boldsymbol{k}(\tau) = -\frac{\partial \omega(\boldsymbol{k}, \boldsymbol{x})}{\partial \boldsymbol{x}}$$

Example reflection of waves by the ionosphere



$$k^2 = \frac{\omega^2}{c^2}\varepsilon_r(z,\omega), \quad \varepsilon_r(z,\omega) = 1 - \frac{\omega_p^2(z)}{\omega^2}$$

$$\omega^2 = k^2c^2 + \omega_p^2(z) \qquad \omega(\boldsymbol{k},z) = \left(k^2c^2 + \omega_p^2(z)\right)^{1/2}$$

$$\frac{dz}{d\tau} = \frac{\partial\omega}{\partial k_z} = \frac{k_z c^2}{\omega}, \qquad \frac{dk_z}{d\tau} = -\frac{\partial\omega}{\partial z} = -\frac{1}{2\omega}\frac{\partial}{\partial z}\omega_p^2(z)$$

$$\frac{dx}{d\tau} = \frac{\partial\omega}{\partial k_x} = \frac{k_x c^2}{\omega}, \qquad \frac{dk_x}{d\tau} = -\frac{\partial\omega}{\partial x} = 0 \qquad k_x = k_{xo}$$

$$\omega^2 = \omega_p^2(z) + k_x^2c^2 + k_z^2c^2 \quad \text{turning point: } k_z = 0$$

$$\omega_p^2(z_{tp}) = \omega^2 - k_{x0}^2c^2$$

Example reflection of waves by the ionosphere