ENEE681
Lecture 04

Radiation

Topics to be covered

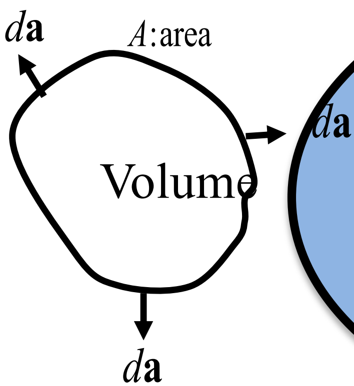
Scalar and Vector Potentials
Green's functions for static and dynamic fields

Review: Static Electric Fields

Coulomb's Law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r'$$

Gauss' Law:



$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho d^3r$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0$$

Poisson Equation:

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{\mathbf{E}} = 0$$

Review: Magnetostatics

Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r'$$

Ampere’s Law:

$$\oint_{loop} \vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} = \mu_0 I_{enclosed}$$

$$= \mu_0 \oint_S \vec{\mathbf{J}}\cdot d\vec{\mathbf{A}}$$

$$\oint_S \vec{\mathbf{B}}\cdot d\vec{\mathbf{A}} = 0$$

Gauss’ Law:

$$\nabla\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}}$$

$$\nabla\cdot\vec{\mathbf{B}} = 0$$

Coulomb's Law for Electrostatic Potential

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r'$$

This can also be written in terms of a scalar potential

$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}),$ Automatically satisfies $\nabla\times\mathbf{E}(\mathbf{r}) = -\nabla\times\nabla\phi(\mathbf{r}) = 0$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3r',$$

Show:

$$-\nabla\phi(\mathbf{r}) = \frac{-1}{4\pi\varepsilon_0}\int_V \frac{\partial}{\partial\mathbf{r}}\frac{\rho(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} d^3r' = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r'$$

Coulomb's Law is Solution to the Following

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{\mathbf{E}} = 0$$

Introduce scalar potential

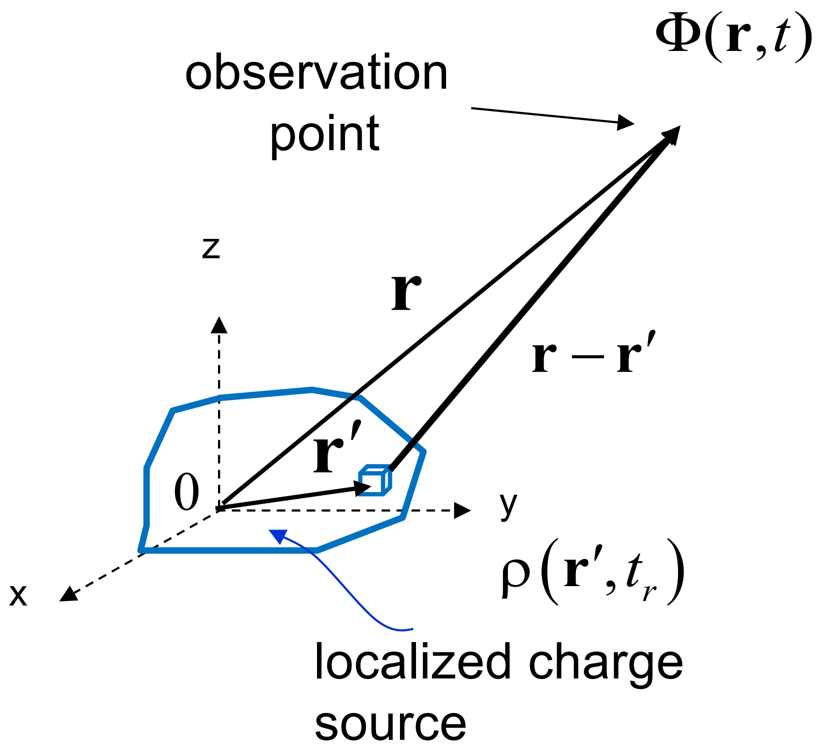
$$\nabla \cdot \vec{\mathbf{E}} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$$

$$-\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$$

Solution is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

Integral

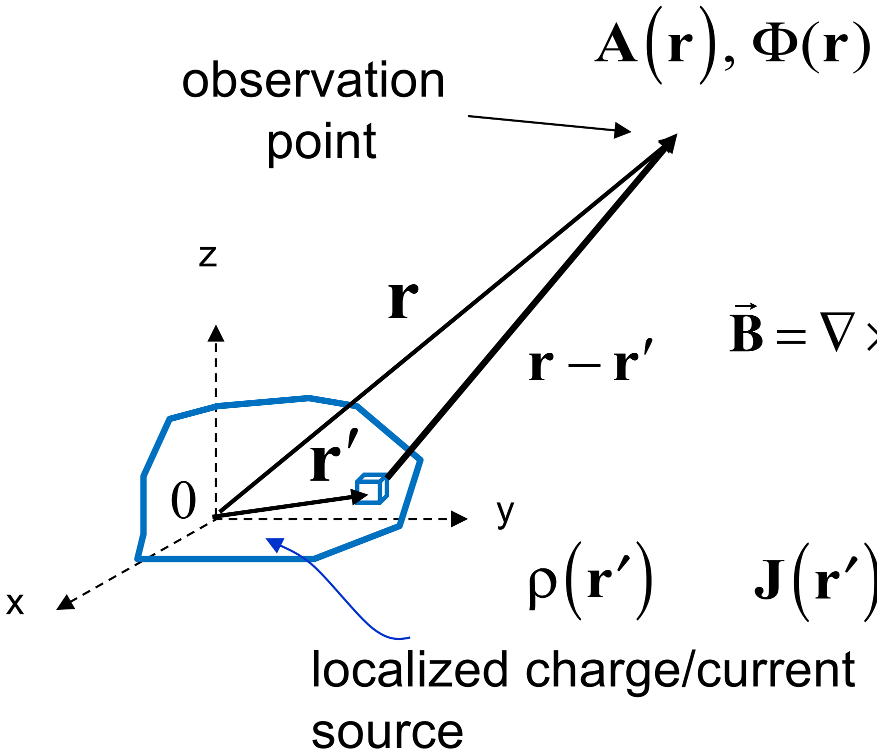


$$\nabla\cdot\vec{\mathbf{E}} = -\nabla\cdot\nabla\phi = -\nabla^2\phi = \frac{\rho}{\varepsilon_0}$$

Solution is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

Magnetic Field



$$\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{Vol} d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{Vol} d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

In static approximation time does not enter.

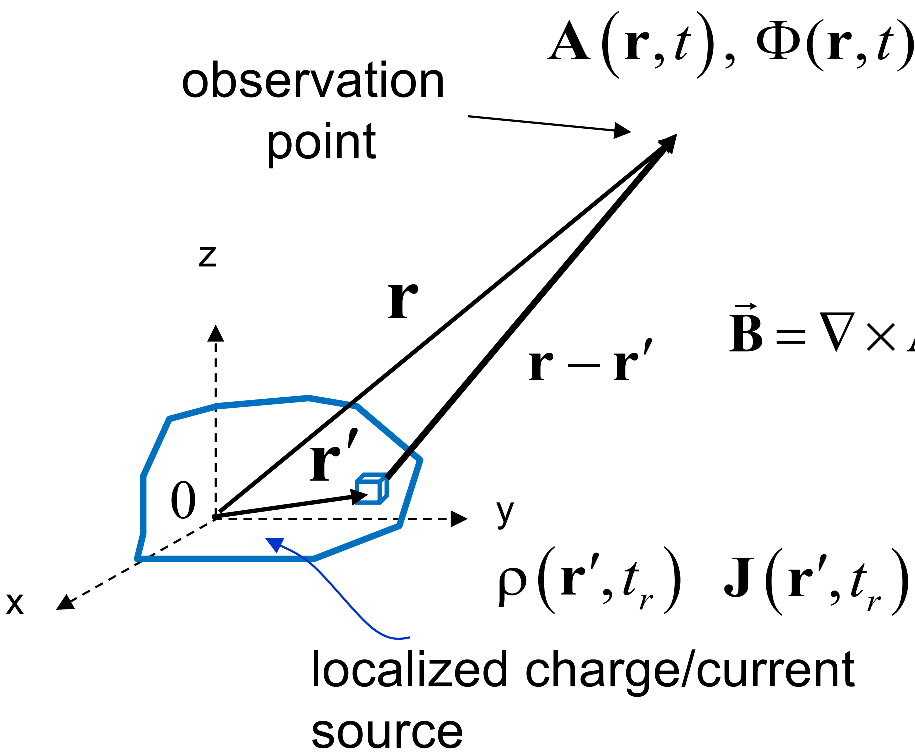
Changes in charge and current densities in time

lead to changes in potentials.

No time delay!?

$$dr' = dx'\, dy'\, dz'$$

Time-retarded potentials



$$\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int_{Vol} d\mathbf{r}' \left. \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} \right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{Vol} d\mathbf{r}' \left. \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} \right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$dr' = dx'\,dy'\,dz'$$

What About Magnetic Field?

Basic Equations $\nabla \cdot \vec{\mathbf{B}} = 0$ $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$

$\nabla \cdot \vec{\mathbf{B}} = 0$ Implies $\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$

$\nabla \cdot \left(\nabla \times \mathbf{A}(\mathbf{r}) \right) = 0$ For any $\mathbf{A}$

$\mathbf{A}$ is not unique.

If $\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$ then $\vec{\mathbf{B}} = \nabla \times \mathbf{A}'(\mathbf{r})$,

where $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi$ for any χ $\nabla \times \nabla\ \chi = 0$

How to understand

$$\mathbf{A} = \left(\hat{\mathbf{x}}A_x(x,y,z) + \hat{\mathbf{y}}A_y(x,y,z) + \hat{\mathbf{z}}A_z(x,y,z)\right)$$ Has three independent functions

$$\mathbf{B} = \left(\hat{\mathbf{x}}B_x(x,y,z) + \hat{\mathbf{y}}B_y(x,y,z) + \hat{\mathbf{z}}B_z(x,y,z)\right)$$ Has only two independent functions

How so?

$$\nabla \cdot \mathbf{B} = \left(\frac{\partial}{\partial x}B_x(x,y,z) + \frac{\partial}{\partial y}B_y(x,y,z) + \frac{\partial}{\partial z}B_z(x,y,z)\right) = 0$$

If I tell you B_x and B_y you can integrate over z to find B_z

Only two independent functions.

Gauges

We need to make up one scalar rule about **A** to determine it uniquely.

Different rules are called different gauges.

Examples:

$$A_x = 0, \text{ No Name Gauge}$$

$$\nabla \cdot \mathbf{A} = 0, \text{ Coulomb Gauge}$$

$$\nabla \cdot \mathbf{A} + \varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0, \text{ Lorenz Gauge}$$

What About Magnetic Field?

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \qquad \nabla \cdot \vec{\mathbf{B}} = 0$$

$\nabla \cdot \vec{\mathbf{B}} = 0$ Implies $\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$

Ampere's Law

$$\nabla \times \vec{\mathbf{B}} = \nabla \times (\nabla \times \mathbf{A}) = \mu_o \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_o \mathbf{J}$$

If Coulomb Gauge $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$,

$$-\nabla^2 \mathbf{A} = \mu_o \mathbf{J}, \qquad -\nabla^2 \phi = \rho / \varepsilon_o$$

Same equation!

Static Scalar and Vector potential

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0}, \qquad -\nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Solutions are:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r', \qquad A(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

Time Dependent Fields

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Introduce Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Insert in Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

Time Dependent Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0\left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \; \mathbf{E} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0\left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \times \; \nabla \times \mathbf{A} = \mu_0\left(\vec{\mathbf{J}} - \varepsilon_0 \frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}\right)\right)$$

$$\nabla \left(\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} \text{ - } \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{A}$$

Now Pick Lorenz Gauge

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho/\varepsilon_0$$

$$\nabla\left(\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}$$

Lorenz Gauge: $\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi = 0$

$$-\left(\nabla^2\phi - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\phi\right) = \rho/\varepsilon_0$$

$$-\left(\nabla^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}\right) = \mu_0\vec{\mathbf{J}}$$

Same Equation
Wave Equation

Maxwell's Equations for Vector and Scalar Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

In the Lorentz gauge $\left(\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} \right)$ the vector and scalar potentials obey wave equations

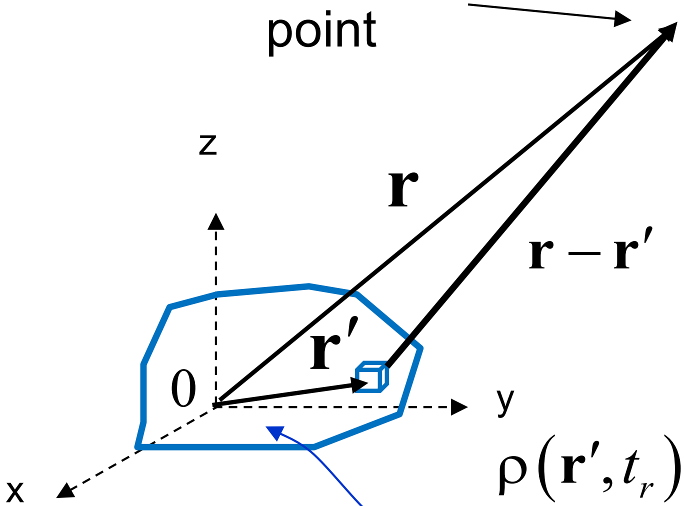
$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \sqrt{\varepsilon_0 \mu_0} = 1/c$$

$$\nabla^2 \Phi - \mu_0 \varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

where $\mathbf{J}$ and ρ are the current and charge densities

The solutions to the wave equations (in the absence of boundaries) are

Solution to Wave Equations



$$\mathbf{A}(\mathbf{r},t),\ \Phi(\mathbf{r},t)$$

$$\mathbf{r} \qquad \mathbf{r}-\mathbf{r}' \qquad \mathbf{r}'$$

$$\rho(\mathbf{r}',t_r) \quad \mathbf{J}(\mathbf{r}',t_r)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}' \left.\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}' \left.\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$d\tau' = dx'\,dy'\,dz'$$

Green's Function Solution

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\boldsymbol{x},t) = -4\pi f(\boldsymbol{x},t)$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\boldsymbol{x},t:\boldsymbol{x}'.t') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$$

$$\psi(\boldsymbol{x},t) = \int d^3x'\, dt' G(\boldsymbol{x},t:\boldsymbol{x}'.t') f(\boldsymbol{x},t)$$

$$G_+ = \begin{cases} R^{-1}\delta(t-t'-R/c) & t > t' \\ 0 & t < t' \end{cases}$$

$$R = \left|\boldsymbol{x}-\boldsymbol{x}'\right|$$