ENEE681

Topics to be covered

Scalar and Vector Potentials
Green's functions

Notes Courtesy of Professor Phil Sprangle

Coulomb's Law

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|^3} d^3 r'$$

This can also be written in terms of a scalar potential

$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r})$$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3 r',$$

Show:

$$-\nabla \phi(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_0} \int_V \frac{\partial}{\partial \mathbf{r}} \frac{\rho(\mathbf{r}')}{\left|\mathbf{r} - \mathbf{r}'\right|} d^3 r' = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|^3} d^3 r'$$

Maxwell's Equations for Vector and Scalar Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

In the Lorentz gauge $\left(\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} \right)$ the vector and scalar potentials obey wave equations

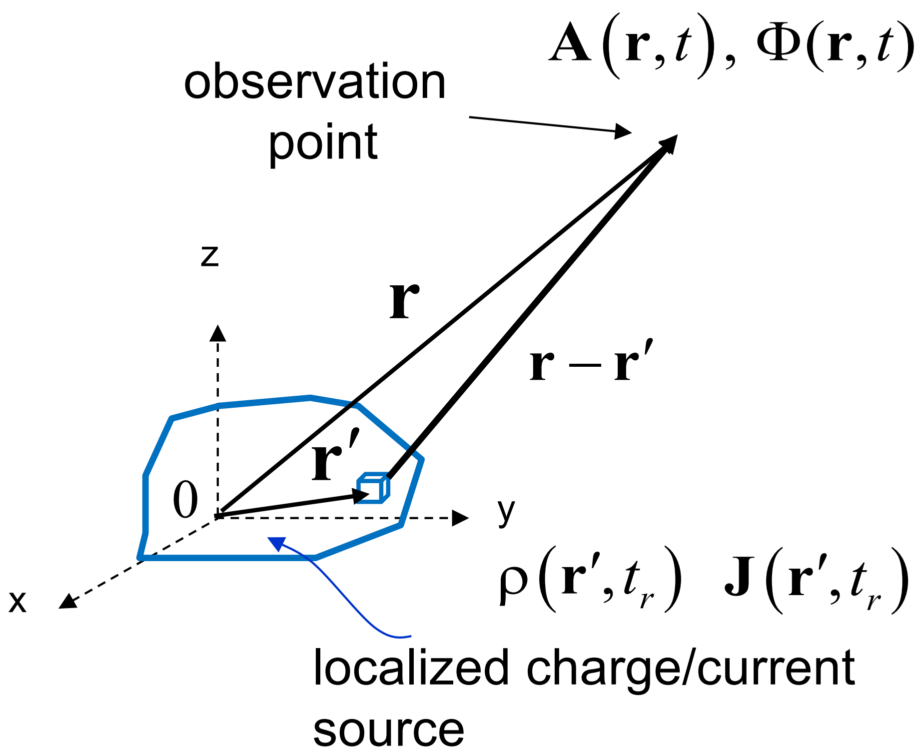
$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \sqrt{\varepsilon_0 \mu_0} = 1/c$$

$$\nabla^2 \Phi - \mu_0 \varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

where $\mathbf{J}$ and ρ are the current and charge densities

The solutions to the wave equations (in the absence of boundaries) are

Solution to Wave Equations



$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{Vol} d\tau' \left.\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \left.\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$d\tau' = dx'\,dy'\,dz'$$

Sinusoidal Dependence on Time

If we assume harmonic (sinusoid dependence on time) for all the fields and sources $e^{-i\omega t_r} = e^{-i\omega t} e^{ik|\mathbf{r}-\mathbf{r}'|}$

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{A}}(\mathbf{r})\, e^{-i\omega t}\right] \qquad \Phi(\mathbf{r},t) = \mathrm{Re}\left[\hat{\Phi}(\mathbf{r})\, e^{-i\omega t}\right]$$

$$\mathbf{J}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{J}}(\mathbf{r})\, e^{-i\omega t}\right] \qquad \rho(\mathbf{r},t) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r})\, e^{-i\omega t}\right]$$

In phasor notation

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{Vol} d\tau' \frac{\hat{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, e^{ik|\mathbf{r}-\mathbf{r}'|}$$

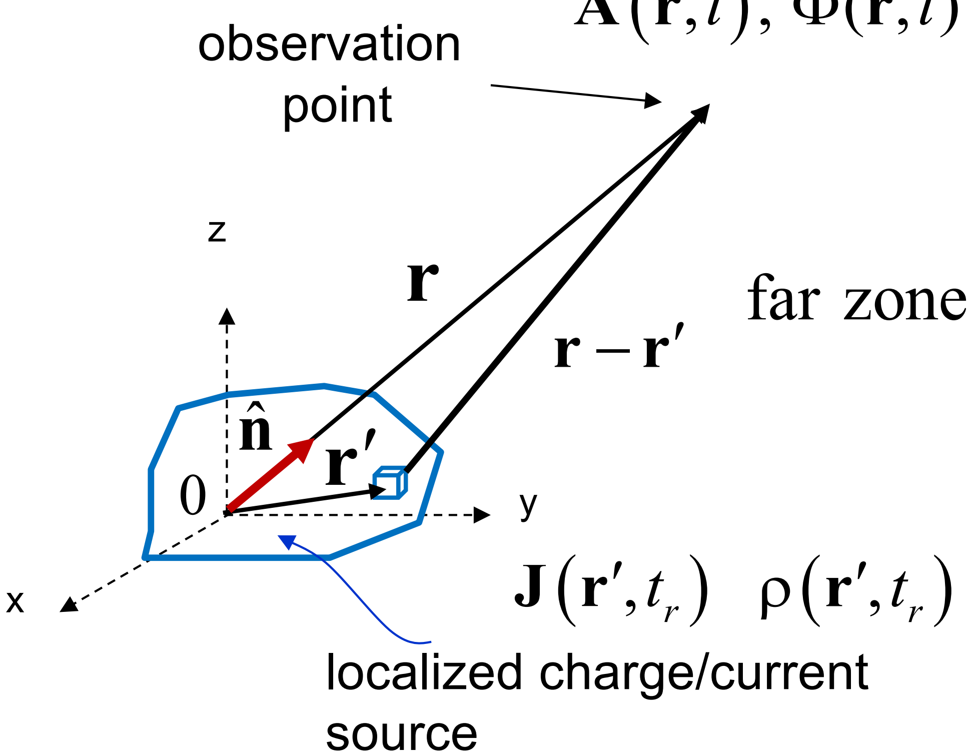
where $k = \omega / c = \dfrac{2\pi}{\lambda}$ is the wavenumber

Far Field Approximation

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \simeq r - \mathbf{r}\cdot\mathbf{r}'/r$$

Assume that the source is localized and the observation point is far away $(r >> r')$

$$\mathbf{A}(\mathbf{r},t),\ \Phi(\mathbf{r},t)$$



$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|}$$

$$\text{far zone } kr = 2\pi\frac{r}{\lambda} >> 1$$

$$|\mathbf{r}-\mathbf{r}'| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}' \quad \text{for } r >> r'$$

$$\hat{\mathbf{n}} = \frac{\mathbf{r}}{|\mathbf{r}|} \text{ unit vector}$$

Far Field Potentials

Using $\left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}'$

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\tau' \frac{\hat{\mathbf{J}}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau' \frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{e^{ikr}}{4\pi\varepsilon_0 r} \int_{Vol} d\tau' \, \hat{\rho}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

where $\mathbf{k} = k\,\hat{\mathbf{n}}$

for $r >> r'$, $\dfrac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \dfrac{1}{r}$ and $k\left|\mathbf{r}-\mathbf{r}'\right| \simeq kr - k\,\hat{\mathbf{n}}\cdot\mathbf{r}'$ in exponent

Calculating Fields from Potentials

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\nabla = \hat{\mathbf{n}} \frac{\partial}{\partial r} \rightarrow i\mathbf{k} \qquad \hat{\mathbf{B}}(\mathbf{r}) = \nabla \times \hat{\mathbf{A}}(\mathbf{r}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = i\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{r})$$

$$\hat{\mathbf{B}}(\mathbf{r}) = i \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \mathbf{k} \times \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$\hat{\mathbf{B}}(\mathbf{r})$ is transvere to $\hat{\mathbf{J}}$, $\mathbf{k} = k\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}(\mathbf{r})$

Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t \qquad \hat{\mathbf{E}}(\mathbf{r}) = -\frac{1}{\varepsilon_0 \omega} \mathbf{k} \times \hat{\mathbf{H}}(\mathbf{r}) \qquad \mu_0 \hat{\mathbf{H}}(\mathbf{r}) = \hat{\mathbf{B}}(\mathbf{r})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\frac{i}{\varepsilon_0 \mu_0 \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{r}) \right) = -i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0 \omega} \int_{Vol} d\tau' \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{J}}(\mathbf{r}') \right) e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

Simplify expressions for the fields

In the far zone $k\,r = 2\pi\,r\,/\,\lambda >> 1$

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx'\,dy'\,dz'$$

The fields in terms of the $F-T$ of the current density are

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi\,r}e^{ikr}\,\hat{\mathbf{C}}(\mathbf{k}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = i\,\frac{\mu_0}{4\pi\,r}e^{ikr}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{ikr}}{4\pi\,r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)$$

Where We Stand

$$\nabla \cdot \vec{\mathbf{D}} = \rho_f$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

Assume the following:

Linear, isotropic, instantaneous, media

Propagation in free space, no free charge or current.

$$\rho_f = 0, \quad \vec{\mathbf{J}}_f = 0$$

$$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$$

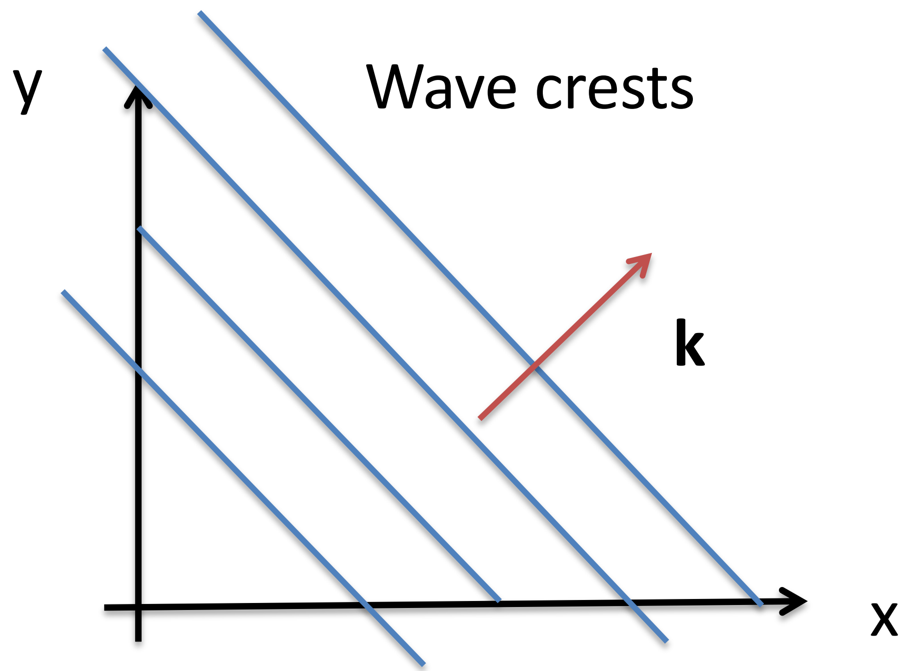
$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$$

Introduce Phasor Notation

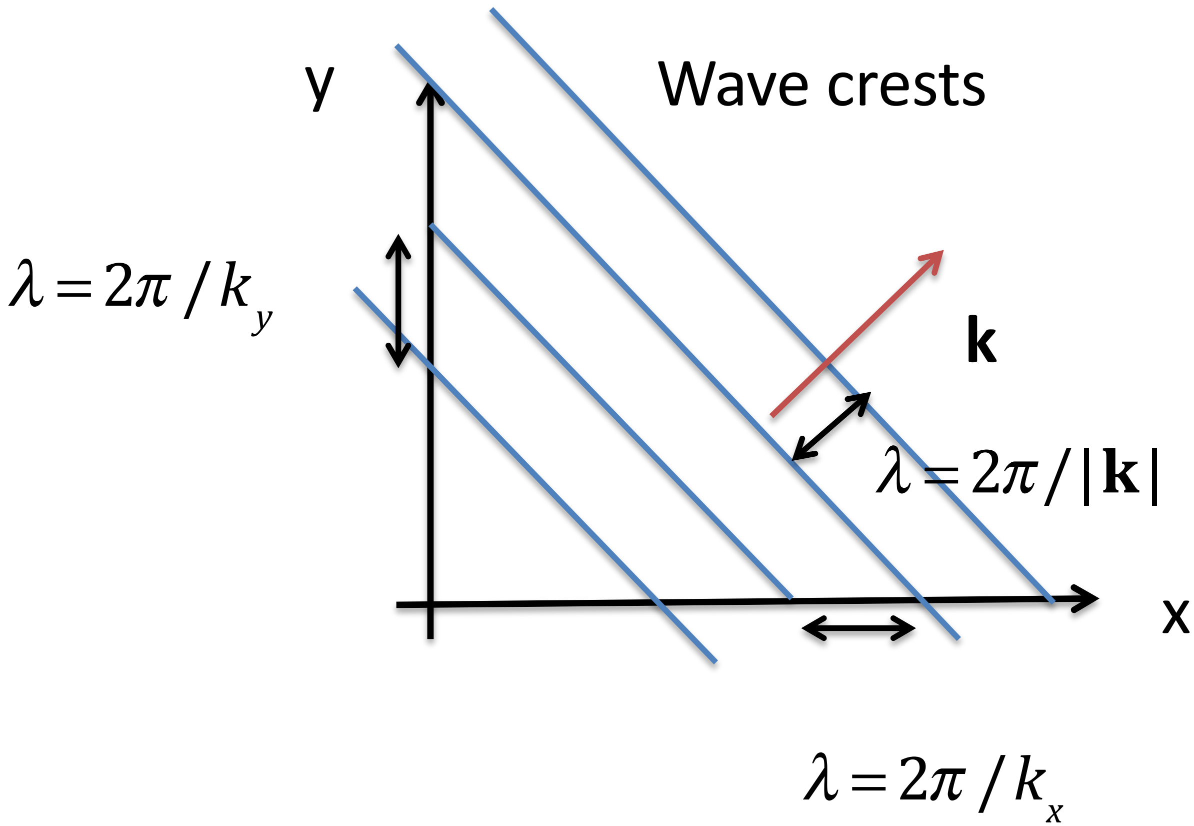
$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right\} \quad \mathbf{H}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right\}$$

Note: two new elements

1. Phasor amplitudes are vectors. Will be independent of space and time.
2. Space and time dependence contained in complex exponential
3. Wave number k is now wave vector **k**.



$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$



$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(k_x x + k_y y - \omega t\right)\right]\right\}$$

When does this work?

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\} \quad \mathbf{H}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

Works when ε and μ are independent of space and time.

$$\nabla\times\vec{\mathbf{E}} = -\mu\frac{\partial\vec{\mathbf{H}}}{\partial t}$$

$$\nabla\times\mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$= \mathrm{Re}\left\{\nabla\times\left(\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

$$= \mathrm{Re}\left\{i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$-\mu\frac{\partial}{\partial t}\mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$= \mathrm{Re}\left\{-\mu\frac{\partial}{\partial t}\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

$$= \mathrm{Re}\left\{-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

Relating phasor amplitudes

$$\mathrm{Re}\left\{i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}=\mathrm{Re}\left\{-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

If the real parts of two complex variables are equal, and there is no restriction on the imaginary parts, then I can make the complex variables equal.

$$i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]=-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)$$

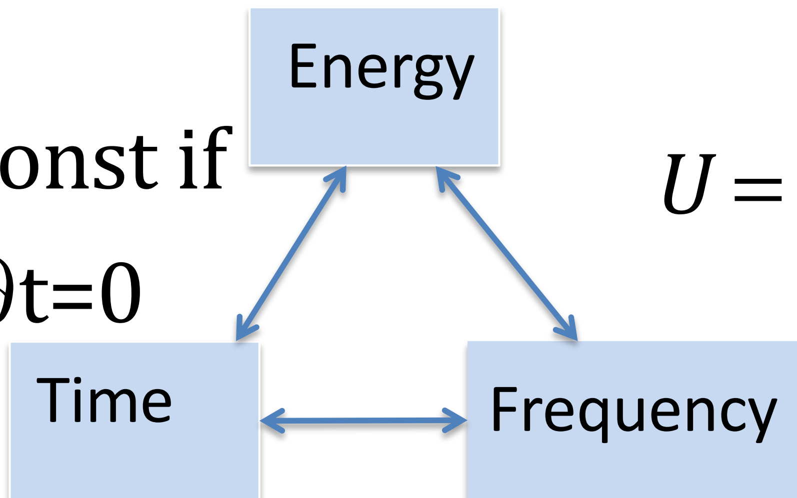
Now cancel the exponential factor from both sides. Result must still hold for all x and t.

$$\boxed{i\mathbf{k}\times\hat{\mathbf{E}}=-\mu(-i\omega)\hat{\mathbf{H}}}$$

Linked Quantities

U const if

$\partial/\partial t=0$



$$U=\hbar\omega$$

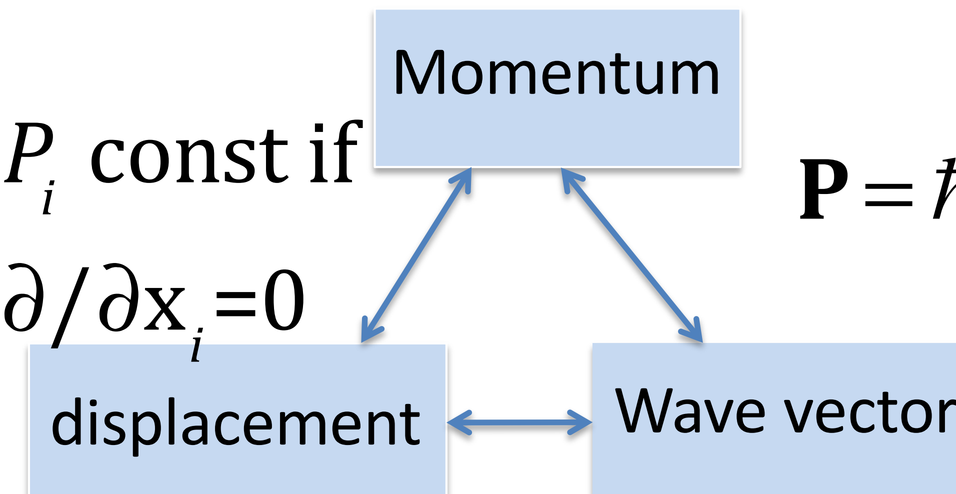
$$1=\Delta t\Delta\omega$$

Sinusoidal waves

$$\exp\left(i\mathbf{k}\cdot\mathbf{x}-i\omega t\right)$$

P_i const if

$\partial/\partial x_i=0$



$$\mathbf{P}=\hbar\mathbf{k}$$

$$1=\Delta x\Delta k$$

Maxwell Eqs. Phasor Amplitudes

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \cdot \vec{\mathbf{H}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad \nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

To get equations for phasor amplitudes $\vec{\mathbf{E}}, \vec{\mathbf{H}} \Rightarrow \hat{\mathbf{E}}, \hat{\mathbf{H}} \quad \frac{\partial}{\partial t}, \nabla \Rightarrow -i\omega, i\mathbf{k}$

$$i\mathbf{k} \cdot \hat{\mathbf{E}} = 0 \qquad i\mathbf{k} \cdot \hat{\mathbf{H}} = 0 \qquad i\mathbf{k} \times \hat{\mathbf{E}} = i\omega\mu\hat{\mathbf{H}} \qquad i\mathbf{k} \times \hat{\mathbf{H}} = -i\omega\varepsilon\hat{\mathbf{E}}$$

Combine

$$i\mathbf{k}\cdot\hat{\mathbf{E}}=0 \qquad i\mathbf{k}\cdot\hat{\mathbf{H}}=0 \qquad i\mathbf{k}\times\hat{\mathbf{E}}=i\omega\mu\hat{\mathbf{H}} \qquad i\mathbf{k}\times\hat{\mathbf{H}}=-i\omega\varepsilon\hat{\mathbf{E}}$$

combine $$i\mathbf{k}\times\left(i\mathbf{k}\times\hat{\mathbf{E}}\right)=i\omega\mu\left(i\mathbf{k}\times\hat{\mathbf{H}}\right)=i\omega\mu\left(-i\omega\varepsilon\hat{\mathbf{E}}\right)=\omega^2\varepsilon\mu\hat{\mathbf{E}}$$

Use “BAC CAB” $$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

$$\left(\mathbf{k}\cdot\mathbf{k}\right)\hat{\mathbf{E}}-\mathbf{k}\left(\mathbf{k}\cdot\hat{\mathbf{E}}\right)=k^2\hat{\mathbf{E}}=\omega^2\varepsilon\mu\hat{\mathbf{E}}$$

Plane waves in 3D

$$\left(k^2 - \omega^2 \varepsilon\mu\right)\hat{\mathbf{E}} = 0, \quad \mathbf{k}\cdot\hat{\mathbf{E}} = 0$$

E can be in any direction perpendicular to k,

Dispersion relation

$$\omega^2 = \frac{k^2}{\varepsilon\mu} = k^2 v^2 \qquad k^2 = |\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$$

Faraday's Law $\mathbf{k}\times\hat{\mathbf{E}} = \omega\mu\hat{\mathbf{H}}$

$$\frac{\mathbf{k}}{|\mathbf{k}|}\times\hat{\mathbf{E}} = \frac{\omega\mu}{|\mathbf{k}|}\hat{\mathbf{H}} = \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{H}}$$

Remember 1D

$$\omega = \pm kv$$

$$\lambda = v / f$$

$$\lambda = 2\pi / k$$

$$\omega = 2\pi f$$

Superposition of Solutions

Maxwell's Eqs. are linear in E & H. Thus, separate solutions can be added together. Consider 2 solutions:

$$\vec{\mathbf{E}}_1(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}_1 \exp\left[i\left(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t\right)\right]\right\} \quad k_1^2 = \omega_1^2 \varepsilon\mu$$

$$\vec{\mathbf{E}}_2(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}_2 \exp\left[i\left(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t\right)\right]\right\} \quad k_2^2 = \omega_2^2 \varepsilon\mu$$

Then $\mathbf{E}_1 + \mathbf{E}_2$ is also a solution of Maxwell's Equations

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\sum_j \hat{\mathbf{E}}_j \exp\left[i\left(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t\right)\right]\right\} \quad k_j^2 = \varepsilon\mu\omega^2(\mathbf{k}_j)$$

Turn the sum into an integral

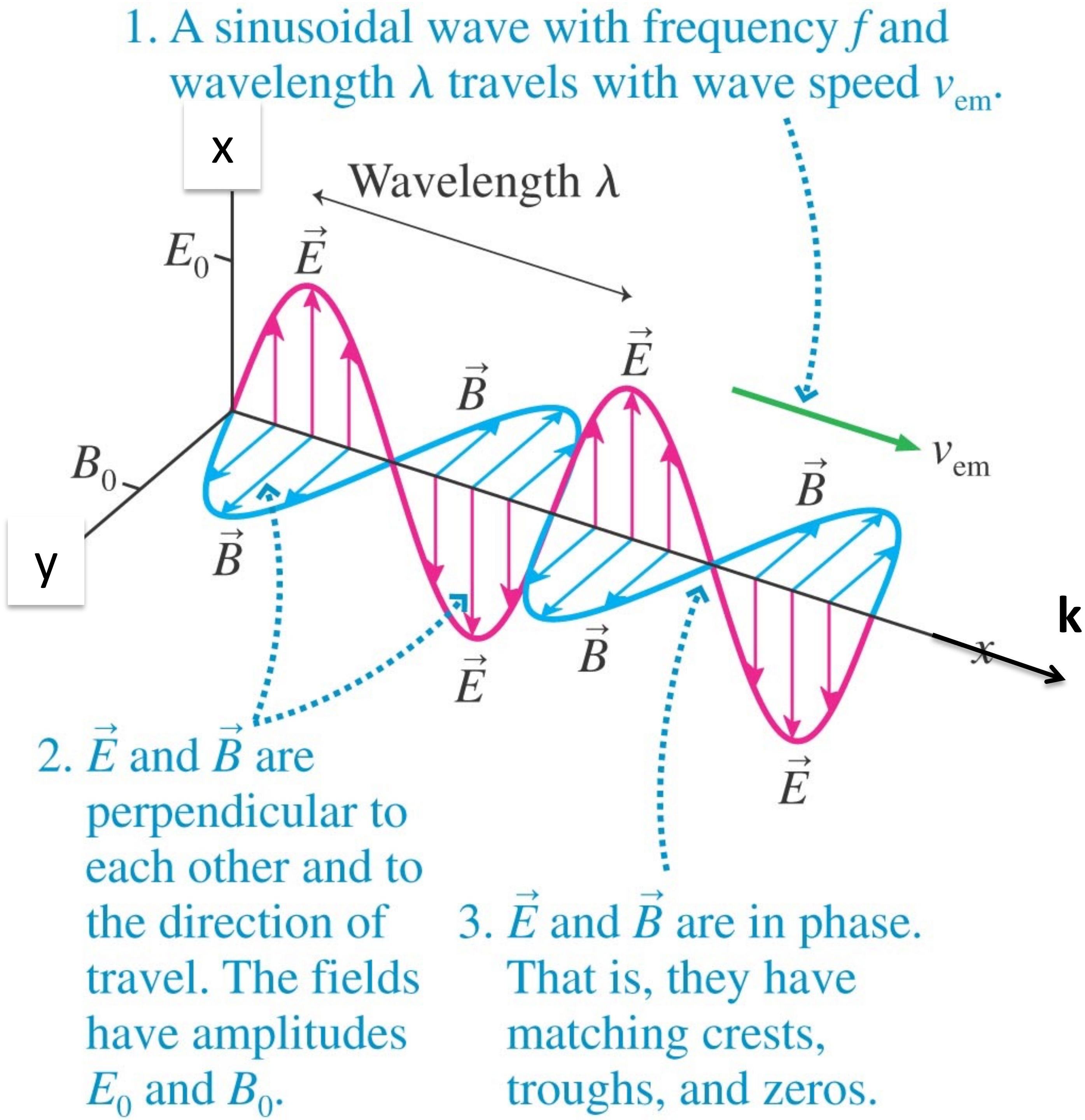
$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\sum_j \hat{\mathbf{E}}_j \exp\left[i\left(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t\right)\right]\right\} \quad k_j^2 = \varepsilon\mu\omega^2(\mathbf{k}_j)$$

Fourier Integral - used to solve diffraction and dispersion

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\int d^3k\, \bar{\hat{\mathbf{E}}}(\mathbf{k}) \exp\left[i\left(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t\right)\right]\right\} \quad k^2 = \varepsilon\mu\omega^2(\mathbf{k})$$

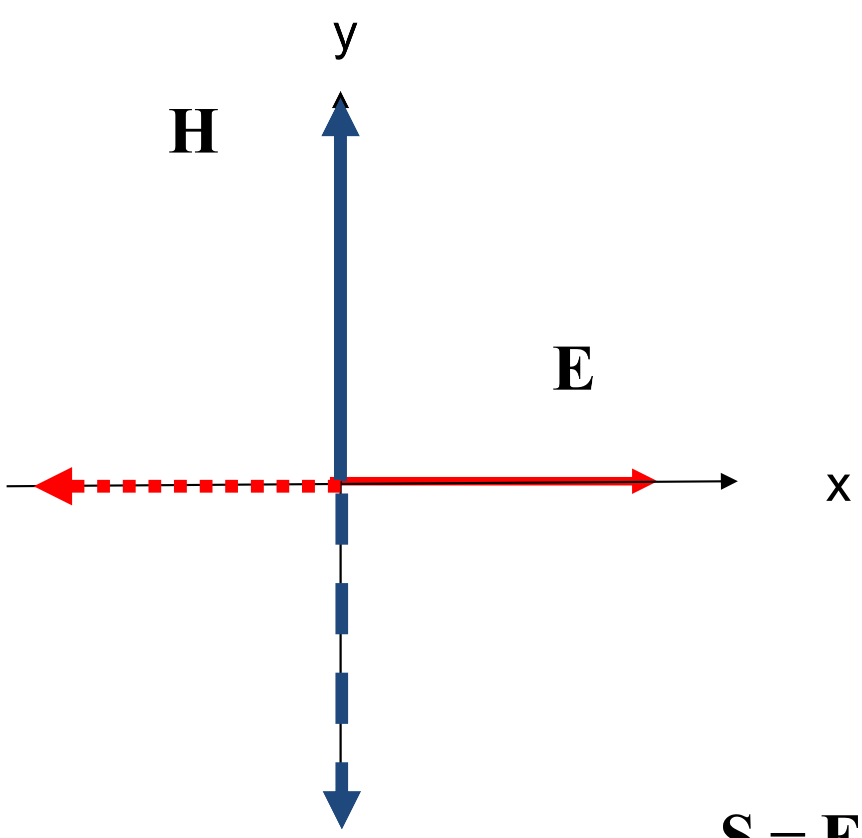
Linearly Polarized Waves

Electric field vector lies in one plane

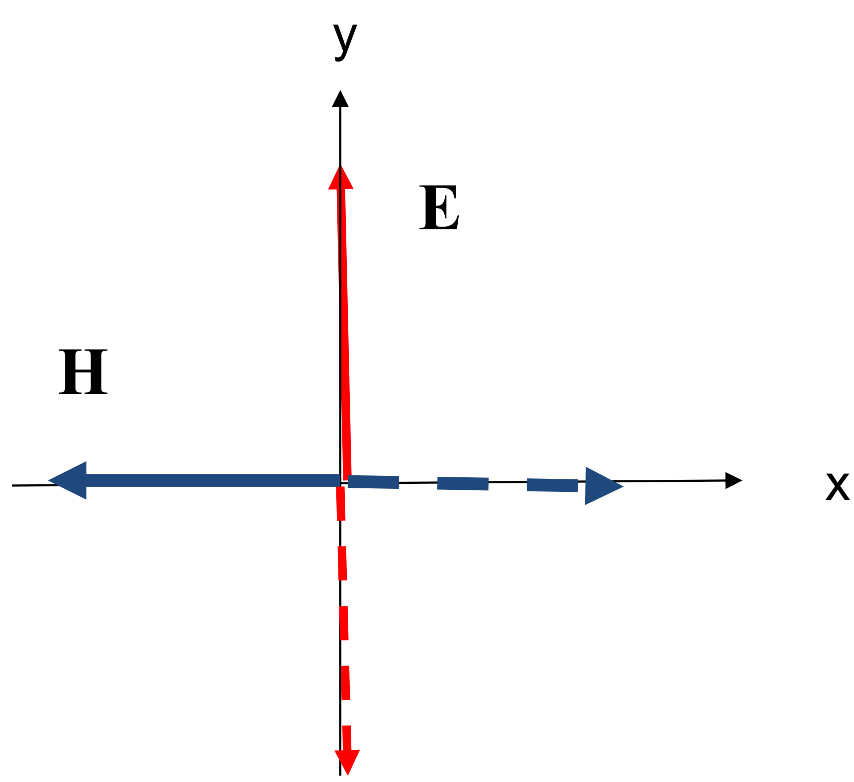


Linear Polarizations

Linearly Polarized in X direction



Linearly Polarized in y direction

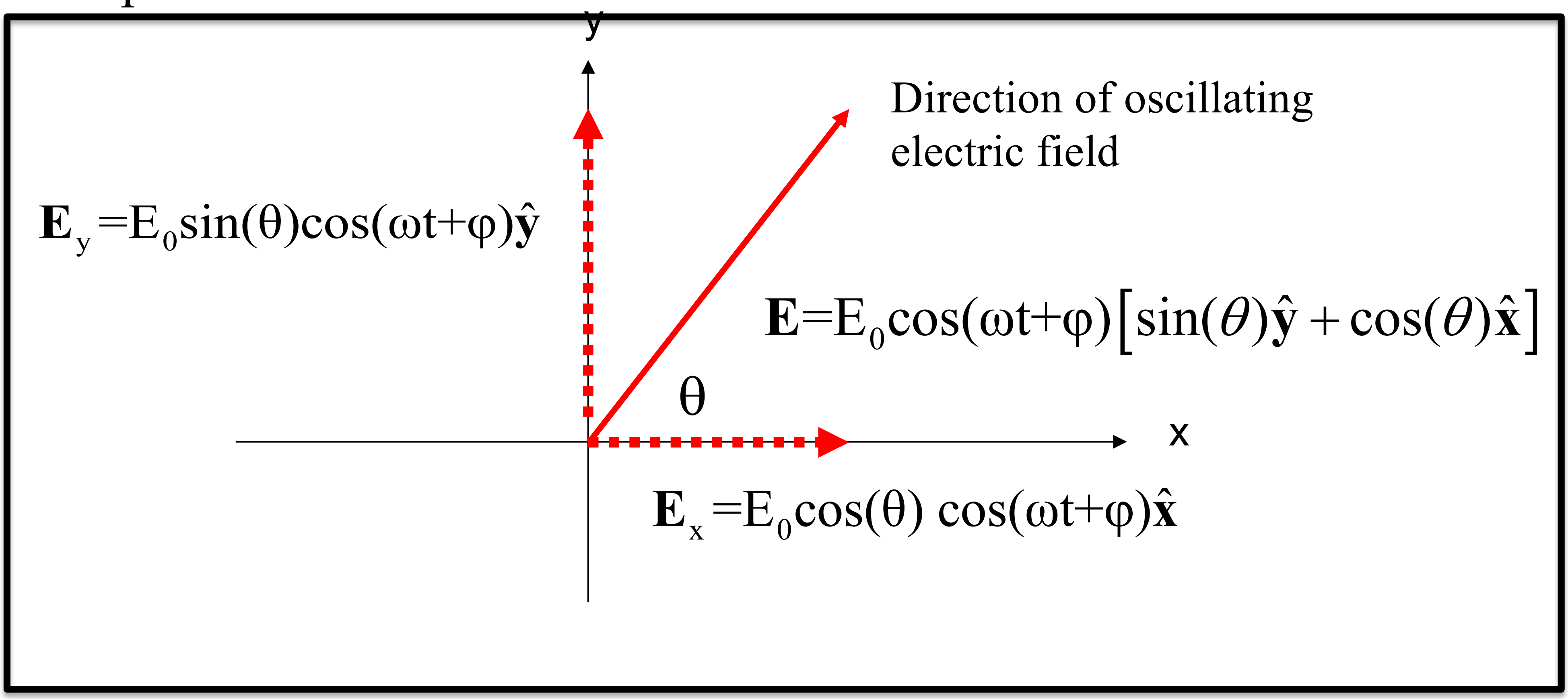


$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = S\hat{\mathbf{z}}$$

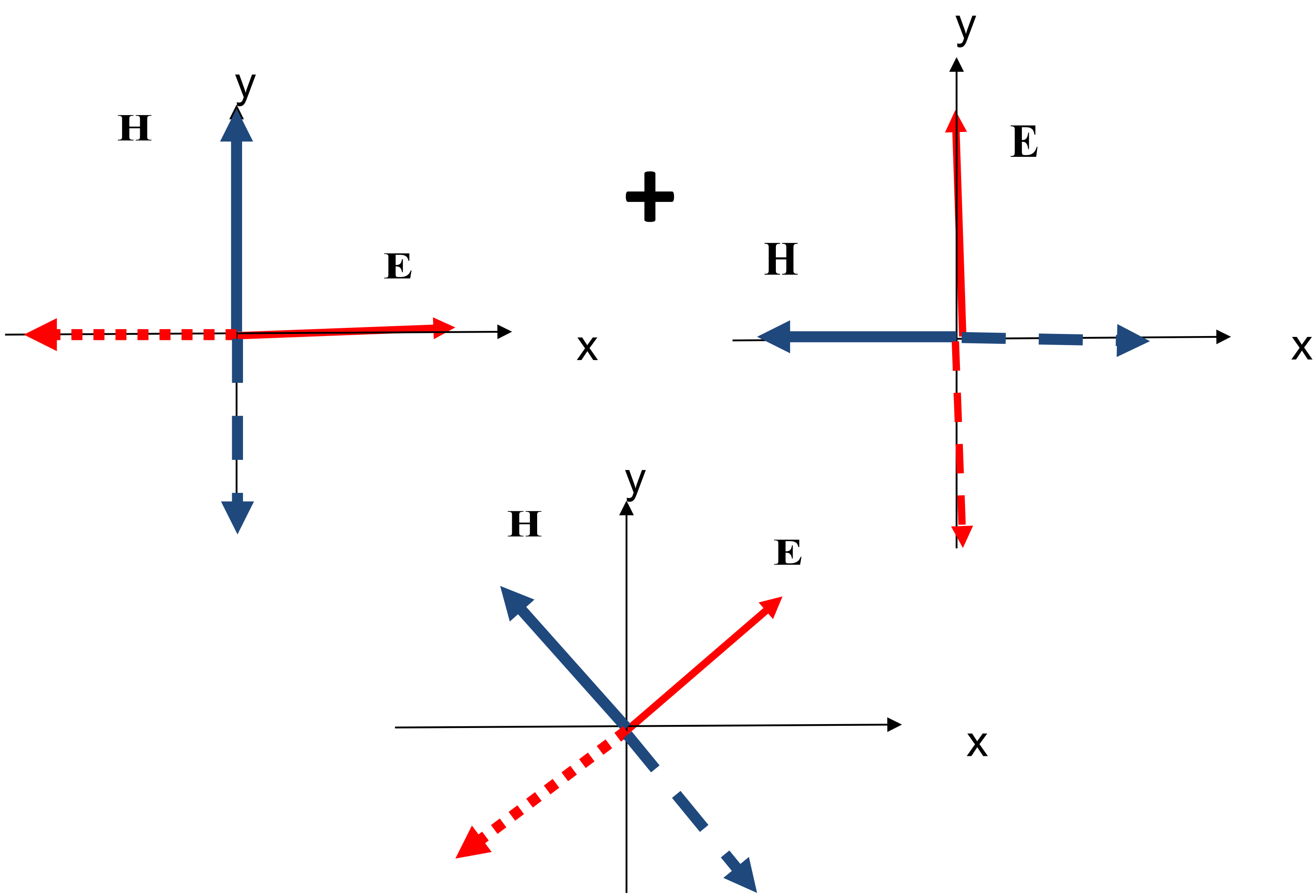
Linear Polarization

For a wave propagating in z direction, a linearly polarized wave has x and y components oscillating in **phase**

z - plane



In phase superposition of two linearly polarized waves

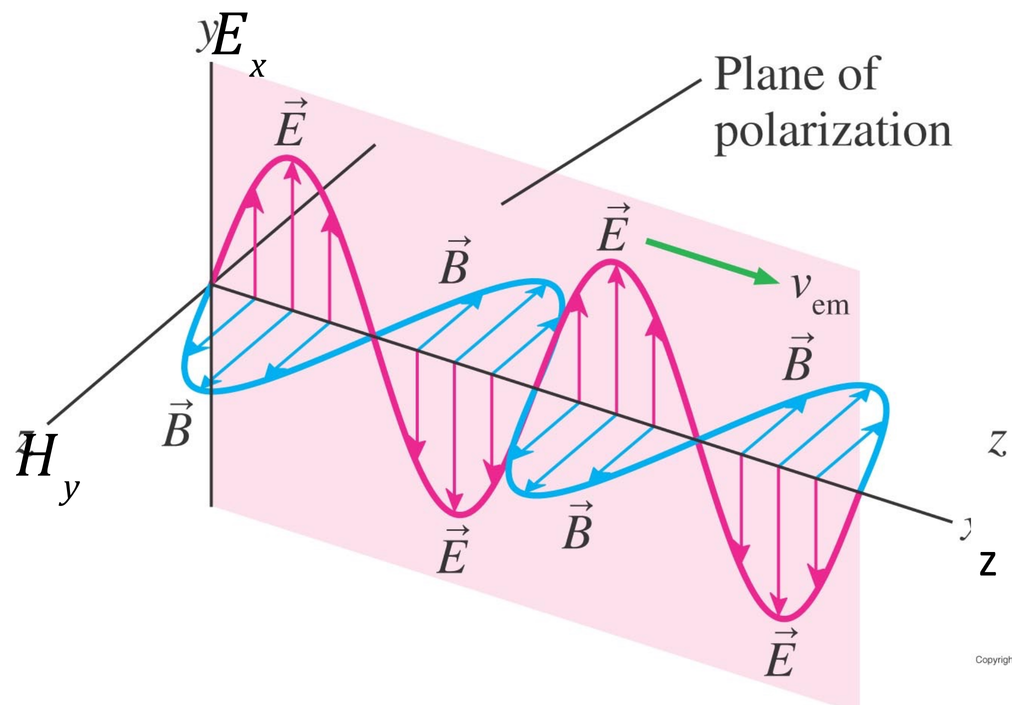


Polarizations

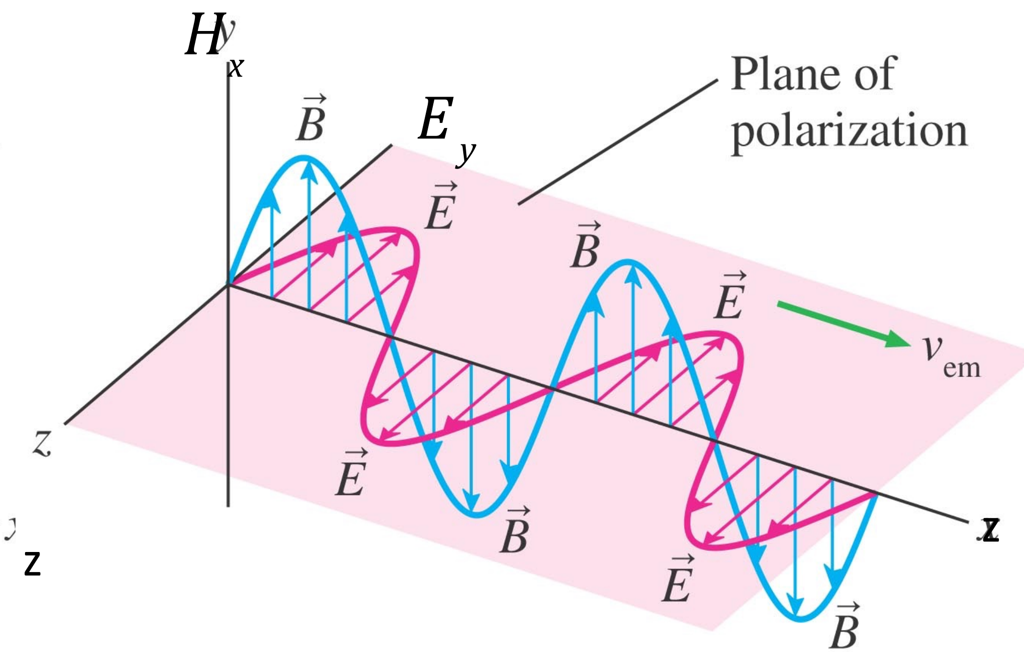
We picked this combination of fields: E_x - H_z

Could have picked this combination of fields: E_y - B_x

(a) Vertical polarization



(b) Horizontal polarization

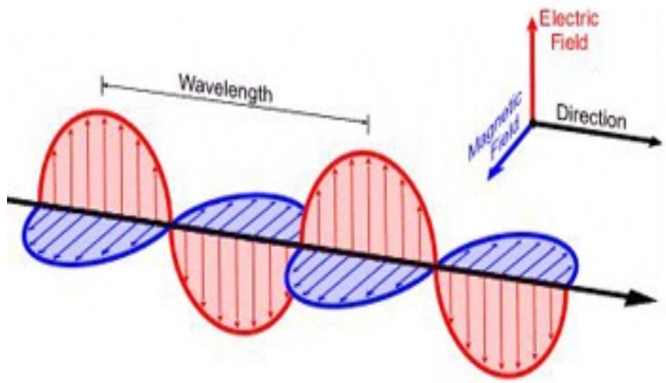


These are called plane polarized. Fields lie in plane

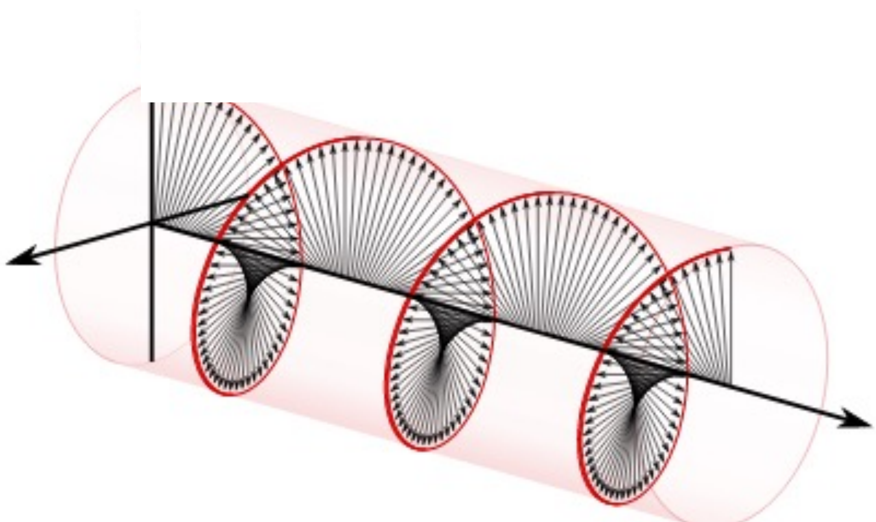
Polarization

Polarization is determined by the direction of **E** field

The wave is **linearly** polarized if the **electric field** oscillates in **one plane**



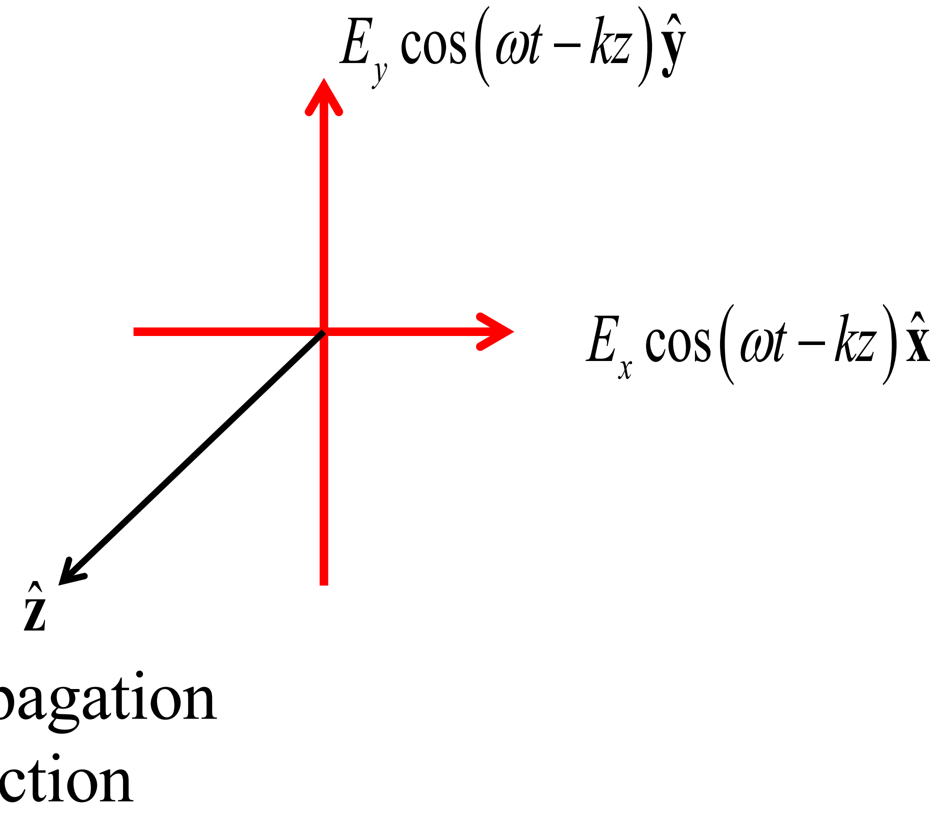
Linear polarization



Circular polarization

Polarization of Fields

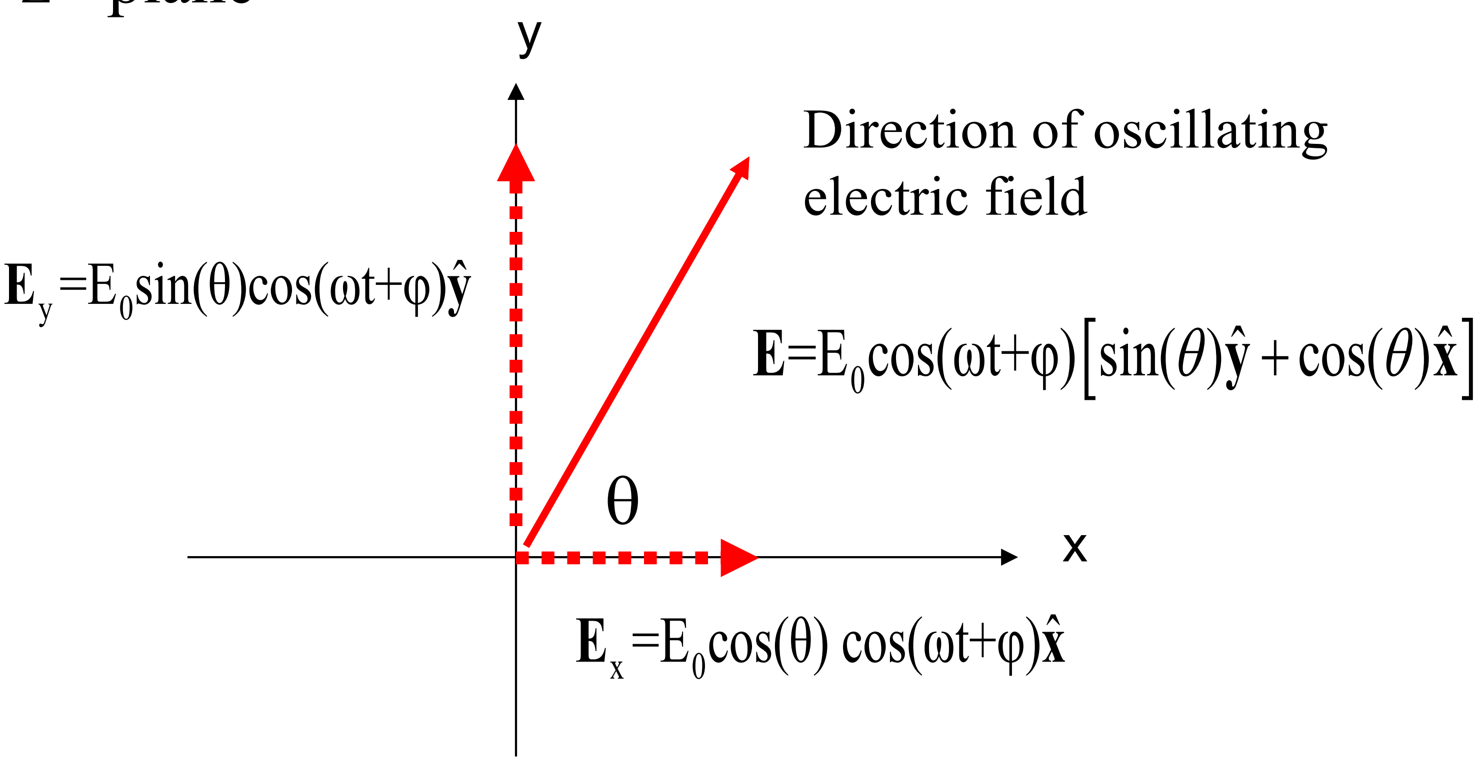
Consider two waves having the same frequency, same directions of propagation, but different orthogonal linear polarizations



Linear Polarization

For a wave propagating in z direction, a linearly polarized wave has x and y components oscillating in **phase**

z - plane



Polarization of Electric Fields

Three parameters determine the state of polarization

Consider a wave propagating in z direction

1. Field strength along x direction

2. Field strength along y direction

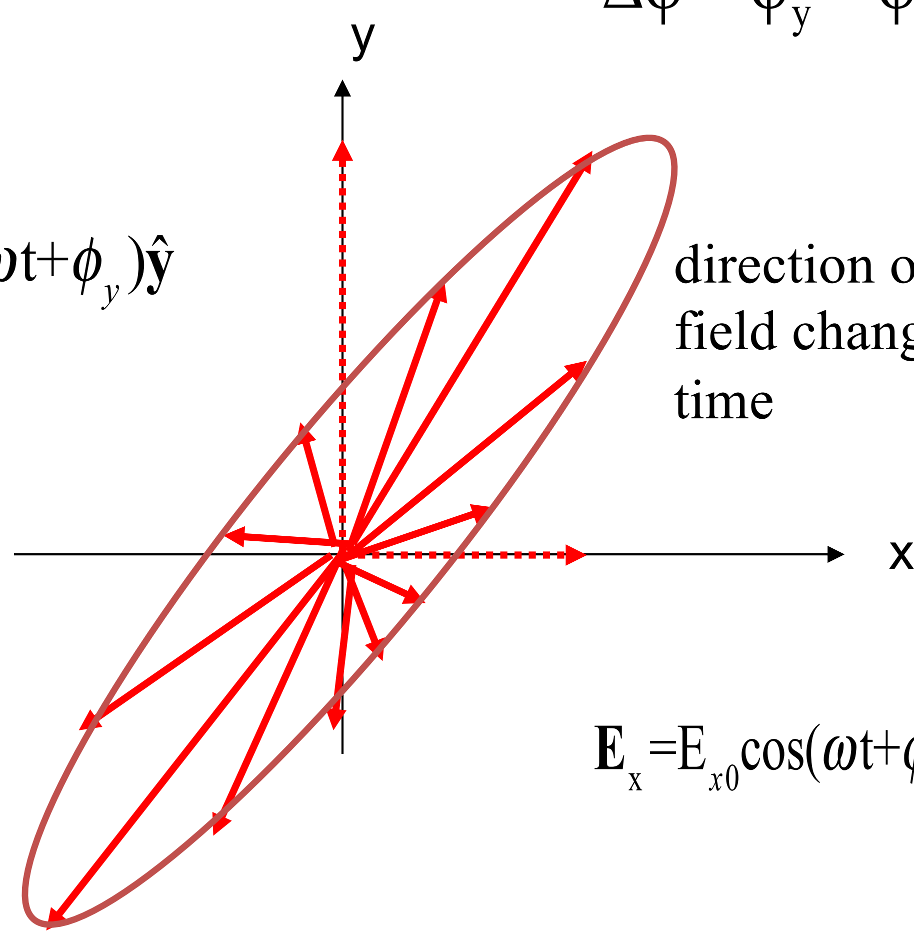
3. Relative phase shift between them

Elliptical Polarization

Elliptically polarized light has x and y field components **not** oscillating **in phase**

$$\Delta\varphi = \varphi_y - \varphi_x \neq 0$$

$$\mathbf{E}_y = E_{y0}\cos(\omega t + \phi_y)\hat{\mathbf{y}}$$



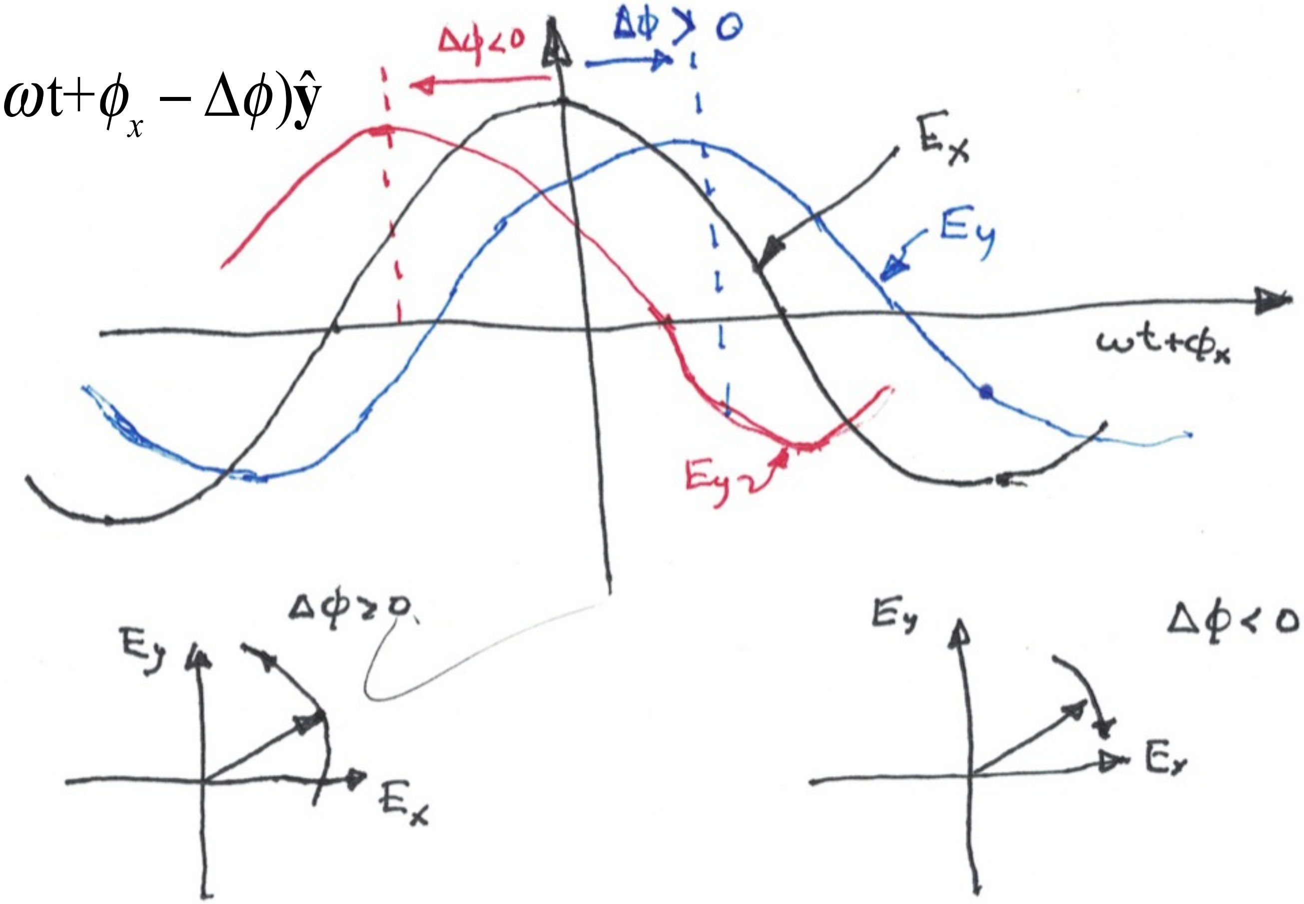
direction of electric field changes in time

$$\mathbf{E}_x = E_{x0}\cos(\omega t + \phi_x)\hat{\mathbf{x}}$$

Phase Relation between E_x and E_y

$$\mathbf{E}_x = E_{x0}\cos(\omega t + \phi_x)\hat{\mathbf{x}} \qquad \mathbf{E}_y = E_{y0}\cos(\omega t + \phi_y)\hat{\mathbf{y}} \qquad \Delta\varphi = \varphi_y - \varphi_x \neq 0$$

$$\mathbf{E}_y = E_{y0}\cos(\omega t + \phi_x - \Delta\phi)\hat{\mathbf{y}}$$



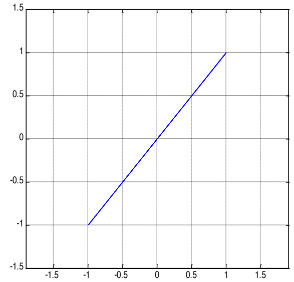
Different States of Polarization

$$\Delta\varphi = \varphi_y - \varphi_x$$

$$E_{x0} = E_{y0}$$

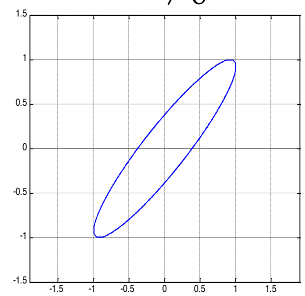
$|E_x| = |E_y|$

$\Delta\varphi = 0$



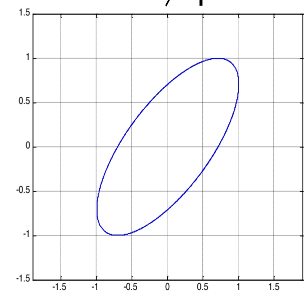
linear

$\Delta\varphi = \pi/8$



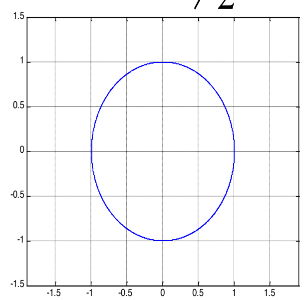
elliptical

$\Delta\varphi = \pi/4$



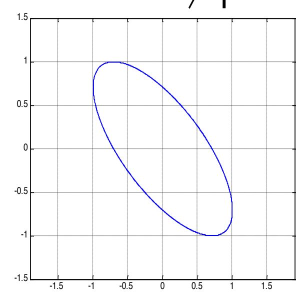
elliptical

$\Delta\varphi = \pi/2$



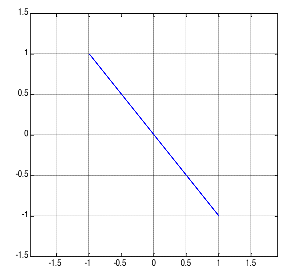
circular

$\Delta\varphi = 3\pi/4$



elliptical

$\Delta\varphi = \pi$



linear

Problem

An electromagnetic wave travelling in vacuum in the +z direction has the real electric field at z=0,

$$\mathbf{E}(z=0,t) = E_{0x}\cos(\omega t + \pi/4)\hat{\mathbf{x}} + E_{0y}\cos(\omega t - \pi/4)\hat{\mathbf{y}}$$

Represent this wave in phasor form:

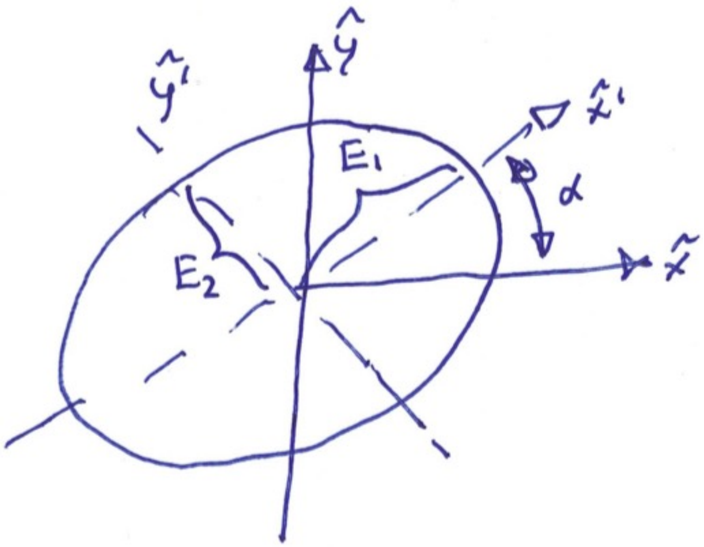
$$\mathbf{E}(z,t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(kz - \omega t\right)\right]\right\}$$

$$\mathbf{H}(z,t) = \mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(kz - \omega t\right)\right]\right\}$$

What is the polarization of the wave? Plane, circular, elliptical?

$$\mathbf{E}(z=0,t)=E_{0x}\cos(\omega t+\pi/4)\hat{\mathbf{x}}+E_{0y}\cos(\omega t-\pi/4)\hat{\mathbf{y}}$$

Elliptical Polarization



In primed coordinate system

$$E_{x'} = E_1 \cos(\omega t)$$
$$E_{y'} = E_2 \sin(\omega t)$$

In unprimed coordinate system

$$E_x = \left(\cos\alpha E_{x'} - \sin\alpha E_{y'}\right) = \left(\cos\alpha E_1 \cos(\omega t) - \sin\alpha E_2 \sin(\omega t)\right) = \left|E_x\right| \cos(\omega t + \phi_x)$$
$$E_y = \left(\sin\alpha E_{x'} + \cos\alpha E_{y'}\right) = \left(\sin\alpha E_1 \cos(\omega t) + \cos\alpha E_2 \sin(\omega t)\right) = \left|E_y\right| \cos(\omega t + \phi_y)$$

Given $\left|E_x\right|, \left|E_y\right|, \phi_x, \phi_y$ find: E_1, E_2, α

$$E_x = \left(\cos\alpha E_1 \cos(\omega t) - \sin\alpha E_2 \sin(\omega t)\right) = \left|\hat{E}_x\right| \cos(\omega t + \phi_x)$$

$$E_y = \left(\sin\alpha E_1 \cos(\omega t) + \cos\alpha E_2 \sin(\omega t)\right) = \left|\hat{E}_y\right| \cos(\omega t + \phi_y)$$

Time average $\langle . \rangle$

$$\left\langle E_x^2 \right\rangle = \frac{1}{2}\left|\hat{E}_x\right|^2 = \frac{1}{2}\left[\cos^2\alpha E_1^2 + \sin^2\alpha E_2^2\right] = \frac{1}{4}\left[\left(E_1^2 + E_2^2\right) + \cos 2\alpha \left(E_1^2 - E_2^2\right)\right]$$

$$\left\langle E_y^2 \right\rangle = \frac{1}{2}\left|\hat{E}_y\right|^2 = \frac{1}{2}\left[\cos^2\alpha E_2^2 + \sin^2\alpha E_1^2\right] = \frac{1}{4}\left[\left(E_1^2 + E_2^2\right) - \cos 2\alpha \left(E_1^2 - E_2^2\right)\right]$$

$$\left\langle E_x E_y \right\rangle = \frac{1}{2}\left|\hat{E}_x\right|\left|\hat{E}_y\right| \cos(\phi_y - \phi_x) = \frac{1}{2}\sin\alpha\cos\alpha\left(E_1^2 - E_2^2\right) = \frac{1}{4}\left(E_1^2 - E_2^2\right)\sin 2\alpha$$

So,

$$\left\langle E_x^2 \right\rangle + \left\langle E_y^2 \right\rangle = \frac{1}{2}\left(E_1^2 + E_2^2\right), \quad \left\langle E_x^2 \right\rangle - \left\langle E_y^2 \right\rangle = \frac{\cos 2\alpha}{2}\left(E_1^2 - E_2^2\right),$$

$$2\left\langle E_x E_y \right\rangle = \frac{\sin 2\alpha}{2}\left(E_1^2 - E_2^2\right)$$

$$\left\langle E_x^2 \right\rangle + \left\langle E_y^2 \right\rangle = \frac{1}{2}\left(E_1^2 + E_2^2 \right), \qquad \left\langle E_x^2 \right\rangle - \left\langle E_y^2 \right\rangle = \frac{\cos 2\alpha}{2}\left(E_1^2 - E_2^2 \right),$$

$$2\left\langle E_x E_y \right\rangle = \frac{\sin 2\alpha}{2}\left(E_1^2 - E_2^2 \right)$$

$$\tan 2\alpha = \frac{2\left\langle E_x E_y \right\rangle}{\left\langle E_x^2 \right\rangle - \left\langle E_y^2 \right\rangle},$$

$$E_1^2 = \left\langle E_x^2 \right\rangle \left(1 + \frac{1}{\cos 2\alpha} \right) + \left\langle E_y^2 \right\rangle \left(1 - \frac{1}{\cos 2\alpha} \right)$$

$$E_2^2 = \left\langle E_x^2 \right\rangle \left(1 - \frac{1}{\cos 2\alpha} \right) + \left\langle E_y^2 \right\rangle \left(1 + \frac{1}{\cos 2\alpha} \right)$$