Introduction

A consequence of the laws of Physics is that certain quantities are conserved once a closed system has been properly defined.

Some of these are:

Charge
Energy (and mass via $E=mc^2$)
Linear Momentum
Angular Momentum

Conservation Laws

Noether's Theorem

Conservation laws in physics are a direct consequence of **symmetries** in nature

Conservation of energy(mass) $\rightarrow$ time invariance

Conservation of linear momentum $\rightarrow$ translation invariance

Conservation of angular momentum $\rightarrow$ rotation invariance

Conservation of electric charge $\rightarrow$ gauge invariance (TBE)

Emmy Noether (Wikipedia)

Born	Amalie Emmy Noether 23 March 1882 Erlangen, Bavaria, German Empire
Died	14 April 1935 (aged 53) Bryn Mawr, Pennsylvania, United States
Nationality	German
Alma mater	University of Erlangen
Known for	Abstract algebra Theoretical physics Noether's theorem
Awards	Ackermann–Teubner Memorial Award (1932)
	Scientific career
Fields	Mathematics and physics
Institutions	University of Göttingen Bryn Mawr College
Thesis	*On Complete Systems of Invariants for Ternary Biquadratic Forms* (1907)

Example: conservation of kinetic + potential energy

$$\frac{d}{dt} m\mathbf{v} = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right]$$

Newton's law of motion (F=ma)

Quasi – Static Fields: $\mathbf{E} = -\nabla\Phi(\mathbf{x}, t)$

$$\mathbf{v} \cdot \frac{d}{dt} m\mathbf{v} = \frac{d}{dt}\frac{m|\mathbf{v}|^2}{2} = q\mathbf{v} \cdot \left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] = -q\mathbf{v} \cdot \nabla\Phi$$

Rate of change of potential following a trajectory

$$\frac{d}{dt} q\Phi(t, \mathbf{x}(t)) = \frac{\partial}{\partial t} q\Phi + q\mathbf{v} \cdot \nabla\Phi$$

$$\frac{d}{dt}\left(\frac{m|\mathbf{v}|^2}{2} + q\Phi\right) = \frac{\partial}{\partial t} q\Phi$$

Kinetic + Potential Energy is conserved only if potential is time independent

Conservation of Linear Momentum

$$\frac{d}{dt} m_i \mathbf{v}_i = q_i \mathbf{E}(\mathbf{x}_i, t) \qquad \mathbf{E}(\mathbf{x}_i, t) = \sum_{j \neq i} \frac{q_j (\mathbf{x}_i - \mathbf{x}_j)}{4\pi\varepsilon_0 |\mathbf{x}_i - \mathbf{x}_j|^3}$$

$$\frac{d}{dt} \sum_i m_i \mathbf{v}_i = \frac{d}{dt} \mathbf{P} = \sum_{i, j \neq i} \frac{q_i q_j (\mathbf{x}_i - \mathbf{x}_j)}{4\pi\varepsilon_0 |\mathbf{x}_i - \mathbf{x}_j|^3} = 0$$

Momentum **P** is constant,
velocity of center of mass is constant

$$\frac{d}{dt} \mathbf{X}_{cm} = \frac{d}{dt} \frac{\sum_i m_i \mathbf{x}_i}{\sum_i m_i} = \frac{\mathbf{P}}{M} = \text{constant}$$

If P =0, $\mathbf{X}_{cm}$ can not change

System is symmetric wrt translation in 3 directions. Three constants of motion: 3 components of **P**.

Conservation of Angular Momentum

$$\frac{d}{dt} m_i \mathbf{v}_i = q_i \mathbf{E}(\mathbf{x}_i, t) \qquad \mathbf{E}(\mathbf{x}_i, t) = \sum_{j \neq i} \frac{q_j \left(\mathbf{x}_i - \mathbf{x}_j \right)}{4\pi\varepsilon_0 \left| \mathbf{x}_i - \mathbf{x}_j \right|^3}$$

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt} \sum_i \mathbf{x}_i \times m_i \mathbf{v}_i = \sum_i \left(\frac{d\mathbf{x}_i}{dt} \times m_i \mathbf{v}_i + \mathbf{x}_i \times \frac{d}{dt} m_i \mathbf{v}_i \right)$$

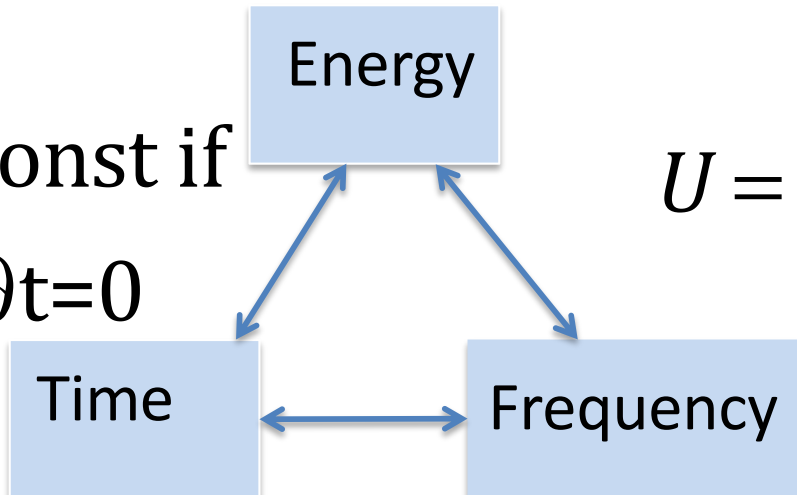
$$= \sum_{i,j \neq i} \mathbf{x}_i \times \frac{q_i q_j \left(\mathbf{x}_i - \mathbf{x}_j \right)}{4\pi\varepsilon_0 \left| \mathbf{x}_i - \mathbf{x}_j \right|^3} = 0$$

System is symmetric wrt rotation in 3 directions. Three constants of motion: 3 components of **L**.

Linked Quantities

U const if

$\partial/\partial t=0$



$$U=\hbar\omega$$

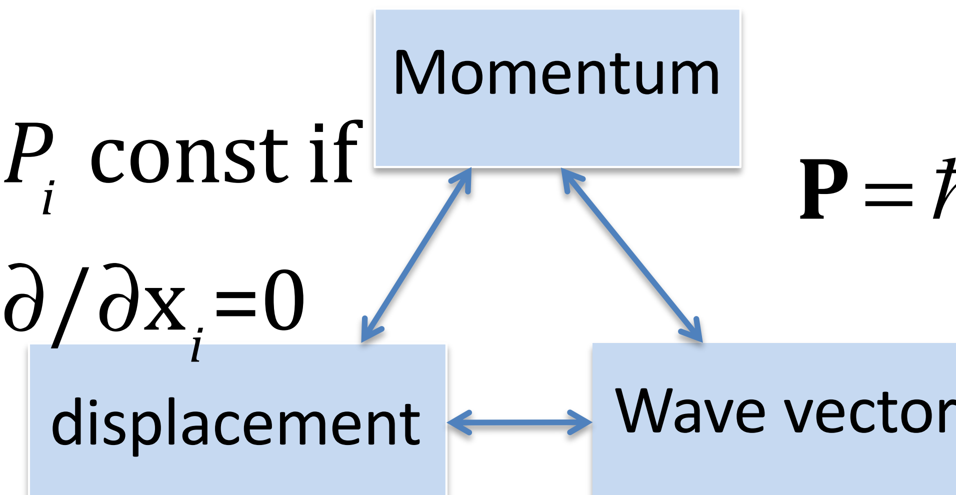
$$1=\Delta t\Delta\omega$$

Sinusoidal waves

$$\exp\left(i\mathbf{k}\cdot\mathbf{x}-i\omega t\right)$$

P_i const if

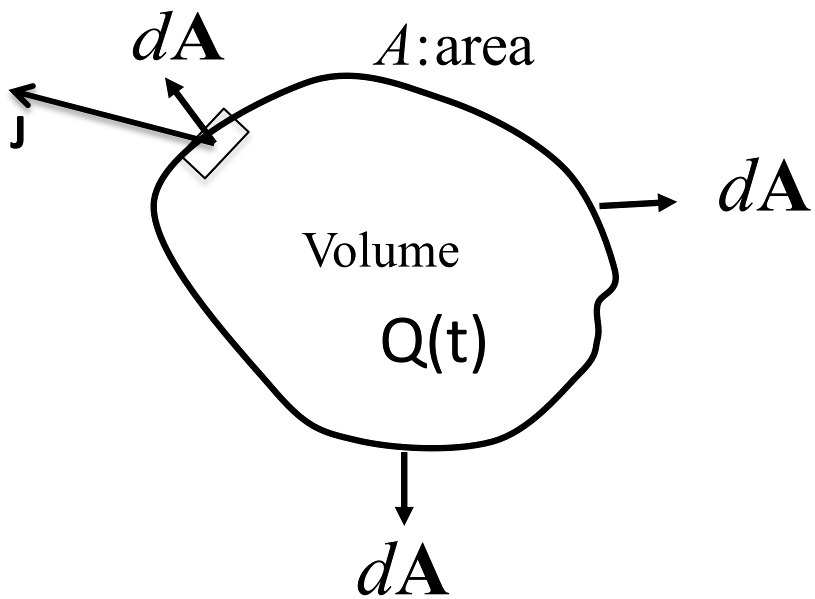
$\partial/\partial x_i=0$



$$\mathbf{P}=\hbar\mathbf{k}$$

$$1=\Delta x\Delta k$$

What does a conservation law for continuous systems look like?



$$\frac{dQ}{dt} + \int_S d\vec{\mathbf{A}} \cdot \vec{\mathbf{J}} = 0$$

$$Q = \int_V d^3 r \rho(\mathbf{r}, t)$$

$$\int_S d\vec{\mathbf{A}} \cdot \vec{\mathbf{J}} = \int_V d^3 r \, \nabla \cdot \vec{\mathbf{J}}$$

Conservation of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = 0$$

Conservation of Energy

$$\frac{\partial}{\partial t}\left[u_E + u_M\right] + \nabla \cdot \vec{\mathbf{S}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$$

Rate at which energy is transferred to current J

$$u_E + u_M = \frac{\varepsilon_0}{2}\mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B}$$

Energy density in fields

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} : \text{ Poynting vector}$$

Flow of local energy density

Conservation of energy

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \nabla \times \vec{\mathbf{B}} = \mu_0 \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

$$\frac{\vec{\mathbf{B}}}{\mu_0} \cdot \nabla \times \mathbf{E} = -\frac{\vec{\mathbf{B}}}{\mu_0} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \mathbf{E} \cdot \nabla \times \frac{\vec{\mathbf{B}}}{\mu_0} = \left[\vec{\mathbf{E}} \cdot \vec{\mathbf{J}} \cdot + \varepsilon_0 \vec{\mathbf{E}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

$$\varepsilon_0 \vec{\mathbf{E}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial t} + \frac{\vec{\mathbf{B}}}{\mu_0} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t} + \frac{\vec{\mathbf{B}}}{\mu_0} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \frac{\vec{\mathbf{B}}}{\mu_0} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$$

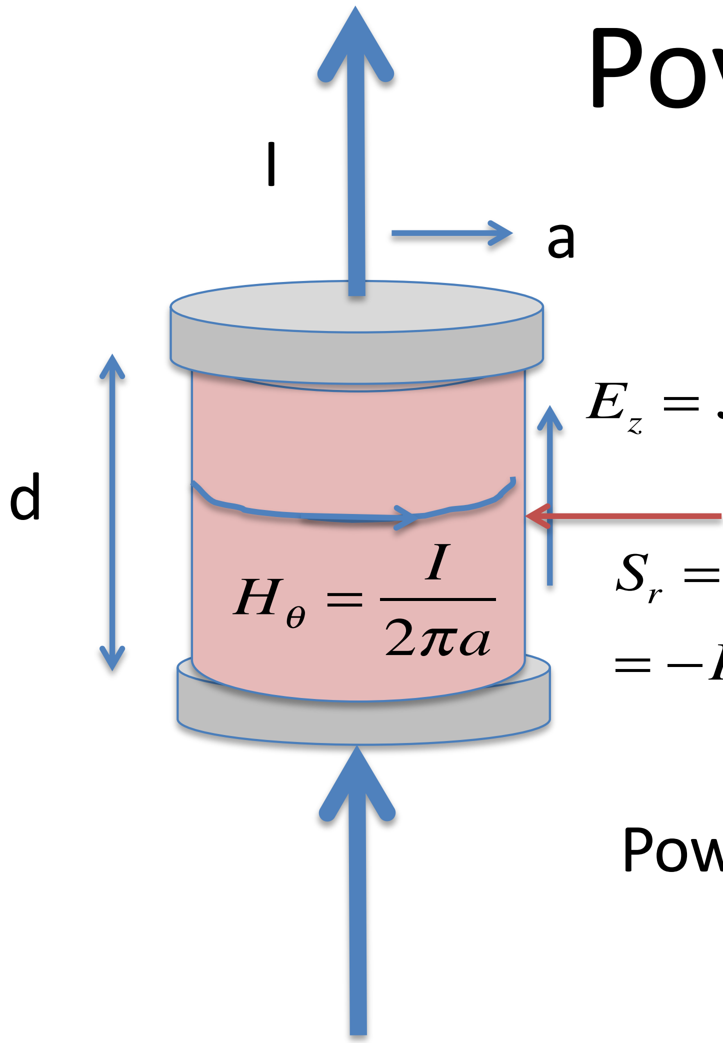
$$\frac{\partial}{\partial t} \left(\frac{\varepsilon_0 \left| \vec{\mathbf{E}} \right|^2}{2} + \frac{\left| \vec{\mathbf{B}} \right|^2}{2\mu_0} \right) + \nabla \cdot \left(\mathbf{E} \times \frac{\vec{\mathbf{B}}}{\mu_0} \right) = -\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$$

Poynting's Theorem

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0|\mathbf{E}|^2}{2}+\frac{\mu_0|\mathbf{H}|^2}{2}\right)+\nabla\cdot(\mathbf{E}\times\mathbf{H})=-\mathbf{E}\cdot\mathbf{J}$$

	Energy density	Power Flux	Rate of work done by E on J
	$\left(\frac{\varepsilon_0\|\mathbf{E}\|^2}{2}+\frac{\mu_0\|\mathbf{H}\|^2}{2}\right)$	$\mathbf{S}=(\mathbf{E}\times\mathbf{H})$	$\mathbf{E}\cdot\mathbf{J}$
Units:	Joules/m^3	Watts/m^2	Watts/m^3

Poynting Example



$$E_z = J_z / \sigma = I / (\pi a^2 \sigma)$$

$$\vec{\mathbf{S}} = \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}\right)$$

$$S_r = -E_z H_\theta$$
$$= -I^2 / (2\pi^2 a^3 \sigma)$$

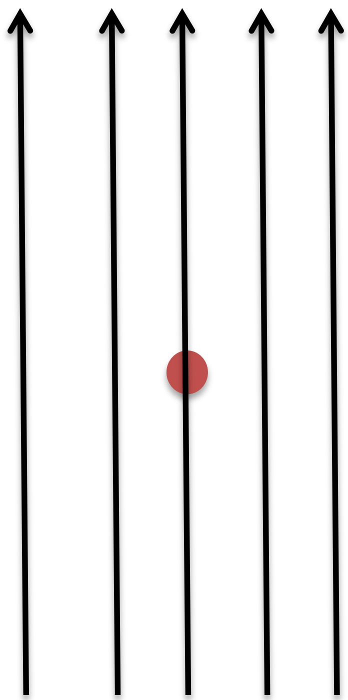
Area of side

Power in: $P = 2\pi a d |S_r| =$

$$= I^2 / (\pi \ a^2 \sigma) = R I^2$$

Resistance

Only divergence of Poynting flux matters



$\vec{\mathbf{B}} = \hat{\mathbf{z}}\ B_0$

Carge q at $\mathbf{r} = 0$

Find S:

What direction?

What does it mean?

Poynting's theorem addresses EM energy, what about mechanical energy?

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0|\mathbf{E}|^2}{2}+\frac{\mu_0|\mathbf{H}|^2}{2}\right)+\nabla\cdot(\mathbf{E}\times\mathbf{H})=-\mathbf{E}\cdot\mathbf{J}$$

Rate of work done by E on J

Newton's Law ma=F

$$m\frac{d}{dt}\mathbf{v}_i=q[\mathbf{E}+\mathbf{v}_i\times\mathbf{B}]$$

$$m\sum_i \mathbf{v}_i\cdot\frac{d}{dt}\mathbf{v}_i=\sum_i \mathbf{v}_i\cdot q[\mathbf{E}+\mathbf{v}_i\times\mathbf{B}]=\sum_i \mathbf{v}_i\cdot q\mathbf{E}=\int_V d^3r\ \mathbf{v}_i\cdot q\mathbf{E}$$

$$\sum_i \frac{d}{dt}\frac{m|\mathbf{v}_i|^2}{2}=\int_V d^3r\ \mathbf{v}_i\cdot q\mathbf{E}=\int_V d^3r\ \mathbf{J}\cdot\mathbf{E}$$

Combining EM and Mechanical Energy

$$\frac{d}{dt}\left\{\int_V d^3r\left(\frac{\varepsilon_0|\mathbf{E}|^2}{2}+\frac{\mu_0|\mathbf{H}|^2}{2}\right)+\sum_i\frac{m|\mathbf{v}_i|^2}{2}\right\}+\int_S d\mathbf{A}\cdot(\mathbf{E}\times\mathbf{H})=0$$

EM + Mechanical Energy   EM power flow

Conservation of EM Momentum

The total EM force on charges in a volume can be written as

$$\frac{d\mathbf{P}_{mech}}{dt} = \sum_i q(\mathbf{E}(\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(\mathbf{x}_i)) = \int_V (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) d^3r$$

After some Math $$\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{EM}}{dt} = \oint_A \overline{\overline{\mathbf{T}}} \cdot \hat{\mathbf{n}} da$$

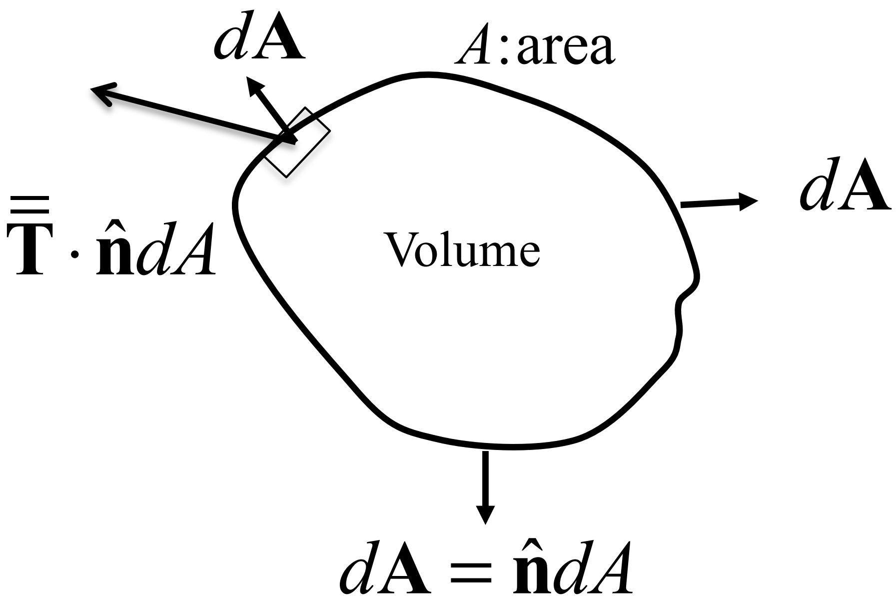
Total EM linear momentum: $\mathbf{P}_{EM} = \varepsilon_0 \mu_0 \int_V \mathbf{E} \times \mathbf{H} \, d^3r$

EM linear momentum density: $\varepsilon_0 \mu_0 \mathbf{E} \times \mathbf{H} = \mathbf{S} / c^2$

Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, $\mu_0 \varepsilon_0 = 1 / c^2$

Maxwell Stress Tensor: $\overline{\overline{\mathbf{T}}} = \varepsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0} \mathbf{B}\mathbf{B} - \frac{1}{2}(\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B}) \overline{\overline{\mathbf{I}}}$

Force on what’s inside



$$\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{EM}}{dt} = \mathbf{F}$$

$$\mathbf{F} = \oint_A \bar{\bar{\mathbf{T}}} \cdot \hat{\mathbf{n}} dA$$

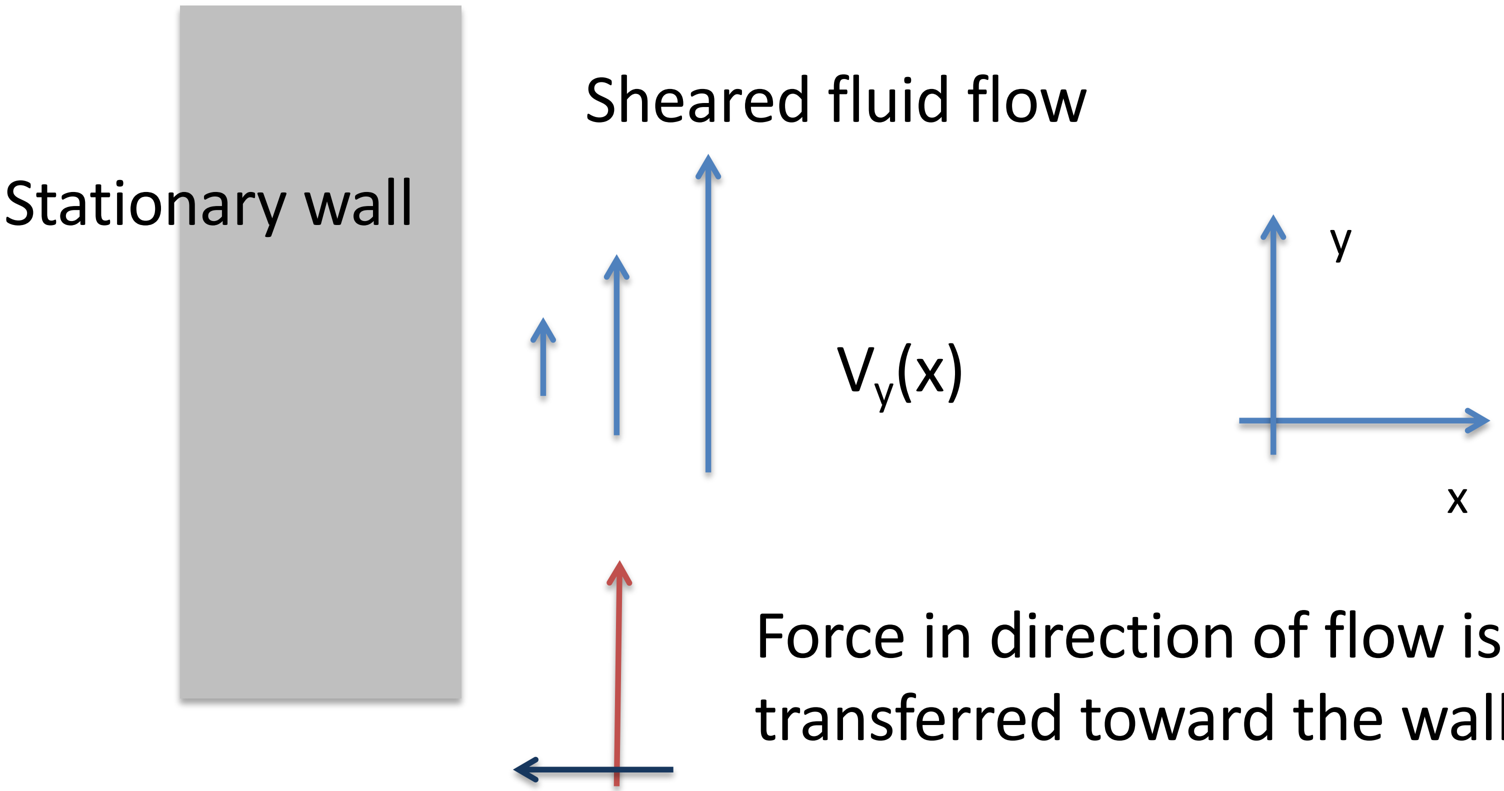
Analogy: pressure

$$\bar{\bar{\mathbf{T}}} = -p \bar{\bar{\mathbf{I}}}$$

$$\bar{\bar{\mathbf{T}}} \cdot \hat{\mathbf{n}} = -p \hat{\mathbf{n}}$$

$$\text{Maxwell Stress Tensor: } \bar{\bar{\mathbf{T}}} = \varepsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\mathbf{B} - \frac{1}{2}(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B})\bar{\bar{\mathbf{I}}}$$

Viscous Fluid Stress



$$T_{xy} \propto \nu \frac{\partial v_y(x)}{\partial x}$$

Energy and Momentum of Light

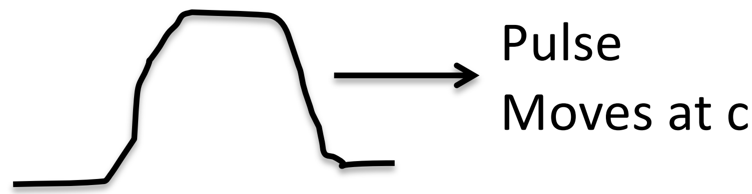
$$\left(\frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{\mu_0 |\mathbf{H}|^2}{2} \right)$$

Energy density

$$\vec{\mathbf{S}} = \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}} \right)$$

Power Flux

Units: Joules/m^3 Watts/m^2



Power Flux = c Energy Density

Pulse also contains momentum

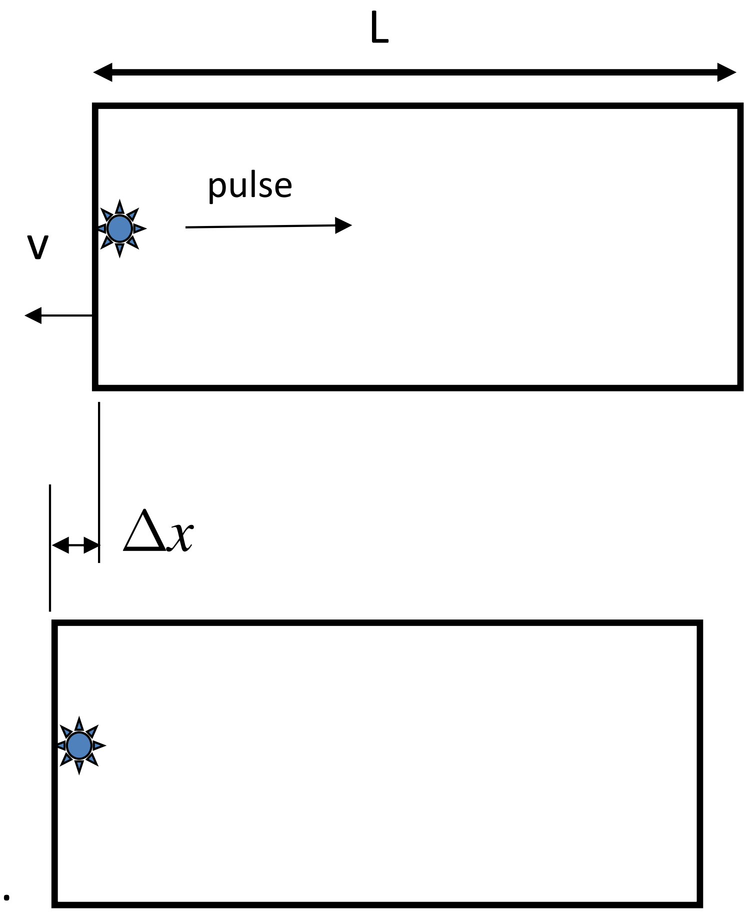
EM linear momentum density: $\varepsilon_0 \mu_0 \mathbf{E} \times \mathbf{H} = \mathbf{S} / c^2$

$$\frac{\text{Energy Density}}{\text{Momentum Density}} = \frac{S / c}{S / c^2} = c$$

A pulse of light carries energy and momentum: ratio = c

Mass Energy Equivalence E = mc^2

Isolated box of mass M and length L in space.
A light on the wall on one side sends out a pulse of energy E toward the right.
The pulse has momentum p=E/c.
The box recoils with velocity v=p/M to the left.
The pulse is absorbed on the other side after a time T=L/c.
The box absorbs the momentum and stops moving.



Displacement of the box $$\Delta x = vT = \frac{EL}{Mc^2}$$

Has the center of mass moved?
We would like to say no.
The box should not be able to move its center of mass.

$$\Delta x M = L\left(E / c^2\right) = Lm$$

We can say that the CM has not moved if the pulse reduced the mass of the left side by m=E/c^2 and increased the right side by the same amount.

$$E = mc^2$$

Stress Tensor

$$\overline{\overline{\mathbf{T}}} = \varepsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\mathbf{B} - \frac{1}{2}(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B})\overline{\overline{\mathbf{I}}}$$

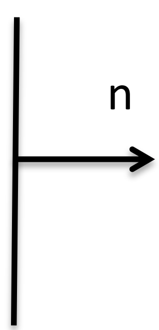
Force transmitted through surface $$\mathbf{F} = \oint_A \overline{\overline{\mathbf{T}}}\cdot\hat{\mathbf{n}}\,da$$

The component normal to the surface is like a pressure force $$\mathbf{n}\cdot\overline{\overline{\mathbf{T}}}\cdot\mathbf{n} = -p$$

$$\mathbf{n}\cdot\overline{\overline{\mathbf{T}}}\cdot\mathbf{n} = \varepsilon_0\left[\frac{1}{2}(\mathbf{n}\cdot\mathbf{E})^2 - \frac{1}{2}|\mathbf{E}_t|^2\right] + \frac{1}{\mu_0}\left[\frac{1}{2}(\mathbf{n}\cdot\mathbf{B})^2 - \frac{1}{2}|\mathbf{B}_t|^2\right]$$

Remember BC's
E_t and B_n are continuous

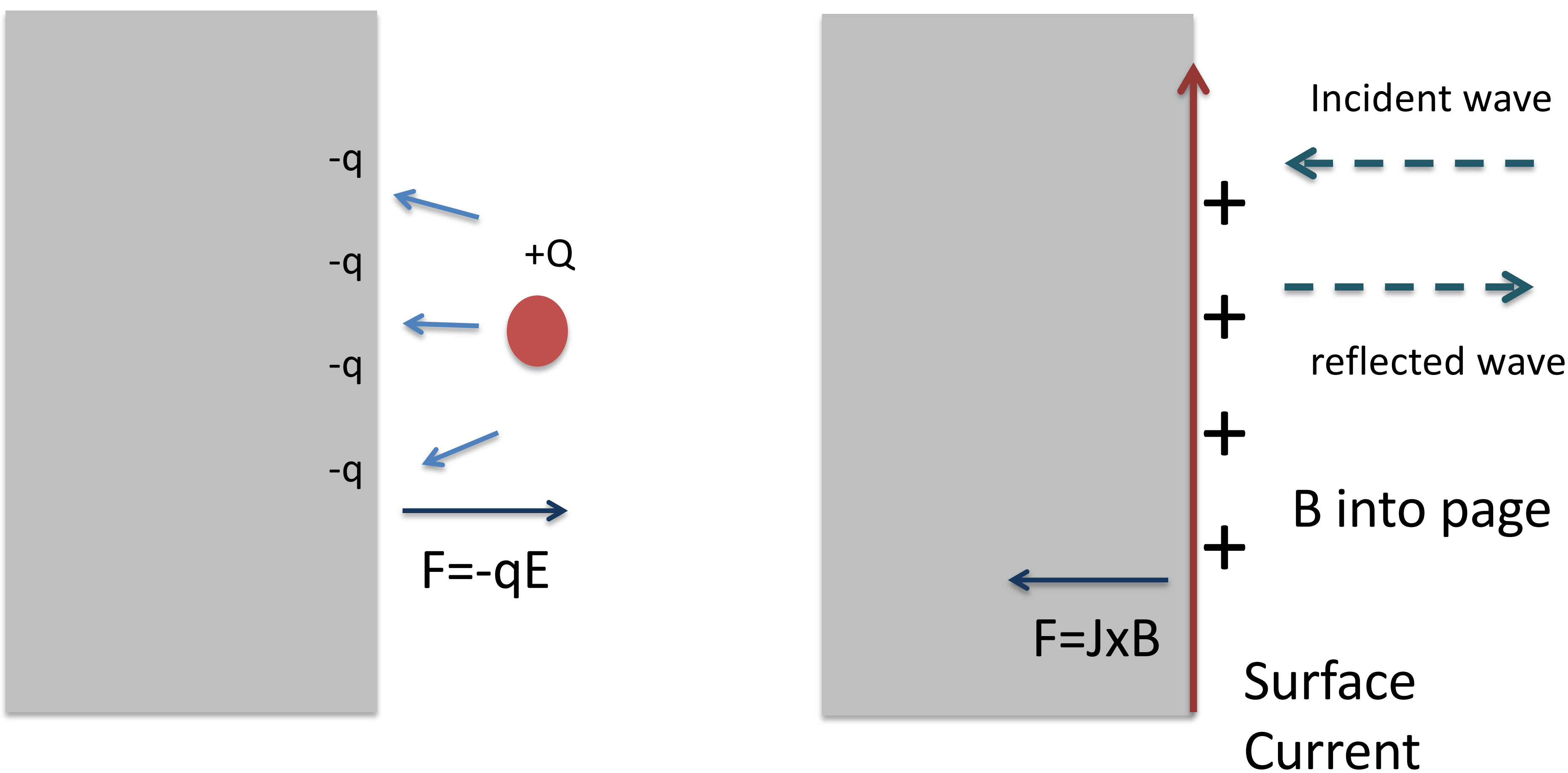
Normal E pulls on surface
Tangential B pushes



Surface of conductor

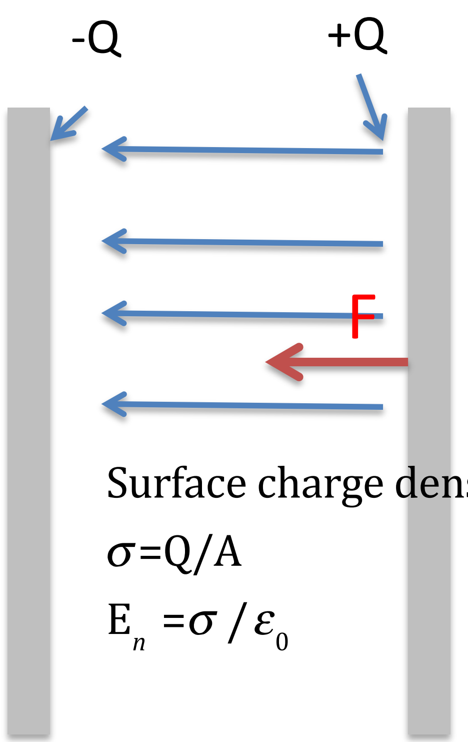
$$\mathbf{n}\cdot\overline{\overline{\mathbf{T}}}\cdot\mathbf{n} = \varepsilon_0\left[\frac{1}{2}(\mathbf{n}\cdot\mathbf{E})^2\right] + \frac{1}{\mu_0}\left[-\frac{1}{2}|\mathbf{B}_t|^2\right]$$

Forces on Conductor



Electric field force on surface charge pulls

Force of attraction between capacitor plates



Area = A

Surface charge density

$\sigma = Q/A$

$E_n = \sigma / \varepsilon_0$

$$\mathbf{F} = \oint_A \overline{\overline{\mathbf{T}}} \cdot \hat{\mathbf{n}}\, da$$

$$\mathbf{n} \cdot \overline{\overline{\mathbf{T}}} \cdot \mathbf{n} = \varepsilon_0 \left[\frac{1}{2} (\mathbf{n} \cdot \mathbf{E})^2 \right] = \frac{1}{2} \frac{Q^2}{A^2 \varepsilon_0}$$

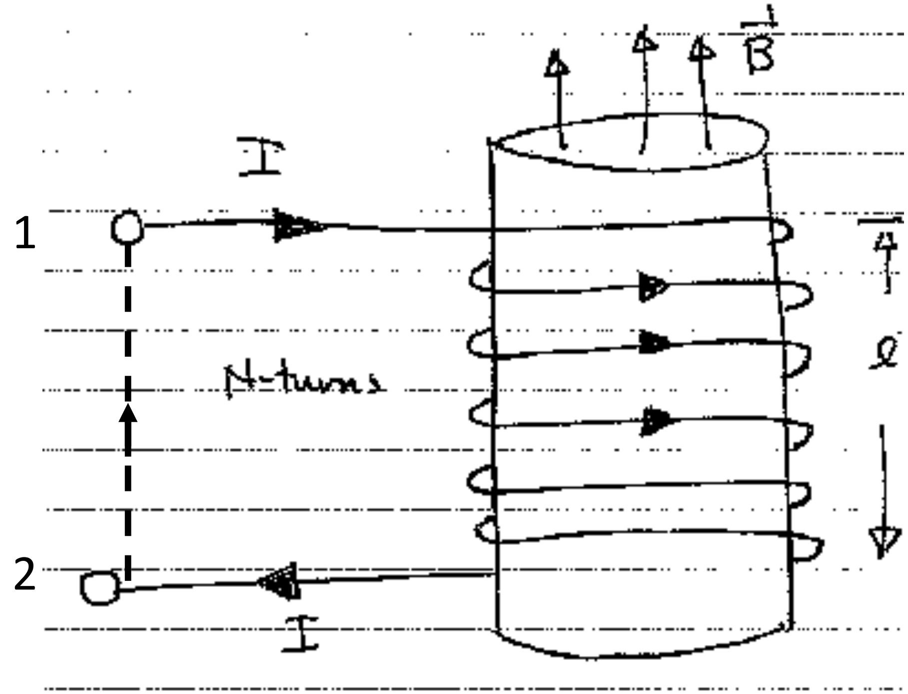
$$F = \frac{1}{2} \frac{Q^2}{A\, \varepsilon_0}$$

How much work must be done to separate plates a distance h?

$$\text{Work} = hF = \frac{h}{2} \frac{Q^2}{A\, \varepsilon_0} = \frac{1}{2} \frac{Q^2}{C}$$

capacitance

What is the force on the windings of a coil?



$$\mathbf{n}\cdot\overline{\overline{\mathbf{T}}}\cdot\mathbf{n}=\varepsilon_0\left[\frac{1}{2}\left(\mathbf{n}\cdot\mathbf{E}\right)^2-\frac{1}{2}\left|\mathbf{E}_t\right|^2\right]+\frac{1}{\mu_0}\left[\frac{1}{2}\left(\mathbf{n}\cdot\mathbf{B}\right)^2-\frac{1}{2}\left|\mathbf{B}_t\right|^2\right]$$

Maxwell's Equations in Matter

Basic Equations (Vacuum)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q / \varepsilon_0$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

Here ρ and J are the <u>total charge and current densities</u>

Includes charge and current densities induced in dielectric and magnetic materials

Separate charge and current densities into "free" and "induced" components

Somewhat arbitrary but very useful

magnetization current

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_m + \mathbf{J}_p$$

polarization current

"Free" current

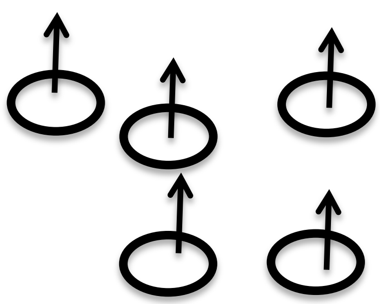
polarization charge density

$$\rho = \rho_f + \rho_p$$

"Free" charge density

polarization density

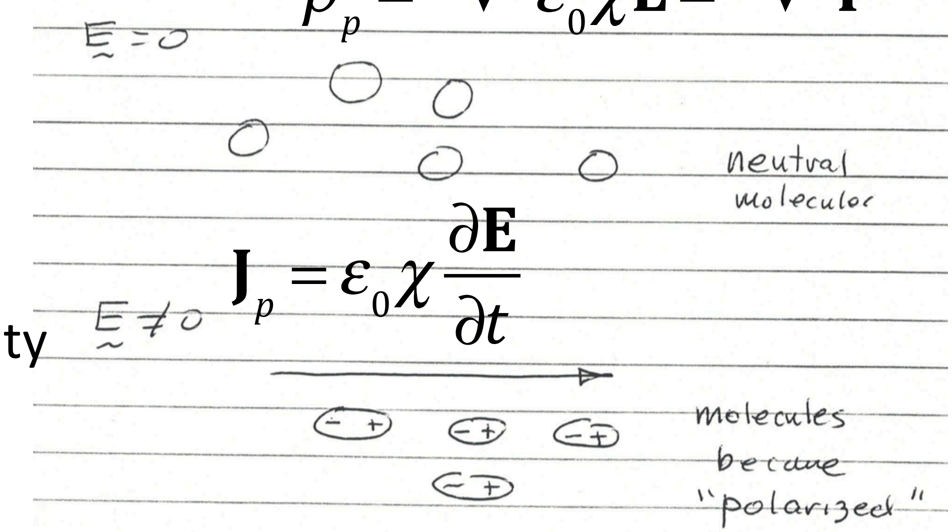
$$\rho_p = -\nabla \cdot \varepsilon_0 \chi \mathbf{E} = -\nabla \cdot \mathbf{P}$$



$$\mathbf{J}_m = \nabla \times \mathbf{M}$$

magnetization density

$$\mathbf{M} = \mu_0 \chi_m \mathbf{H}$$



$$\mathbf{J}_p = \varepsilon_0 \chi \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell's Equations in Matter

Equations in linear media

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu \mathbf{H}$$

$$\mathbf{P} = \varepsilon_0 \chi_E \mathbf{E}$$

$$\mathbf{M} = \mu_0 \chi_M \mathbf{H}$$

$$\oint \vec{\mathbf{D}} \cdot d\vec{\mathbf{A}} = Q_{free}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\oint_{Loop} \vec{\mathbf{H}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}}_{free} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \right]$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho_{free}$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{free} + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

Energy Density in a Linear Medium

Field Energy

Energy density

$$\left(\frac{\varepsilon_0\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\left|\vec{\mathbf{B}}\right|^2}{2\mu_0}\right)$$

Power Flux

$$\vec{\mathbf{S}}=\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}\right)$$

Rate of work done by E on J

$$\vec{\mathbf{E}}\cdot\vec{\mathbf{J}}$$

$$\vec{\mathbf{B}}=\mu\vec{\mathbf{H}} \qquad \vec{\mathbf{D}}=\varepsilon\vec{\mathbf{E}}$$

Energy density

$$\left(\frac{\varepsilon\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\mu\left|\vec{\mathbf{H}}\right|^2}{2}\right)$$

Power Flux

$$\vec{\mathbf{S}}=\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}\right)$$

Rate of work done by E on J

$$\vec{\mathbf{E}}\cdot\vec{\mathbf{J}}$$

Almost always wrong

$$\frac{\partial}{\partial\omega}\left(\frac{\omega\varepsilon\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\omega\mu\left|\vec{\mathbf{H}}\right|^2}{2}\right)$$