ENEE681

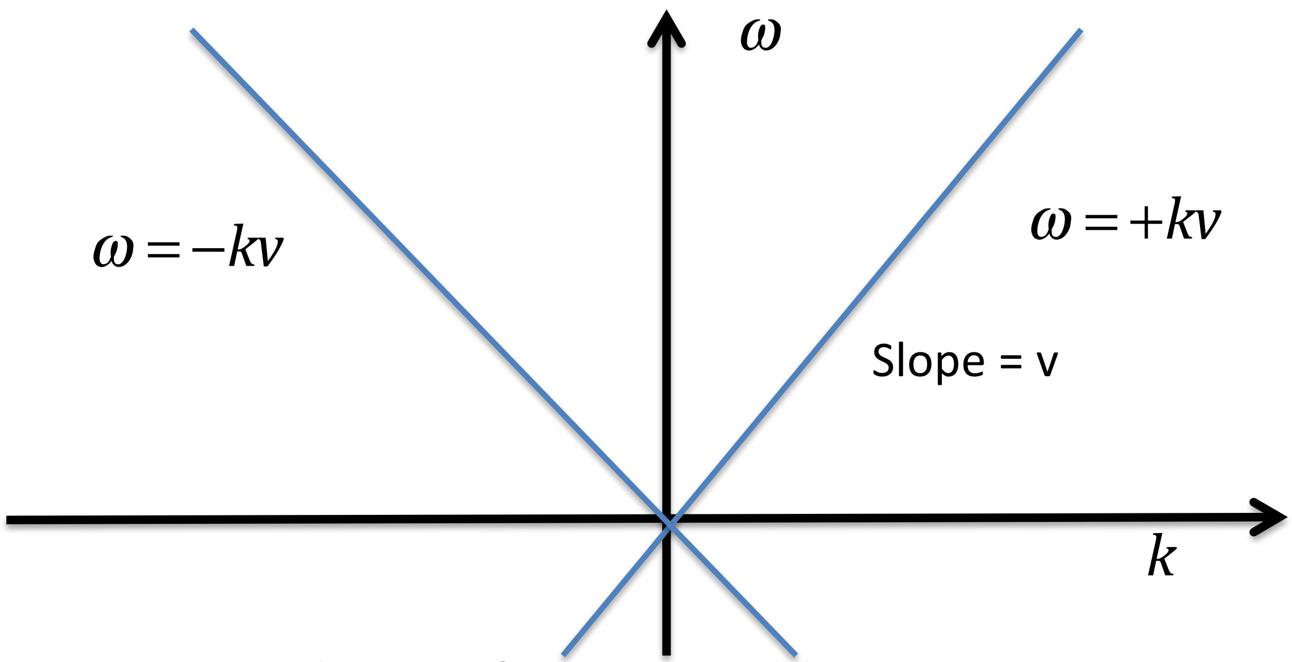
Lecture 7
Dispersion
Absorption
Skin Effect
Group and Phase Velocity

Dispersion Relation

Plane waves in a nondispersive medium

$v = 1/\sqrt{\varepsilon\mu}$

ε, μ independent of frequency



$\omega = +kv$

$\lambda = v/f$

Slope of omega vs k is independent of k or omega

$\lambda = 2\pi/k$

$\omega = 2\pi f$

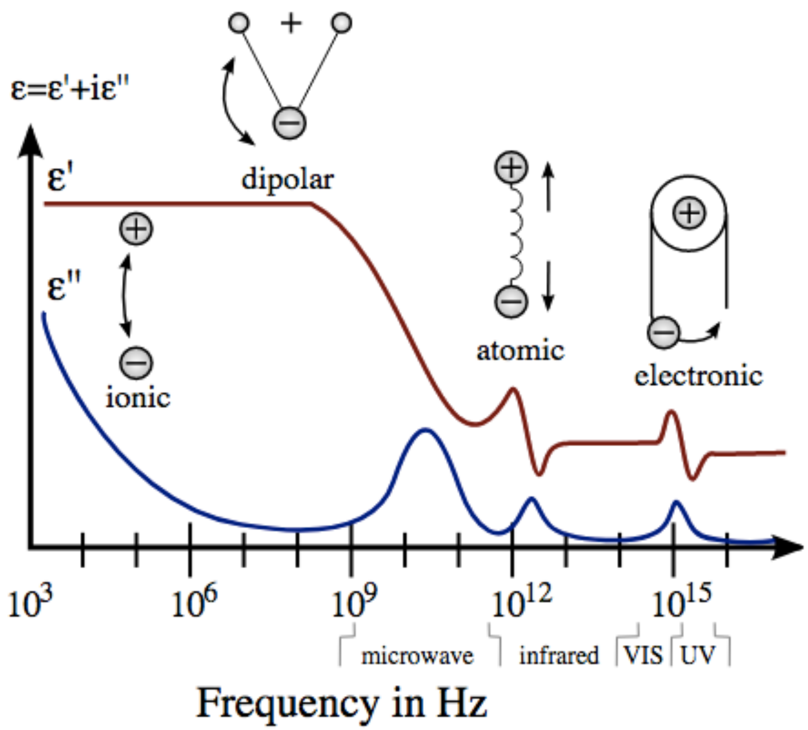
Pulses maintain their shape while propagating

Dispersion

$$\omega^2 = \frac{k^2}{\varepsilon\mu} = k^2 v^2$$

But in reality

$$\varepsilon = \varepsilon(\omega)$$
$$\mu = \mu(\omega)$$
$$v = v(\omega)$$



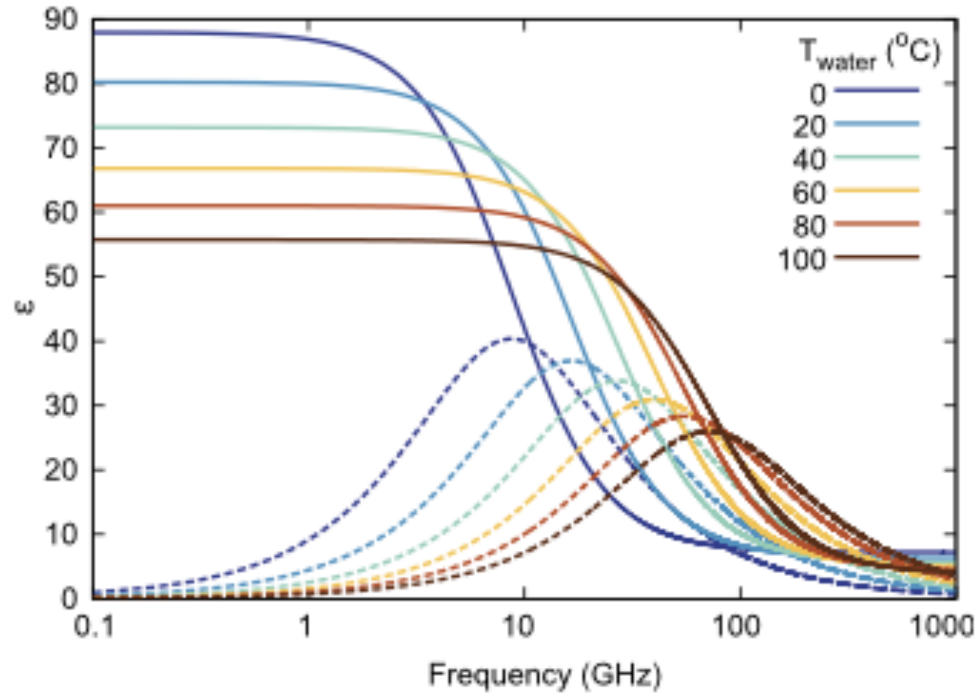
Different frequencies propagate with different speeds

Group velocity dispersion - GVD

Dielectric function is complex
Absorption

https://en.wikipedia.org/wiki/Permittivity
"Dielectric Spectroscopy". Archived from the original on 2006-01-18. Retrieved 2018-11-20.

Dielectric Function for H_2O



Water: Promising Opportunities For Tunable All-dielectric Electromagnetic Metamaterials

Article *in* Scientific Reports · August 2015

DOI: 10.1038/srep13535

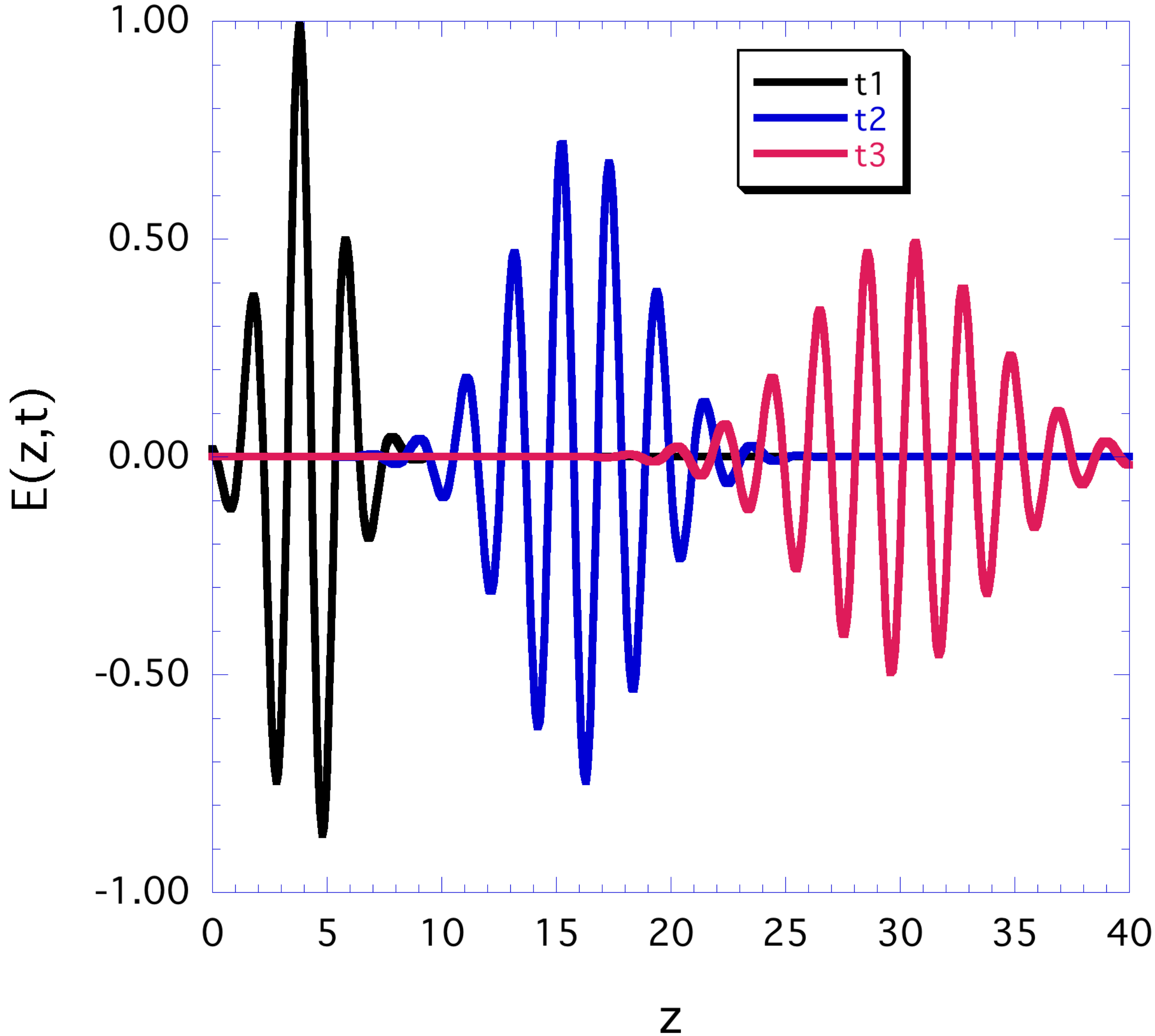
Dispersion and Attenuation

Pulses contain a spectrum of frequencies.

In dispersive media different frequency components propagate with different speeds.

Pulses spread out.

Losses lead to attenuation



Simple Models

Suppose you have a dielectric material that also has a conductivity?

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{free} + \frac{\partial \varepsilon \vec{\mathbf{E}}}{\partial t}$$

$$\vec{\mathbf{J}}_{free} = \sigma \vec{\mathbf{E}}$$ Ohm's Law

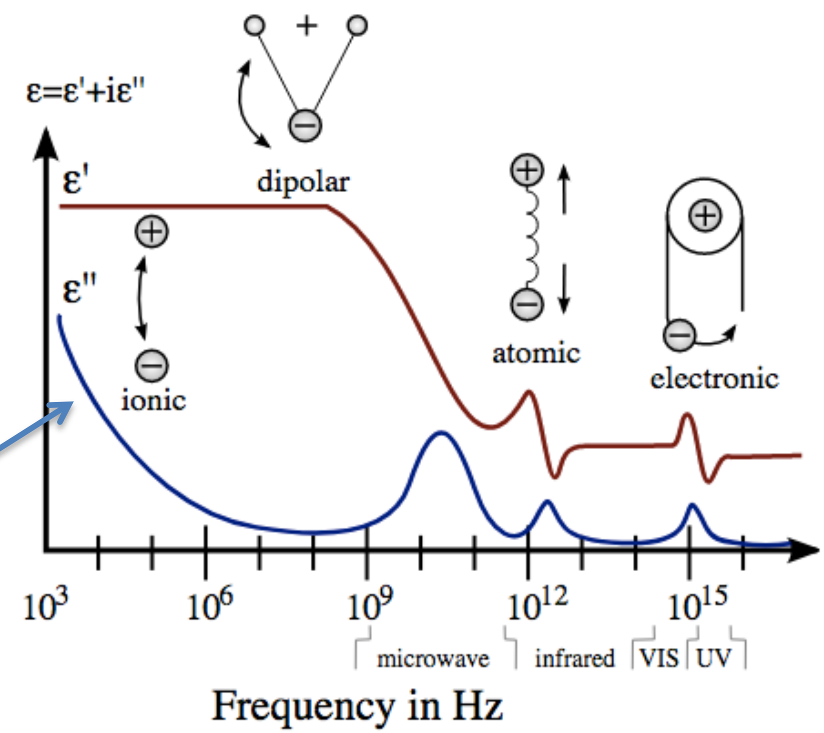
Pass to Phasor Representation

$$i\mathbf{k} \times \hat{\mathbf{H}} = (\sigma - i\omega\varepsilon)\hat{\mathbf{E}}$$

$$= i\omega\left(\varepsilon + i\frac{\sigma}{\omega}\right)\hat{\mathbf{E}}$$

Effective Dielectric function

$$\varepsilon' + i\varepsilon'' = \left(\varepsilon + i\frac{\sigma}{\omega}\right)$$



Simple model of polarizable material

Atom/molecule

Displacement

$\mathbf{x}_d(t)$  q, m

Spring constant- $m\omega_0^2$

Restoring force — friction force — driving force

Newton's law

$$m\frac{d^2}{dt^2}\mathbf{x}_d(t) = -m\omega_0^2\mathbf{x}_d(t) - m\nu\frac{d}{dt}\mathbf{x}_d(t) + q\mathbf{E}(\mathbf{x},t)$$

Phasors

$$\left[\omega_0^2 - i\omega\nu - \omega^2\right]\hat{\mathbf{x}}_d = \frac{q}{m}\hat{\mathbf{E}}(\mathbf{x})$$

Current density

$$\mathbf{J} = qN\frac{d}{dt}\mathbf{x}_d(t) \qquad \hat{\mathbf{J}} = -i\omega qN\hat{\mathbf{x}}_d \qquad \hat{\mathbf{J}} - i\omega\varepsilon_0\hat{\mathbf{E}} = -i\omega\varepsilon_0\varepsilon_r\hat{\mathbf{E}}$$

Number density, m^{-3}

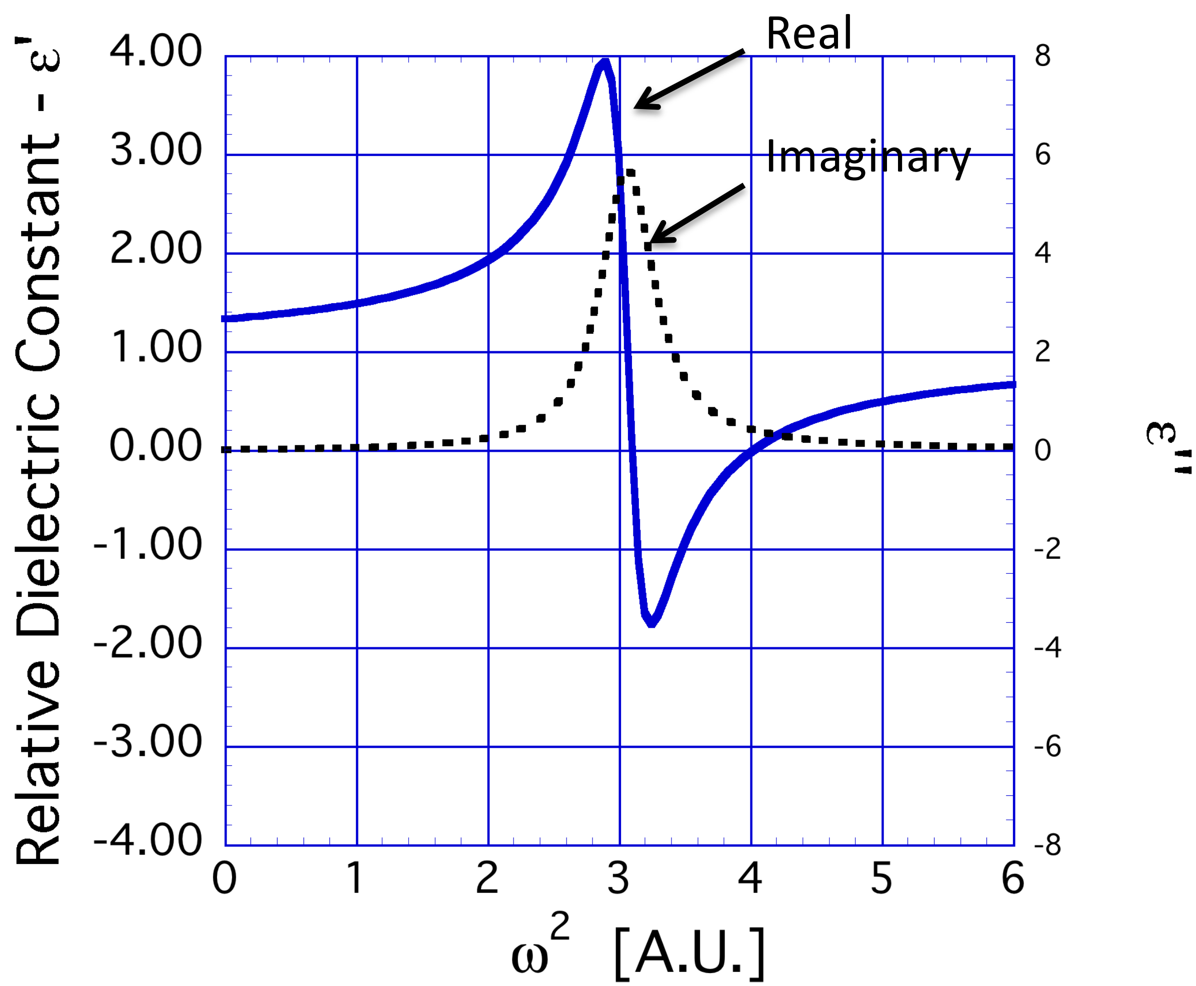
Complex relative dielectric function

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\nu - \omega^2}\right]$$

$$\omega_p^2 = \frac{Nq^2}{\varepsilon_0 m}$$

Model relative dielectric function

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\nu - \omega^2} \right]$$

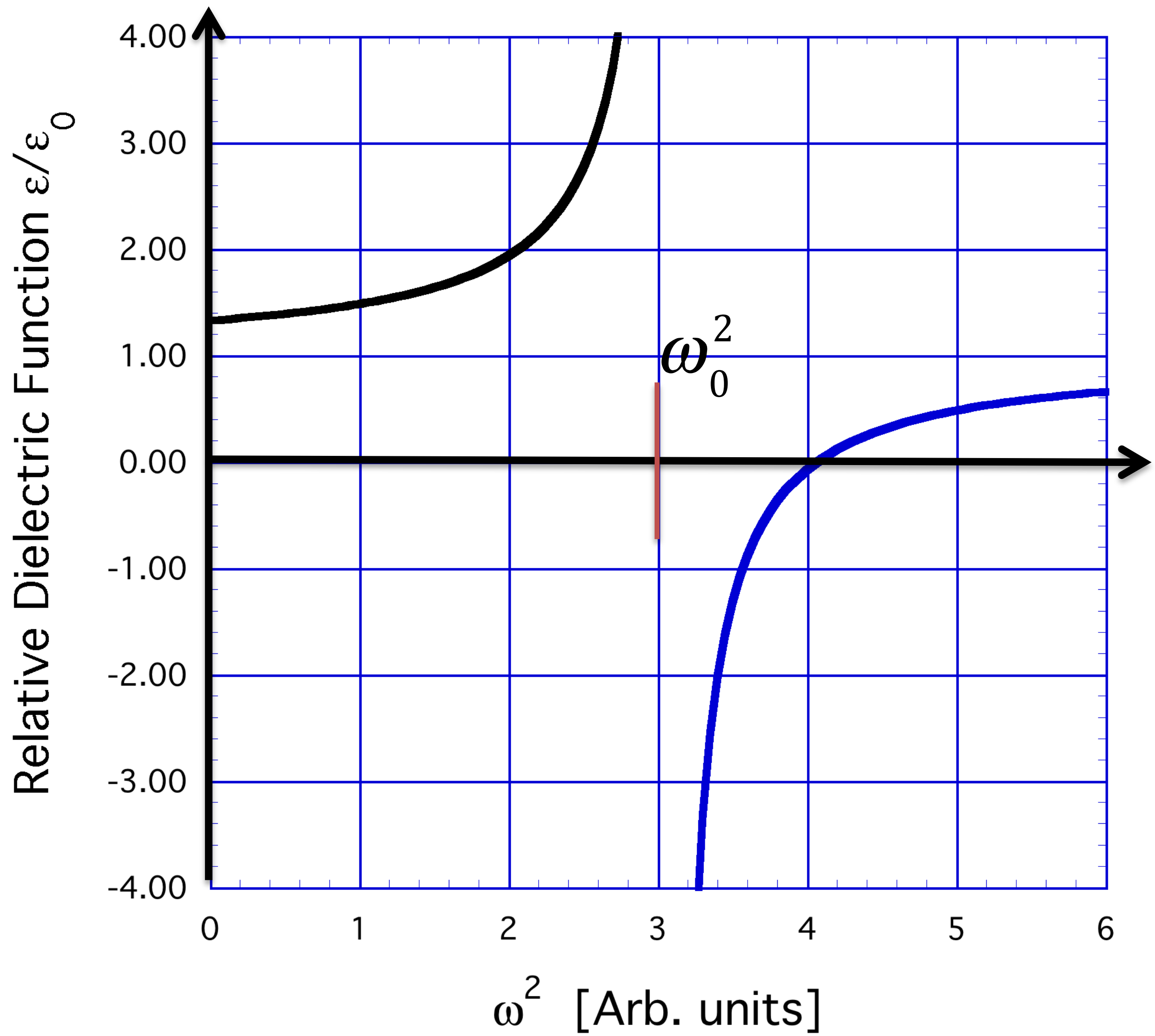


Simple Model for Dielectric Function

Loss free case

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right]$$

Restoring force



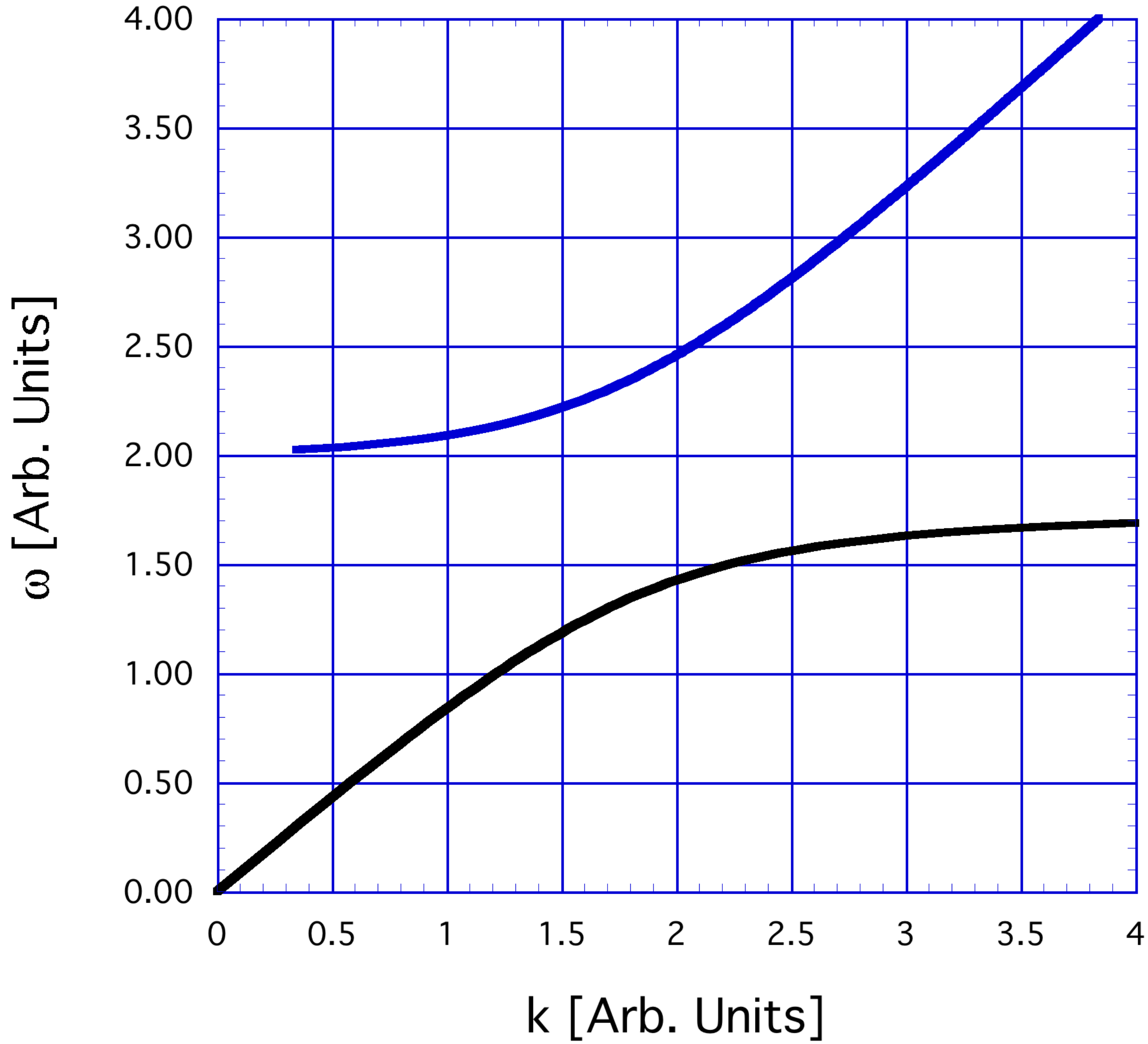
Dispersion Relation

$$\omega^2\left[1+\frac{\omega_p^2}{\omega_0^2-\omega^2}\right]=k^2c^2$$

Two modes

$\omega(k)$ vs k

not a straight line



Phase and Group Velocity

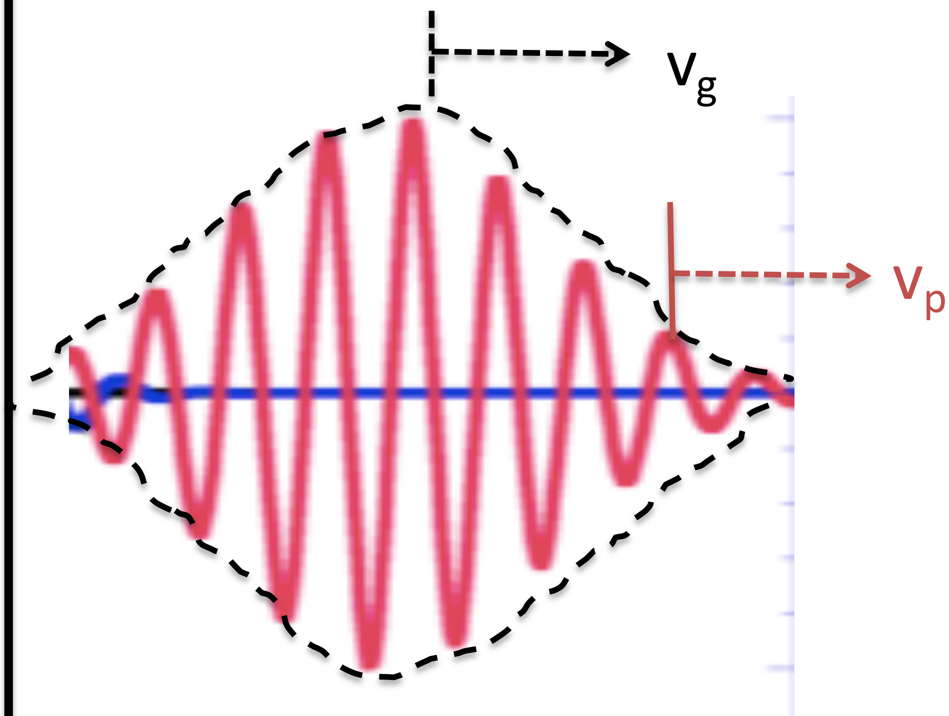
$$\omega\sqrt{\varepsilon(\omega)\mu_0} = k$$

The crests travel at the phase velocity

$$\mathrm{v}_p = \frac{\omega_c}{k(\omega_c)}$$

The envelope travels at the group velocity

$$\mathrm{v}_g = \left.\frac{d\omega}{dk}\right|_{\omega_c}$$



Group and Phase Velocity

Consider a superposition of two waves:

$$E = E_0 \cos\left(k_1 z - \omega_1 t\right) + E_0 \cos\left(k_2 z - \omega_2 t\right)$$

$$\omega_1 = \omega(k_1), \quad \omega_2 = \omega(k_2)$$

Trig identity::

$$\cos\left(A\right) + \cos\left(B\right) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

$$E = 2E_0 \cos\left(\bar{k}z - \bar{\omega}t\right)\cos\left(\Delta k z - \Delta\omega t\right)$$

$$\bar{k} = \frac{1}{2}\left(k_1 + k_2\right), \qquad \bar{\omega} = \frac{1}{2}\left(\omega_1 + \omega_2\right) = \frac{1}{2}\left(\omega(k_1) + \omega(k_2)\right)$$

$$\Delta k = \frac{1}{2}\left(k_1 - k_2\right), \qquad \Delta\bar{\omega} = \frac{1}{2}\left(\omega(k_1) - \omega(k_2)\right) \approx \Delta k \frac{d\omega}{d\bar{k}}$$

Carrier and Envelope

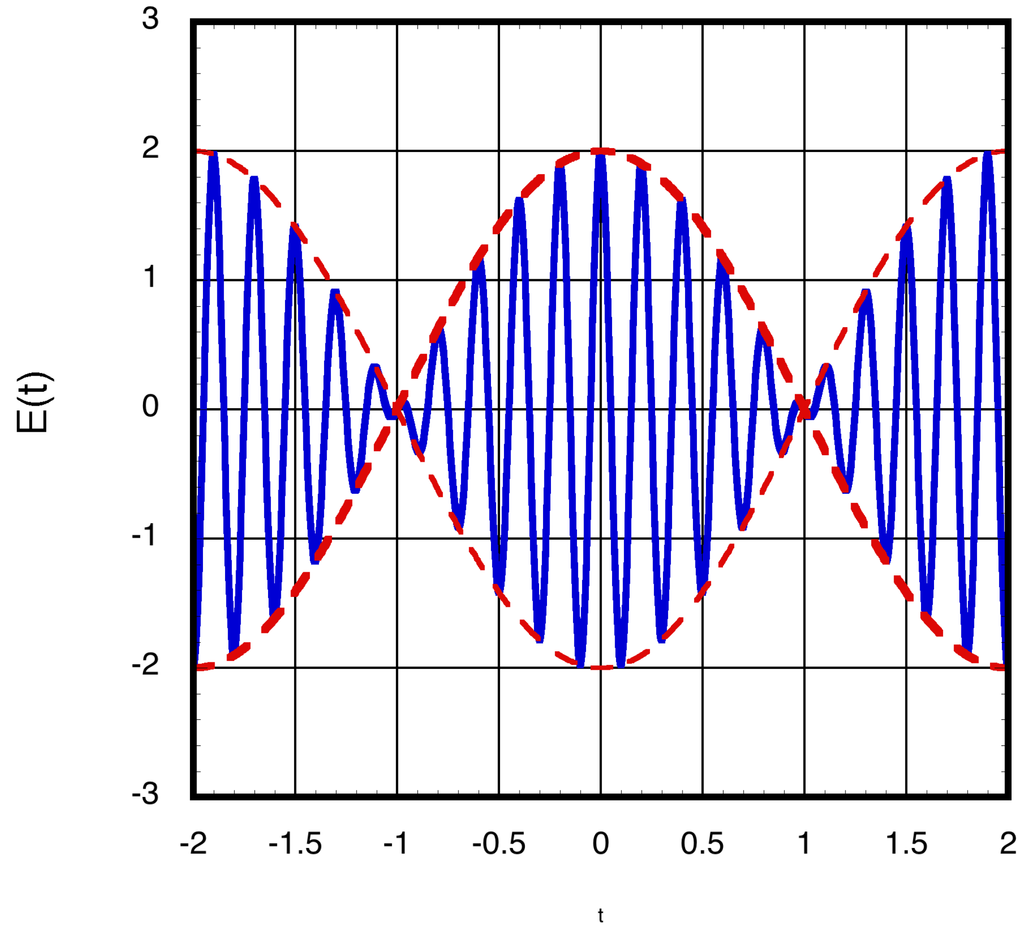
$$E = 2E_0 \cos\left(\bar{k}z - \bar{\omega}t\right)\cos\left(\Delta k z - \Delta\omega t\right)$$

Carrier envelope

Speed of carrier: $\bar{\omega} / \bar{k} = v_{phase}$

Speed of envelope: $\Delta\omega / \Delta \mathrm{k}$

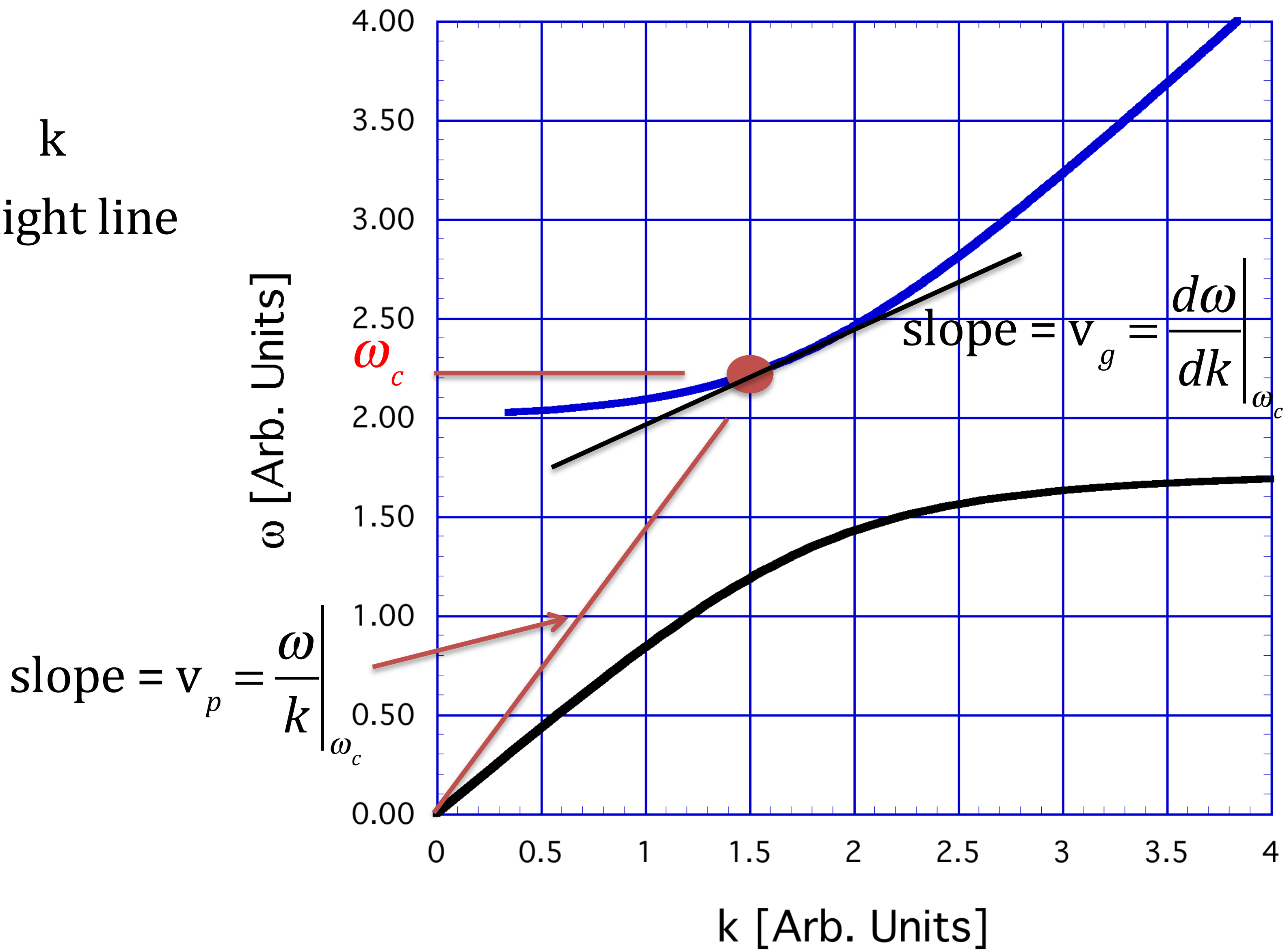
$\simeq \mathrm{d}\omega(\mathrm{k})/\mathrm{dk} = \mathrm{v}_{group}$



Dispersion Relation

$\omega(k)$ vs k

not a straight line



Problem

The dielectric function for a plasma is

$$\varepsilon(\omega)=\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right),\quad \text{where } \omega_p^2=\frac{ne^2}{m\varepsilon_0},\ n \text{ is the electron number density.}$$

ω_p is called the plasma frequency.

Find the dispersion relation $\omega(k)$, also find v_p, v_g

Problem

The dielectric function for a plasma is

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ω_p is called the plasma frequency.

Find the dispersion relation $\omega(k)$, also find v_p, v_g

$$k^2 = \omega^2 \varepsilon\mu_0 = \omega^2\varepsilon_0\mu_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \frac{1}{c^2}\left(\omega^2 - \omega_p^2\right)$$

$$v_p = \frac{\omega}{k} = \frac{c\omega}{\left(\omega^2 - \omega_p^2\right)^{1/2}} = \frac{c}{\left(1 - \omega_p^2/\omega^2\right)^{1/2}}$$

rewriting the dispersion rellation

$$\omega = \left(k^2c^2 + \omega_p^2\right)^{1/2}$$

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\left(k^2c^2 + \omega_p^2\right)^{1/2}} = c\left(1 - \omega_p^2/\omega^2\right)^{1/2}$$

How to represent a pulse

At z=0,

$$E(z=0,t)=\mathrm{Re}\left\{\hat{E}_e(z=0,t)\exp\left[-i\omega_c t\right]\right\}$$

$\hat{E}_e(z=0,t)$ - time varying envelope

ω_c - carrier frequency

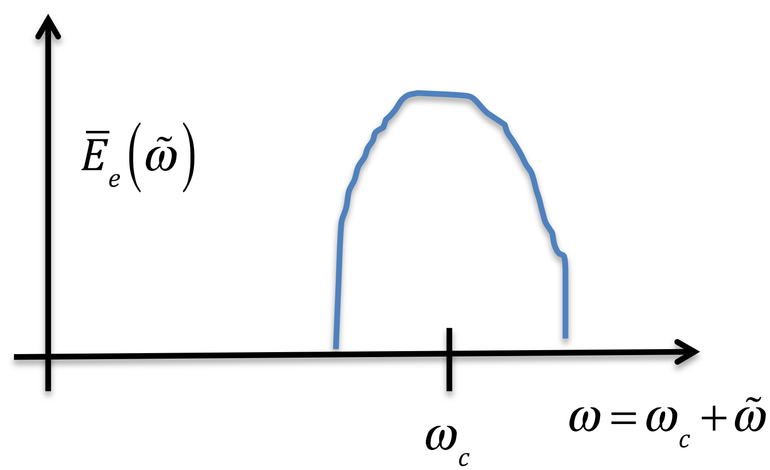
The same waveform can be represented as a Fourier integral

$$\hat{E}_e(z=0,t)\exp\left[-i\omega_c t\right]=\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Fourier transform of time varying envelope

$\omega_c+\tilde{\omega}$ - frequency

Spectrum is narrow and peaked at the carrier frequency



After Propagation

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right] = \int \frac{d\tilde{\omega}}{2\pi} \bar{E}_e(\tilde{\omega}) \exp\left[ik(\omega_c+\tilde{\omega})z - i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Peaked at $\tilde{\omega}=0$

Taylor expand $k(\omega_c+\tilde{\omega}) = k(\omega_c) + \left.\frac{dk}{d\omega}\right|_{\omega_c} \tilde{\omega} + ...$

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right] = \exp\left[i\left(k(\omega_c)z - \omega_c t\right)\right] \int \frac{d\tilde{\omega}}{2\pi} \bar{E}_e(\tilde{\omega}) \exp\left[-i\tilde{\omega}\left(t - \left.\frac{dk}{d\omega}\right|_{\omega_c} z\right)\right]$$

Carrier Wave

Envelope

$\hat{E}_e(z=0, t-z/v_g)$,

$v_g = d\omega/dk\big|_{\omega_c}$ the group velocity

The pulse envelope retains its shape, but travels at a different speed than the crests of the carrier.

Energy and Momentum of Light

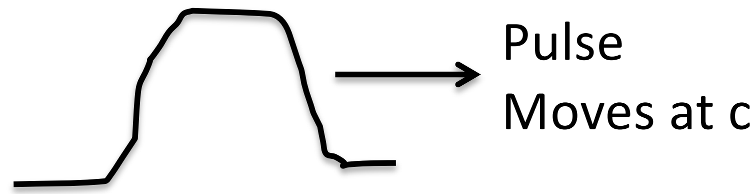
$$\left(\frac{\varepsilon_0\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\left|\vec{\mathbf{B}}\right|^2}{2\mu_0}\right)$$

Energy density

$$\vec{\mathbf{S}}=\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}\right)$$

Power Flux

Units: Joules/m^3 Watts/m^2



Power Flux = c Energy Density

EM linear momentum density: $\varepsilon_0\mathbf{E}\times\mathbf{B}=\mathbf{S}/c^2$

$$\frac{\text{Energy Density}}{\text{Momentum Density}}=\frac{S/c}{S/c^2}=c$$

A pulse of light carries energy and momentum: ratio = c

Energy and Momentum

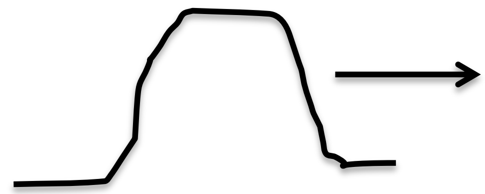
$$\left(\frac{\varepsilon|\vec{\mathbf{E}}|^2}{2}+\frac{\mu|\vec{\mathbf{H}}|^2}{2}\right)$$

Energy density

$$\vec{\mathbf{S}}=\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}\right)$$

Power Flux

Units: Joules/m^3 Watts/m^2



Pulse
Moves at v_g

Power Flux = v_g Energy Density

Energy Density

$$U=\int d^3x\left(\frac{\partial(\omega\varepsilon(\omega))}{\partial\omega}\frac{|\vec{\mathbf{E}}|^2}{2}+\frac{\partial(\omega\mu(\omega))}{\partial\omega}\frac{|\mathbf{H}|^2}{2}\right)$$

Energy Stored in Medium

magnetization current

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_m + \mathbf{J}_p$$

polarization current

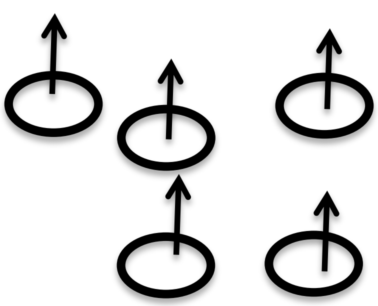
"Free" current

polarization charge density

$$\rho = \rho_f + \rho_p$$

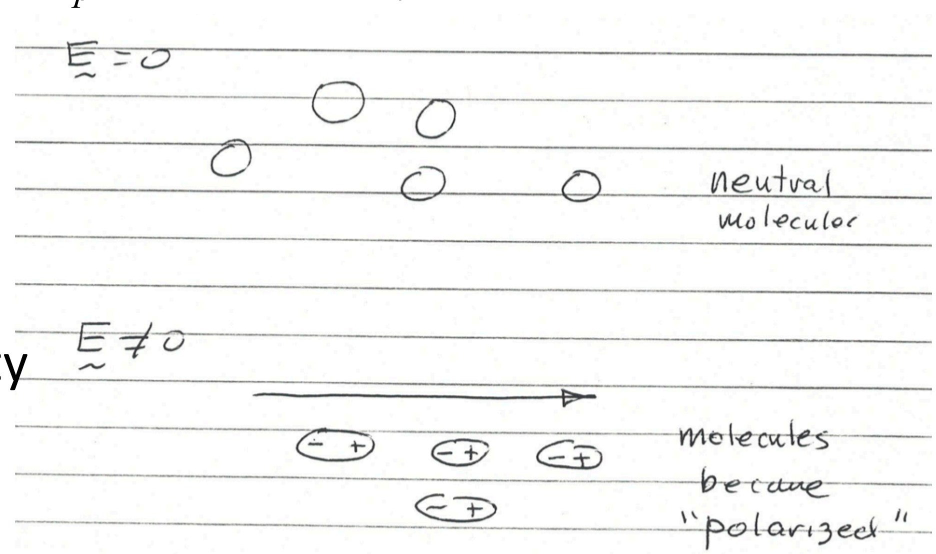
"Free" charge density

$$\bar{\mathbf{J}}_p(\omega) = -i\omega\varepsilon_0\chi(\omega)\bar{\mathbf{E}}(\omega)$$



$$\mathbf{J}_m = \nabla \times \mathbf{M}$$

magnetization density



Rate of change of Energy in Dielectric

$\mathbf{E}\cdot\mathbf{J}_p$ = rate at which energy is transferred to polarization current.

Energy density stored in medium $\quad u_\chi = \int_{-\infty}^{0} dt\, \mathbf{E}(t)\cdot\mathbf{J}_p(t)$

Now suppose E field grows in time $\mathbf{E}(t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[-i\left(\omega_r + i\gamma\right)t\right]\right\} \quad \gamma > 0$

Then $\quad \mathbf{J}_p(t) = \mathrm{Re}\left\{-i\left(\omega_r + i\gamma\right)\varepsilon_0\chi\left(\omega_r + i\gamma\right)\hat{\mathbf{E}}\exp\left[-i\left(\omega_r + i\gamma\right)t\right]\right\}$

$$\bar{\mathbf{J}}_p(\omega) = -i\omega\varepsilon_0\chi(\omega)\bar{\mathbf{E}}(\omega)$$

$$u_\chi = \int_{-\infty}^{0} dt\, \mathbf{E}(t)\cdot \mathbf{J}_p(t)$$

$$u_\chi = \frac{1}{2}\varepsilon_0 \operatorname{Re}\left\{ \int_{-\infty}^{0} dt\, \hat{\mathbf{E}}^* \exp\left[+i\left(\omega_r - i\gamma\right)t\right]\cdot\left[-i\left(\omega_r + i\gamma\right)\chi\left(\omega_r + i\gamma\right)\hat{\mathbf{E}}\exp\left[-i\left(\omega_r + i\gamma\right)t\right]\right]\right\}$$

$$u_\chi = \frac{1}{2}\varepsilon_0 \operatorname{Re}\left\{ \int_{-\infty}^{0} dt\, \hat{\mathbf{E}}^* \cdot \hat{\mathbf{E}} \exp\left[2\gamma t\right]\cdot\left[-i\left(\omega_r + i\gamma\right)\chi\left(\omega_r + i\gamma\right)\right]\right\} = \frac{1}{2}\varepsilon_0 \left|\hat{\mathbf{E}}\right|^2 \operatorname{Re}\left\{ \frac{\left[-i\left(\omega_r + i\gamma\right)\chi\left(\omega_r + i\gamma\right)\right]}{2\gamma}\right\}$$

Now assume slow growth $\gamma \to 0$

$$\frac{\left[-i\left(\omega_r + i\gamma\right)\chi\left(\omega_r + i\gamma\right)\right]}{2\gamma} \to -\frac{i\omega_r}{2\gamma} + \frac{1}{2}\frac{\partial}{\partial\omega_r}\left(\omega_r \chi\left(\omega_r\right)\right)$$

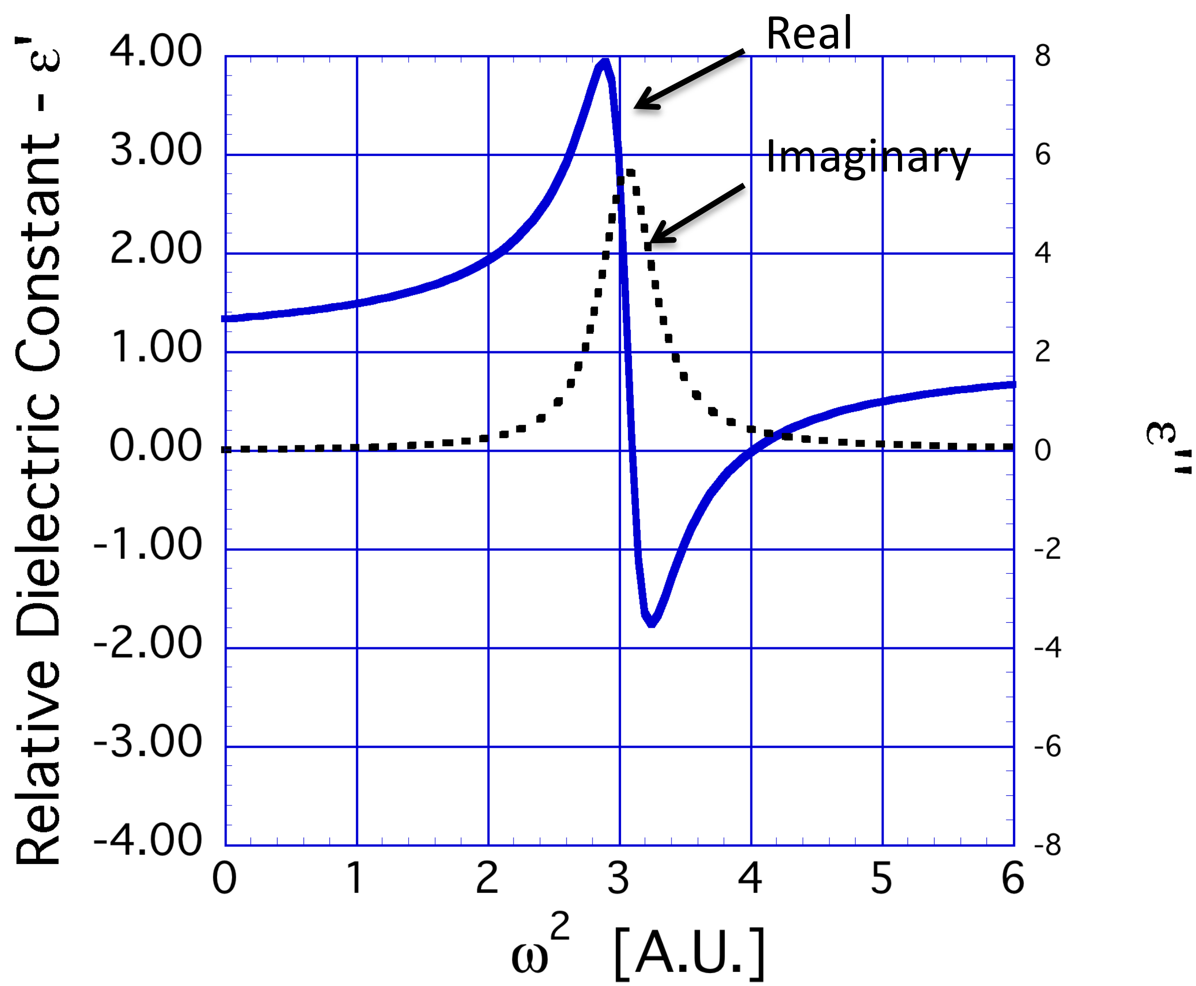
$$u_\chi = \frac{1}{4}\left|\hat{\mathbf{E}}\right|^2 \frac{\partial}{\partial\omega_r}\left(\omega_r \varepsilon_0 \chi\left(\omega_r\right)\right)$$

Energy stored in E-field $u_E = \frac{1}{4}\varepsilon_0 \left|\hat{\mathbf{E}}\right|^2 = \frac{1}{4}\left|\hat{\mathbf{E}}\right|^2 \frac{\partial}{\partial\omega_r}\left(\omega_r \varepsilon_0\right)$

$$u_\chi + u_E = \frac{1}{4}\left|\hat{\mathbf{E}}\right|^2 \frac{\partial}{\partial\omega_r}\left(\omega_r \varepsilon\left(\omega_r\right)\right)$$

Model relative dielectric function

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\nu - \omega^2} \right]$$



Krammers-Kronig Relation

Real and imaginary parts of dielectric function are related

Convolution Response $D(t) = \int_{-\infty}^{\infty} dt' E(t') K(t-t'),$ Causality requires $K = 0, \ t - t' < 0$

Fourier Transform (physics) $\bar{D}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} D(t) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' e^{i\omega t'} E(t') e^{i\omega t - i\omega t'} K(t-t') = \varepsilon(\omega)\bar{\mathrm{E}}(\omega)$

$$\varepsilon(\omega) = \int_{-\infty}^{\infty} dt K(t) e^{i\omega t} = \int_{0}^{\infty} dt K(t) e^{i\omega t}$$ If Im(ω)>0 Integral converges for bounded $K(t)$.

Thus, $\varepsilon(\omega)$ has no poles in upper half ω- plane.

Complex Analytic Functions

Suppose $f(z)$ is an analytic function of complex variable $z = x + iy$

If $f(z)$ has no singularities (poles, branch cuts, etc) in a region of the complex plane of z then

$\oint f(z)\,dz = 0$ where the integral is taken around a closed contour

If $f(z)$ has simple pole at z_1 $f(z) \sim \dfrac{N_1}{z - z_1}$

$\oint f(z)\,dz = 2\pi i N_1$ where N_1 is called the residue, Cauchy Residue Theorem-Integral closed in upper half plane

$$\varepsilon(\omega) = \oint \frac{d\omega'}{2\pi i}\frac{\varepsilon(\omega')}{\omega'-\omega} = P\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi i}\frac{\varepsilon(\omega')}{\omega'-\omega} + \frac{\pi i}{2\pi i}\varepsilon(\omega) \quad \text{Thus: } \varepsilon(\omega) = P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi i}\frac{\varepsilon(\omega')}{\omega'-\omega}$$

$$\varepsilon = \varepsilon' + i\varepsilon'' \qquad \varepsilon'(\omega) = P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\varepsilon''(\omega')}{\omega'-\omega} \qquad \varepsilon''(\omega) = -P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\frac{\varepsilon'(\omega')}{\omega'-\omega}$$

Modification to dispersion relation

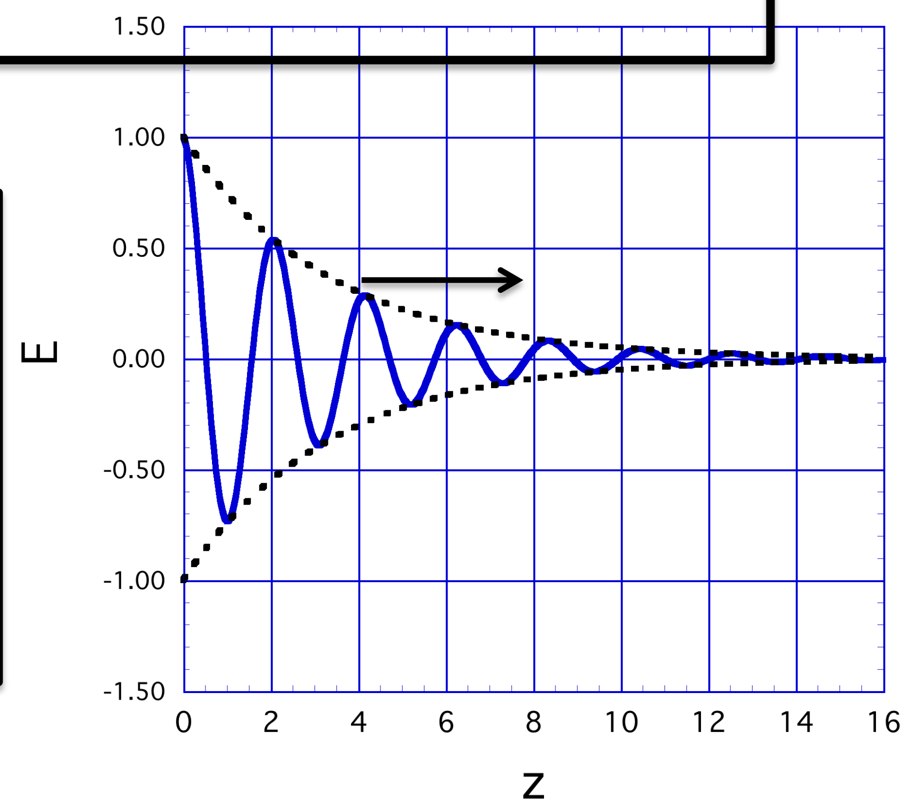
$$k^2 = \omega^2 \varepsilon\mu = \omega^2 \mu\left(\varepsilon' + i\varepsilon''\right)$$

If frequency is real, (controlled by source of waves), k must be complex.

$$k = k' + ik''$$

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(k' + ik''\right)z - i\omega t\right]\right\}$$

$$= \exp\left[-k''z\right]\mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[ik'z - i\omega t\right]\right\}$$



Two limiting cases: 1. weak damping $\varepsilon' >> \varepsilon''$

2. good conductor $\varepsilon'' = \dfrac{\sigma}{\omega} >> \varepsilon$

Weak damping $k^2 = \omega^2 \mu \varepsilon' \left(1 + i\dfrac{\varepsilon''}{\varepsilon'}\right)$ $\quad 1 >> \dfrac{\varepsilon''}{\varepsilon'}$ loss tangent

$$k = k' + ik'' \qquad k'' << k'$$

$$k^2 = (k' + ik'')^2 = k'^2 - k''^2 + 2ik'k'' \qquad \simeq k'^2 + 2ik'k''$$

$$k'^2 = \omega^2 \mu \varepsilon' \qquad 2k'k'' = \omega^2 \mu \varepsilon''$$

$$\mathbf{E} = \mathrm{Re}\left\{\hat{\mathbf{E}} \exp\left[i(k' + ik'')z - i\omega t\right]\right\}$$
$$= \exp[-k''z] \mathrm{Re}\left\{\hat{\mathbf{E}} \exp\left[ik'z - i\omega t\right]\right\}$$

Decay per wavelength determined by loss tangent

$$\frac{k''}{k'} = \frac{1}{2}\frac{\varepsilon''}{\varepsilon'}$$

Good Conductor

$$k^2 = \omega^2 \mu \left(\varepsilon + i \frac{\sigma}{\omega} \right) \simeq i\omega\mu\sigma$$

$$k = k' + ik'' \qquad k'' = k' = \sqrt{\frac{\omega\mu\sigma}{2}}$$

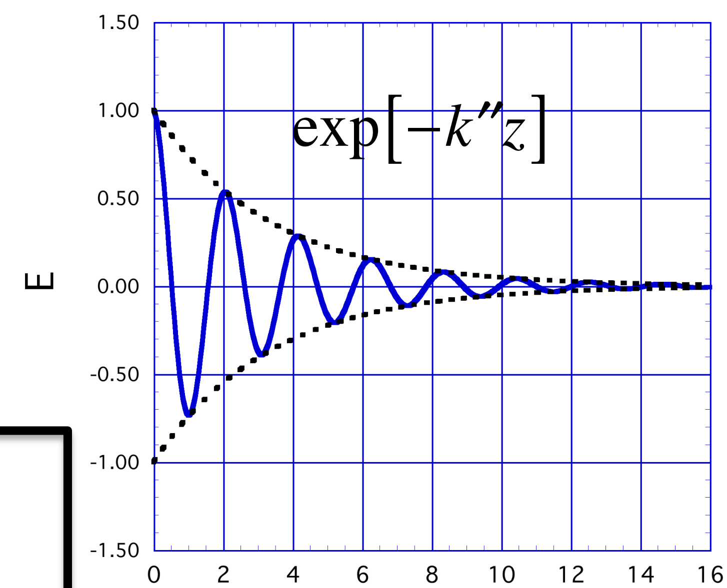
$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

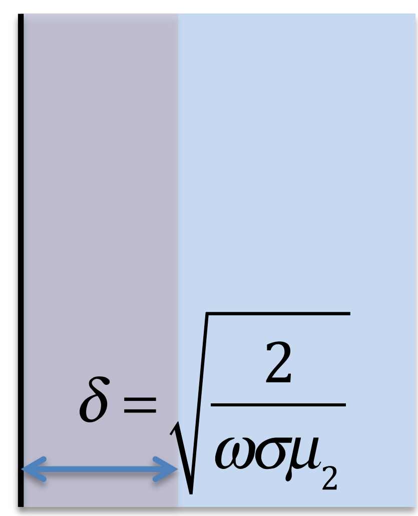
Current flows in a layer

$$\delta = k''^{-1} = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

Surface Impedance

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$





Skin Depth

$$\delta = k''^{-1} = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

TABLE 8–1
Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

† The ϵ of seawater is approximately $72\epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega\epsilon \cong 1$ (not $\gg 1$). Under these conditions, seawater is not a good conductor, and Eq. (8–57) is no longer applicable.

Surface Impedance

Ratio of tangential E to tangential H at surface

$$\frac{E_{\tan}}{H_{\tan}} = \eta_2$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

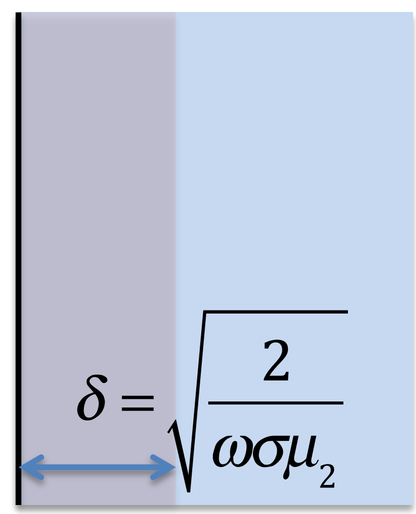
$$\eta_2 = Z_s = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

$$Z_s = (1-i)\sqrt{\frac{\omega\mu_2}{2\sigma}} = (1-i)R_s$$

Resistance + i Reactance

Poynting Flux

$$S_z = \frac{1}{2}\mathrm{Re}\left\{E^*_{\tan} H_{\tan}\right\} = \frac{1}{2}R_s\left|H_{\tan}\right|^2$$

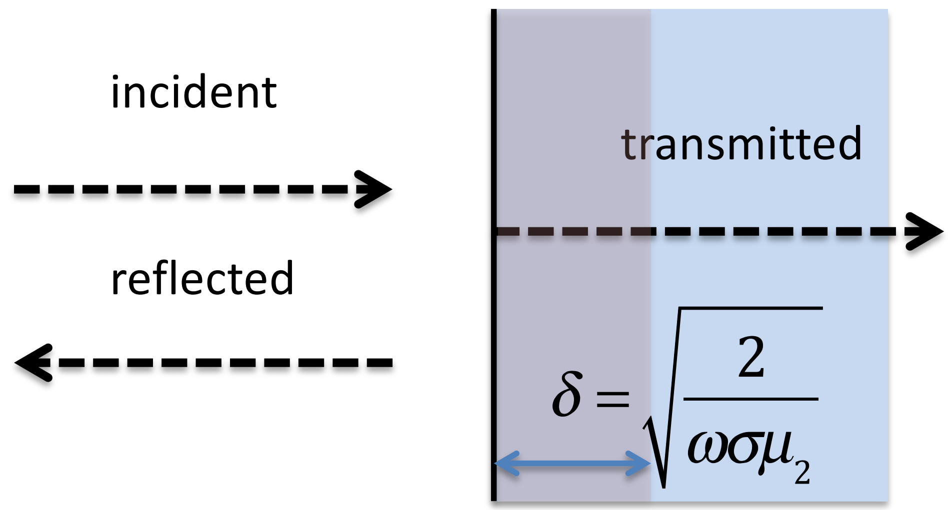


$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

Reflection from conductor

Formally the same as reflection from dielectric. Just use the surface impedance

$$\rho = \frac{Z_s - \eta_1}{Z_s + \eta_1}$$



$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

$$\eta_2 = Z_s = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

$$Z_s = (1-i)\sqrt{\frac{\omega\mu_2}{2\sigma}} \equiv (1-i)R_s$$

Calculate power transmission coefficient

$$T = 1 - R = 1 - \left|\rho\right|^2$$

Assume $\left|Z_s\right| << \eta_1$

$$S_z = \frac{1}{2}\text{Re}\left\{E^*_{\tan} H_{\tan}\right\} = \frac{1}{2} R_s \left|H_{\tan}\right|^2$$

$$H_{\tan} = 2\hat{H}_{inc}$$

$$\boxed{S_z = \frac{1}{2}\frac{R_s}{\eta_1} 4\eta_1 \left|\hat{H}_{inc}\right|^2 = 4\frac{R_s}{\eta_1} P_{inc}}$$

Group Velocity Dispersion (GVD)

Different frequencies propagate with different group velocities
Evolution of pulse envelope

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right]=\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[ik(\omega_c+\tilde{\omega})z-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Peaked at $\tilde{\omega}=0$

Taylor expand $k(\omega_c+\tilde{\omega})=k(\omega_c)+\left.\frac{dk}{d\omega}\right|_{\omega_c}\tilde{\omega}+....$

Remember, we stopped at first order

Let's go one order higher

$$k(\omega_c+\tilde{\omega})=k(\omega_c)+\left.\frac{dk}{d\omega}\right|_{\omega_c}\tilde{\omega}+\frac{1}{2}\tilde{\omega}^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}$$

$$\frac{d^2k}{d\omega^2}=-\frac{1}{v_g^2}\frac{d}{d\omega}v_g$$

Gaussian Pulse Envelope

Pulse width - τ

Chirp- Ω'

$$\hat{E}_e = E_0 \exp\left[-\frac{t^2}{\tau^2} - i\Omega' \frac{t^2}{2} \right]$$

Instantaneous frequency - $\frac{d}{dt}\Omega'\frac{t^2}{2} = \Omega' t$

If $\Omega' > 0$ Low frequencies come before high frequencies

If $\Omega' < 0$ High frequencies come before low frequencies

Fourrier Transform

$$\bar{E}(\tilde{\omega}) = \int_{-\infty}^{\infty} dt \hat{E}_e(t) \exp\left[i\tilde{\omega} t \right] = \pi^{1/2} \tau_c E_0 \exp\left[-\frac{\tilde{\omega}^2 \tau_c^2}{4} \right]$$

$$\tau_c^2 = \frac{\tau^2}{1 + i\Omega' \tau^2 / 2}$$

Inverse Transform

$$\hat{E}_e(z,t)\exp\left[ik(\omega_c)z - i\omega_c t\right] =$$

$$\int \frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[iz\left(\frac{dk}{d\omega}\tilde{\omega} + \frac{d^2k}{d\omega^2}\frac{\tilde{\omega}^2}{2}\right) - i\left(\omega_c + \tilde{\omega}\right)t\right]$$

$$\hat{E}_e(z,t) = \frac{\tau_c}{\tau_s}E_0\exp\left[-\frac{\left(t - z/v_g\right)^2}{\tau_s^2}\right]$$

$$\tau_c^2 = \frac{\tau^2}{1 + i\Omega'\tau^2/2}$$

$$\tau_s^2 = \tau_c^2 - 2iz\frac{d^2k}{d\omega^2}$$

You will be asked to make plots for HW

Simple Case: no initial chirp

$$\hat{E}_e(z,t) = \frac{\tau_c}{\tau_s} E_0 \exp\left[-\frac{\left(t - z/v_g\right)^2}{\tau_s^2} \right]$$

$$\tau_c^2 = \frac{\tau^2}{1 + i\Omega'\tau^2/2} = \tau^2$$

$$\tau_s^2 = \tau_c^2 - 2iz\frac{d^2k}{d\omega^2} = \tau^2 - i(2zk'')$$

$$\hat{E}_e(z,t) = \frac{\tau_c}{\tau_s} E_0 \exp\left[-\frac{\left(t - z/v_g\right)^2}{\tau^4 + (2zk'')^2}\left(\tau^2 + i(2zk'')\right) \right]$$

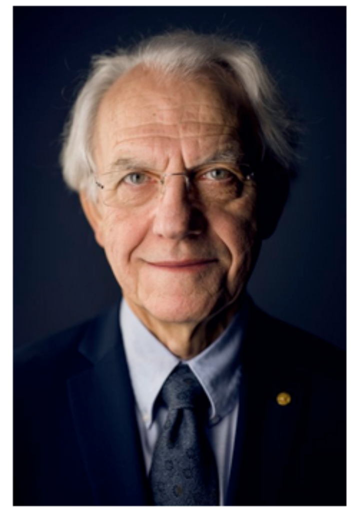
Pulse Width: $\tau_w(z) = \sqrt{\tau^2 + (2zk'')^2/\tau^2}$

Chirp: $\Omega'(\mathrm{z}) = 4(zk'')/\left(\tau^4 + (2zk'')^2\right)$

CPA – 2018 Nobel Prize in Physics

Gérard Mourou and Donna Strickland “for their method of generating high-intensity, ultra-short optical pulses”

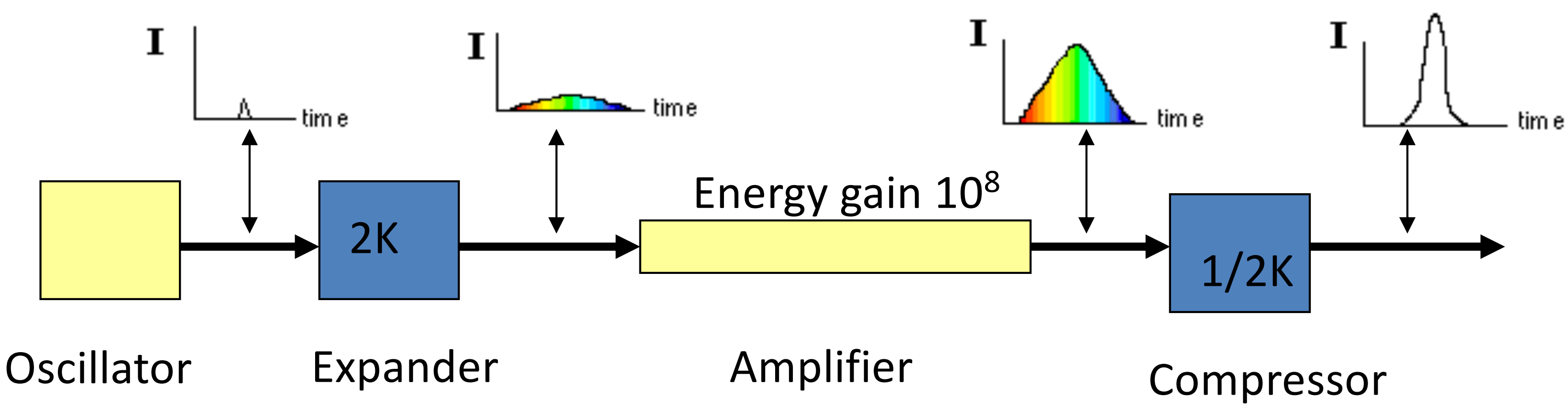
https://www.nobelprize.org/prizes/physics/2018/strickland/facts/

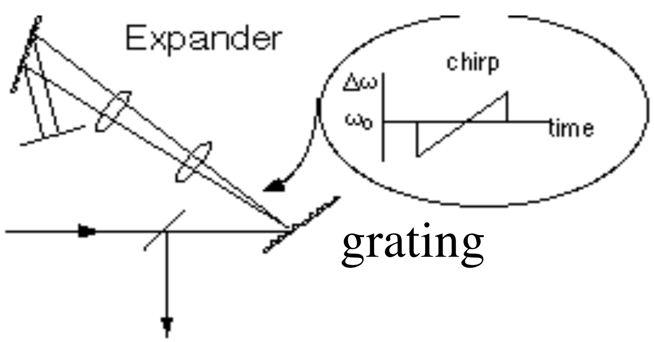


https://www.nobelprize.org/prizes/physics/2018/mourou/facts/

T^3 Lasers - (Table Top Terawatt) Ultra- High Intensity -CPA

CPA: Chirped Pulse Amplification,

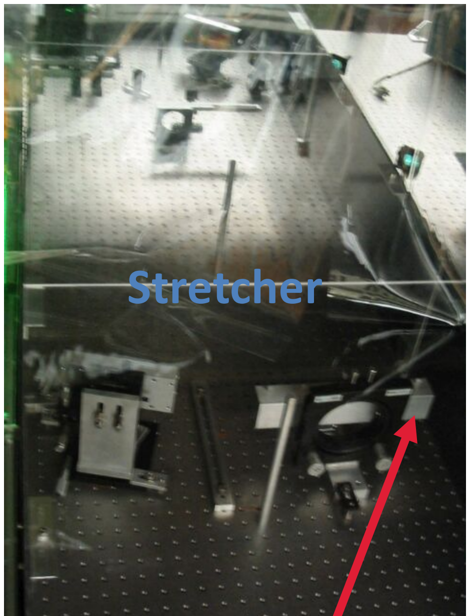




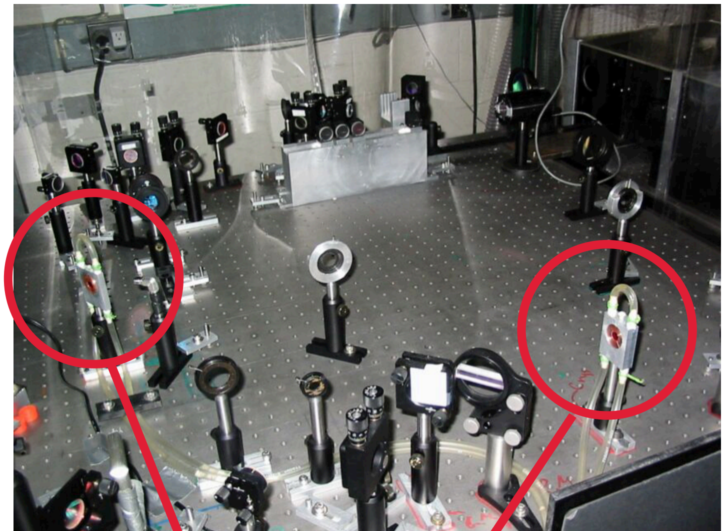
Sample parameters:

Pulse energy	1 Joule		
Pulse Duration	100 fsec	=	1×10^{-13} sec
Power	10 TW	=	$1 \times 10^{+13}$ W
Wave Length	1 μm	=	1×10^{-4} cm
Spot Size	10 μm	=	1×10^{-3} cm
Power Density	$6.4 \times 10^{+18}$ watts/cm^2		

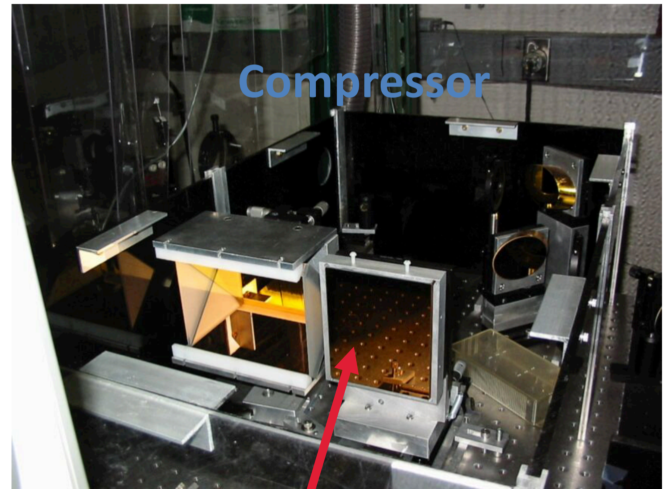
Realization at UMD – H. Milchberg



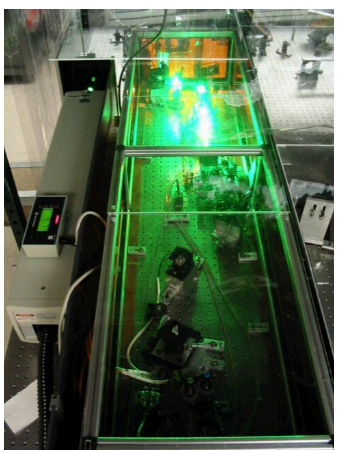
Grating



Power Amplifiers



Grating



Oscillator

Experimental chamber

Applications of Short Pulse Lasers

Review: G. A. Mourou, C. P. J. Barty, and M. D. Perry, Physics Today **51**, 22 (1998).

- **Particle Acceleration:** Use laser as a driver of plasma waves. Ultra High gradient 50 Gev/m (50 Mev/mm)

- **Ultra-short wavelength radiation:** generate high harmonics in nonlinear media
  soft X-rays in cluster gasses (Ditmire, Milchberg)

- **ICF fast Igniter:** Inertial fusion approach, use short pulse to burn a hole for subsequent long pulse to heat and compress target

- **X ray lasers:** create population inversions in partially ionized gasses

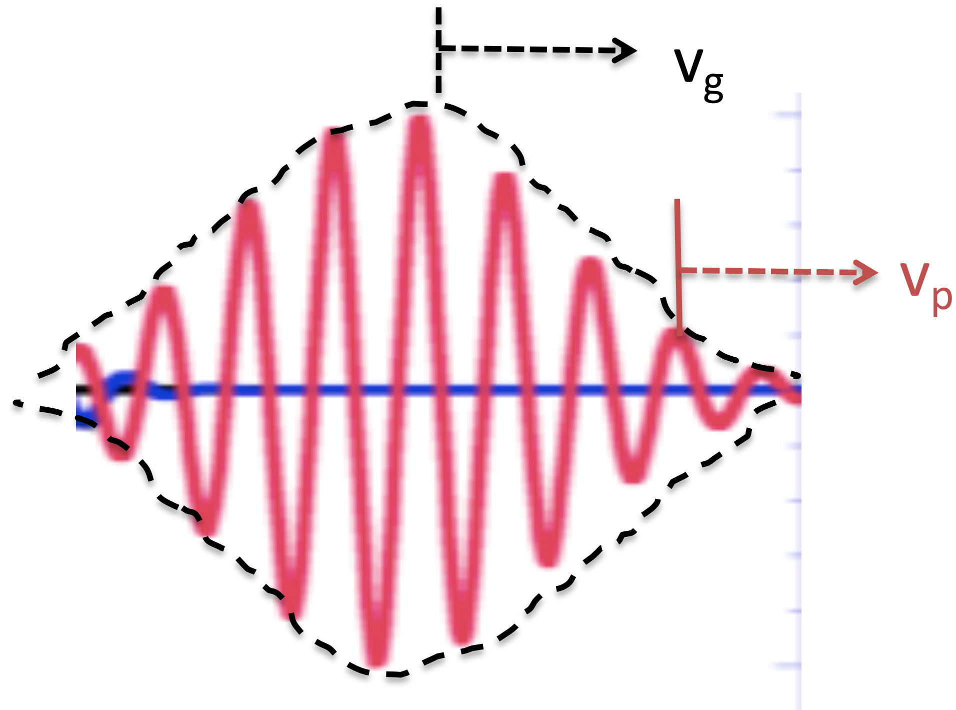
Phase and Group Velocity

The crests travel at the phase velocity

$$v_p = \frac{\omega_c}{k(\omega_c)}$$

The envelope travels at the group velocity

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega_c}$$

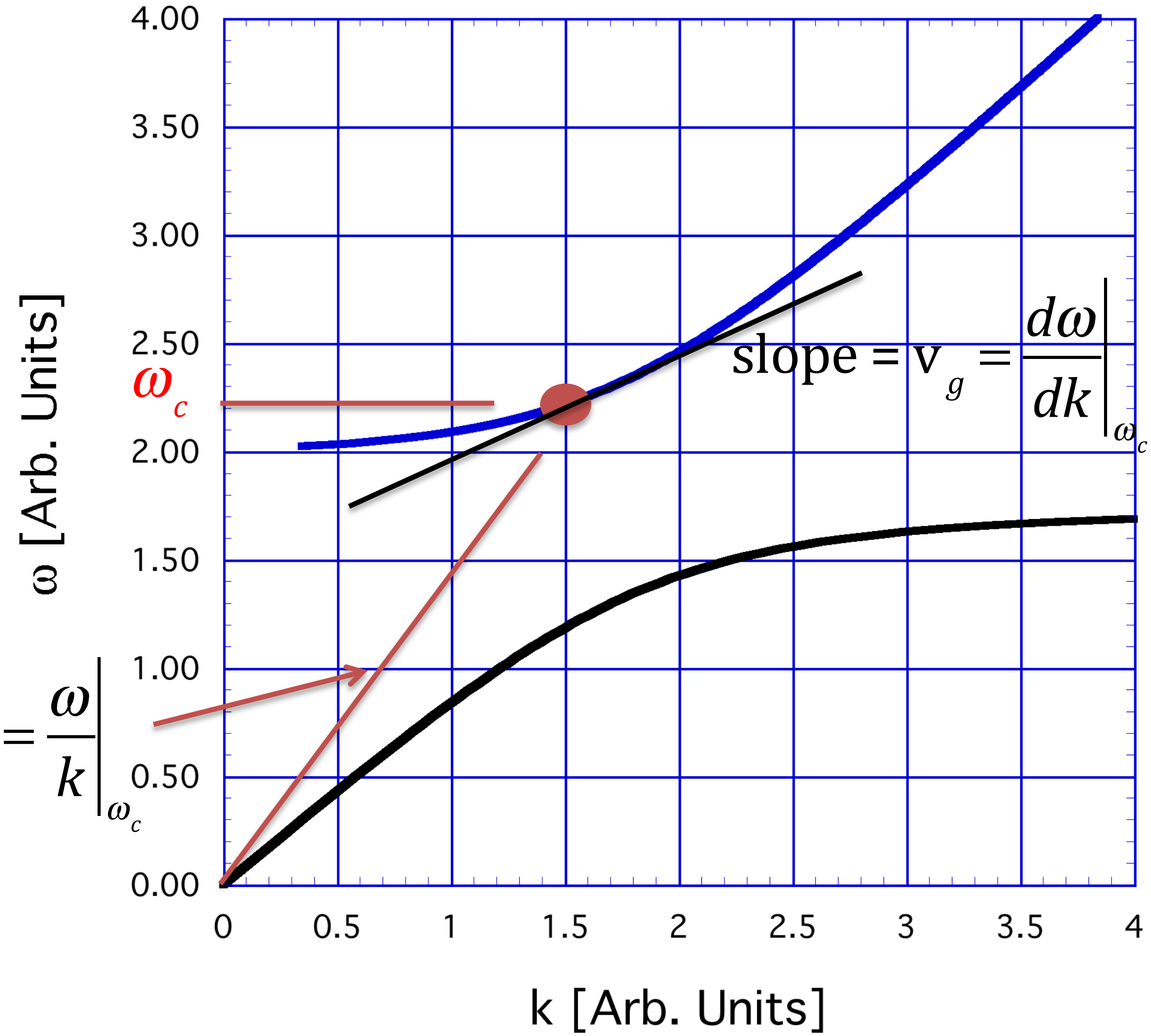


Dispersion Relation

Two modes

$\omega(k)$ vs k

not a straight line



$$\text{slope} = \mathrm{v}_g = \left.\frac{d\omega}{dk}\right|_{\omega_c}$$

$$\text{slope} = \mathrm{v}_p = \left.\frac{\omega}{k}\right|_{\omega_c}$$