ENEE681

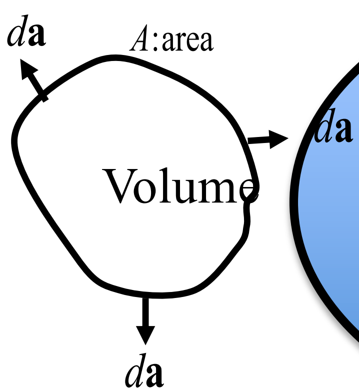
Lecture 2
Inductance
Mutual Inductance
Skin effect

Electrostatics

Coulomb's Law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r'$$

Gauss' Law:



$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho d^3r$$

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0$$

Poisson Equation:

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{\mathbf{E}} = 0$$

Magnetostatics

Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\times\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r'$$

Ampere's Law:

$$\oint_{loop} \vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} = \mu_0 I_{enclosed}$$

$$= \mu_0 \oint_S \vec{\mathbf{J}}\cdot d\vec{\mathbf{A}}$$

Gauss' Law:

$$\oint_S \vec{\mathbf{B}}\cdot d\vec{\mathbf{A}} = 0$$

$$\nabla\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}}$$

$$\nabla\cdot\vec{\mathbf{B}} = 0$$

Dynamic Fields

Faraday's Law

Maxwell's Displacement Current

Integrals around closed loops

Faraday's Law:

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Ampere's Law:

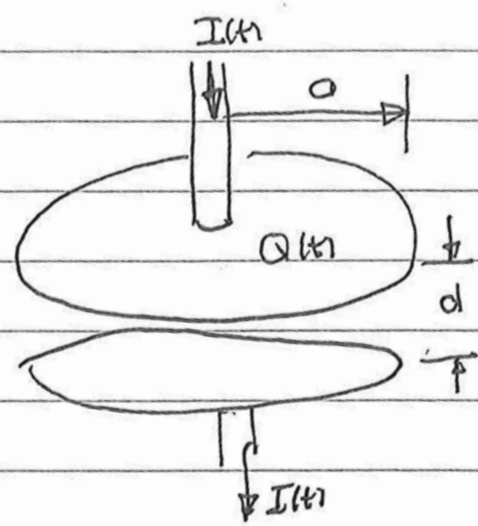
$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

$$\nabla \times \vec{\mathbf{B}}(\vec{\mathbf{r}}) = \mu_0 \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$

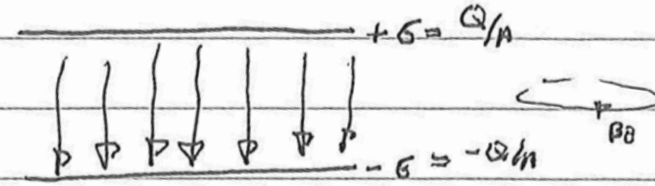
Displacement Current

Problem [redacted]

Find field inside



Start with the assumption that fields are electrostatic



$-\epsilon_0 E_z = \sigma = Q/A \qquad A = \pi a^2$

$E_z = -\frac{Q(t)}{\epsilon_0 A} \qquad \frac{dQ}{dt} = I$

Ampere's Law with Displacement Current

$$(\nabla \times B)_z = \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t} \qquad \frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \epsilon_0 \mu_0 \frac{\partial E_z}{\partial t}$$

$$B_\theta = \frac{r}{2}\epsilon_0\mu_0 \frac{\partial E_z}{\partial t} = \frac{r}{2}\mu_0\left(-\frac{1}{A}\frac{dQ}{dt}\right) = -\frac{\mu_0 r}{2A} I(t)$$

Faraday's Law

$$-\frac{\partial B_\theta}{\partial t} = (\nabla \times E)_\theta = -\frac{\partial}{\partial r} E_z$$

$$E_z(r) = E_z(0) + \int_0^r dr' \frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{r'}{2} \frac{\partial}{\partial t} E_z\right) \simeq E_z(0) + \frac{\epsilon_0 \mu_0 r^2}{4} \frac{\partial^2 E_z(0)}{\partial t^2}$$

Suppose $E_z(0) = E_0 \sin \omega t$

$$E_z(r) = E_0 \sin\omega t \left\{ 1 - \frac{\epsilon_0 \mu_0 r^2 \omega^2}{4} \right\}$$

$\lambda = c/f$

Require $\frac{\epsilon_0 \mu_0 a^2 \omega^2}{4} \ll 1$

$\frac{a^2\omega^2}{4c^2} \ll 1 \qquad \left(\frac{\pi a}{\lambda}\right)^2 \ll$

Actually we should solve

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \epsilon_0 \mu_0 \frac{\partial E_z}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial B_\theta}{\partial t} = \epsilon_0 \mu_0 \frac{\partial^2 E_z}{\partial t^2} \qquad \frac{\partial B_\theta}{\partial t} = \frac{\partial}{\partial r} E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial t} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \qquad E_z = \sout{\hat{E}_z(r)\cos\omega t} \quad Re\{\hat{E}_z(r) e^{-i\omega t}\}$$

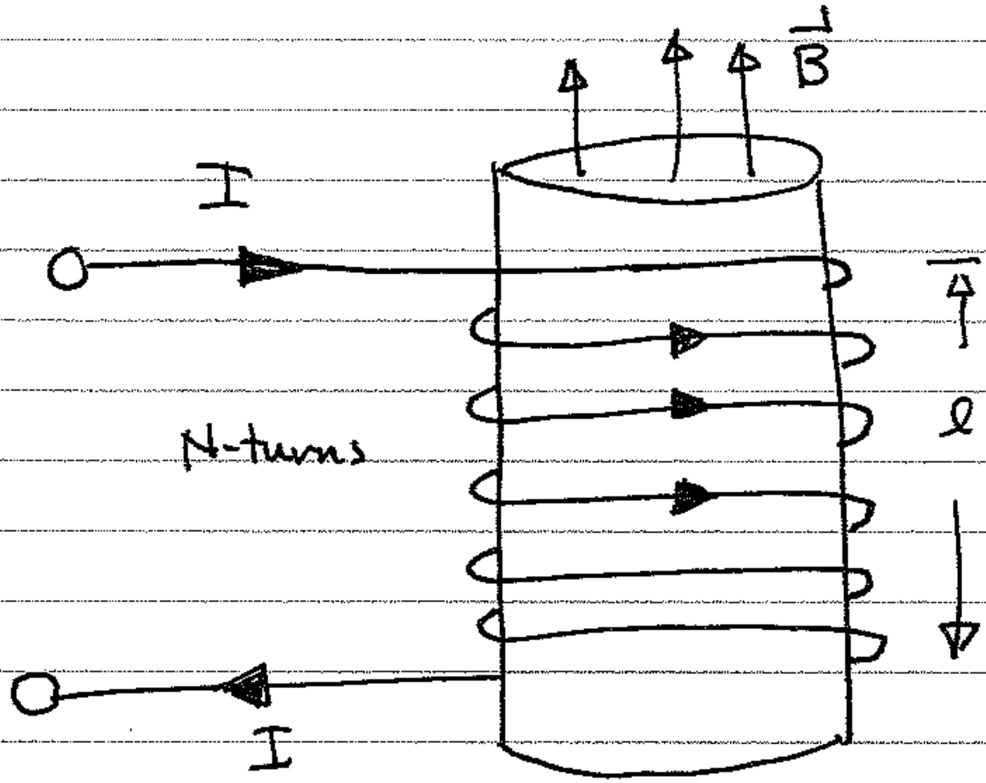
$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \hat{E}_z(r) + \frac{\omega^2}{c^2}\hat{E}_z(r) = 0$$ Bessel's Eqn. (3.7)

Bessel's Eqn $\quad \tilde{\hat{E}}_z = E_0 J_0(\frac{\omega}{c} r) \qquad J_0(x) \simeq 1 - \frac{1}{4}x^2 + O(x^4)$

$$J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{2m}}{(m!)^2}$$

$$\boxed{\frac{1}{x}\frac{\partial}{\partial x} x \frac{\partial}{\partial x} J_0 + J_0 = 0}$$

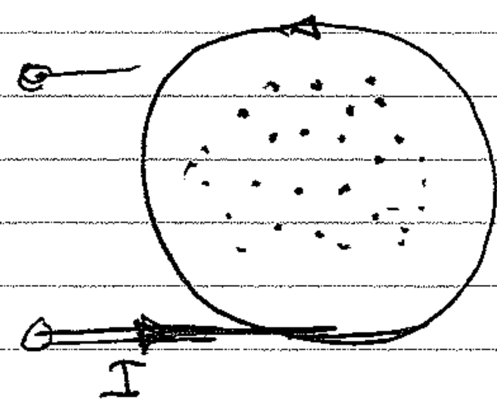
Consider a solenoid with N turns



Put your right thumb in direction of I. Fingers give direction of $\vec{B}$ (up inside)

$$|\vec{B}| = \frac{\mu_0 I N}{l}$$

VIEW FROM ABOVE



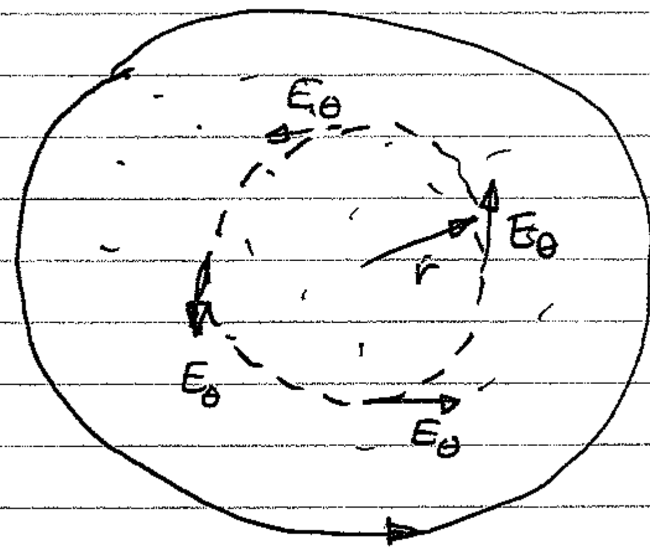
$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I$$

$$lB = \mu_0 N I$$

$\vec{B}$ out of page

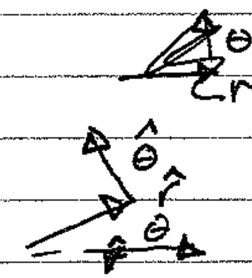
Calculate induced $\vec{E}$-field as a function of r

Consider a loop of radius r



$\vec{B}$ – out of page

$\vec{B} = B_z\hat{k}$



Q: Which direction is $\vec{E}$? $+\hat{\theta}$ or $-\hat{\theta}$

Ans: We don't know. Is B increasing or decreasing?

$$E_\theta = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

Faraday's Law for Stationary Loops

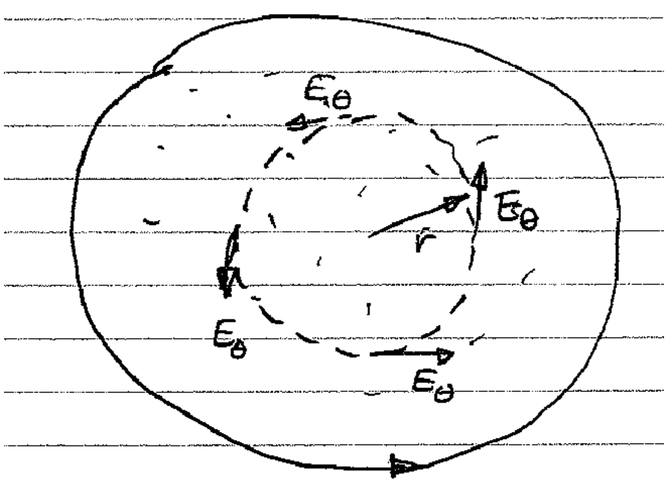
ccw

Out of page (+z)

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\mathbf{l} == -\int_{Area} \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}}$$

Only time derivative of B enters

Call component of E in τηετα direction $E_\theta(r,t)$



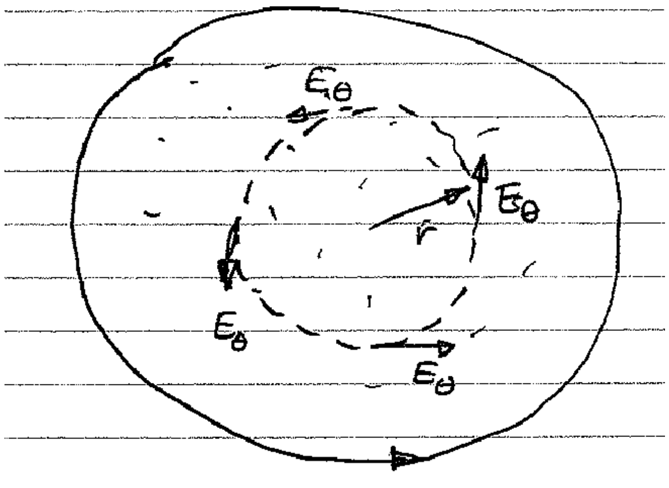
$$\oint_{loop} \vec{\mathbf{E}} \cdot d\mathbf{l} = 2\pi r E_\theta(r,t)$$

$$\int_{Area} \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} = \pi r^2 \frac{\partial B_z}{\partial t}$$

Therefore:

$$E_\theta(r,t) = -\frac{r}{2} \frac{\partial B_z}{\partial t}$$

Is Lenz's law satisfied ????

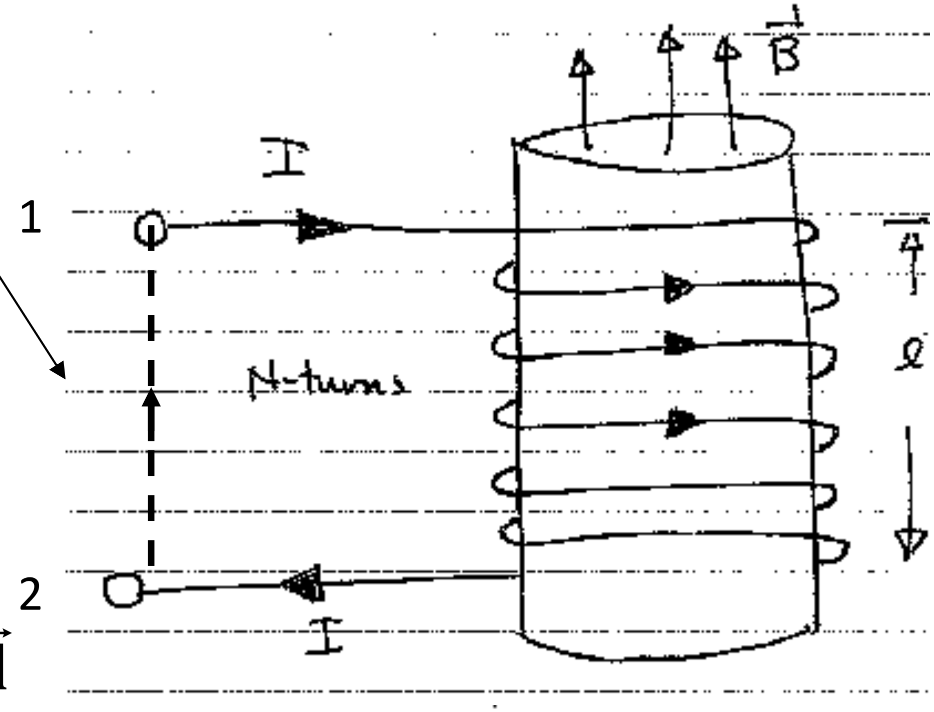


$$E_\theta(r,t) = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

B_z - out of page and increasing

An induced current would flow Counterclockwise

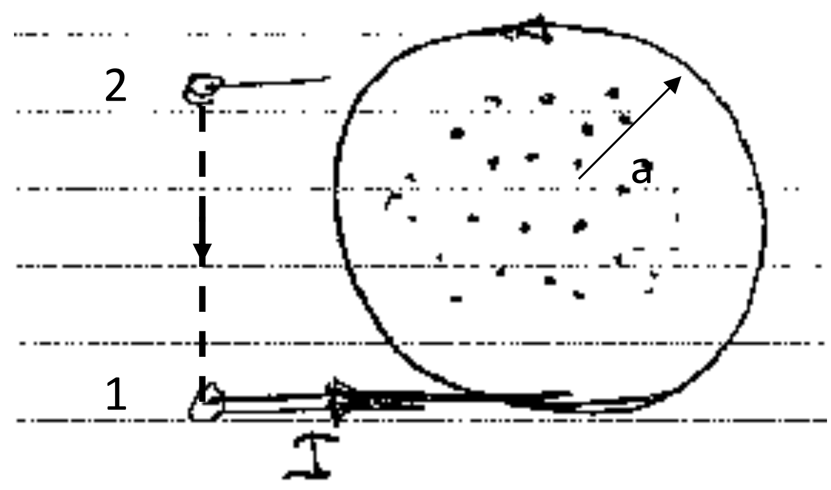
What is $\int_{2}^{1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$



N turns

$$\int_{2}^{1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} + \int_{1,\ \mathrm{wire}}^{2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

$$= \oint_{Loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -N\pi a^2 \frac{\partial B_z}{\partial t}$$

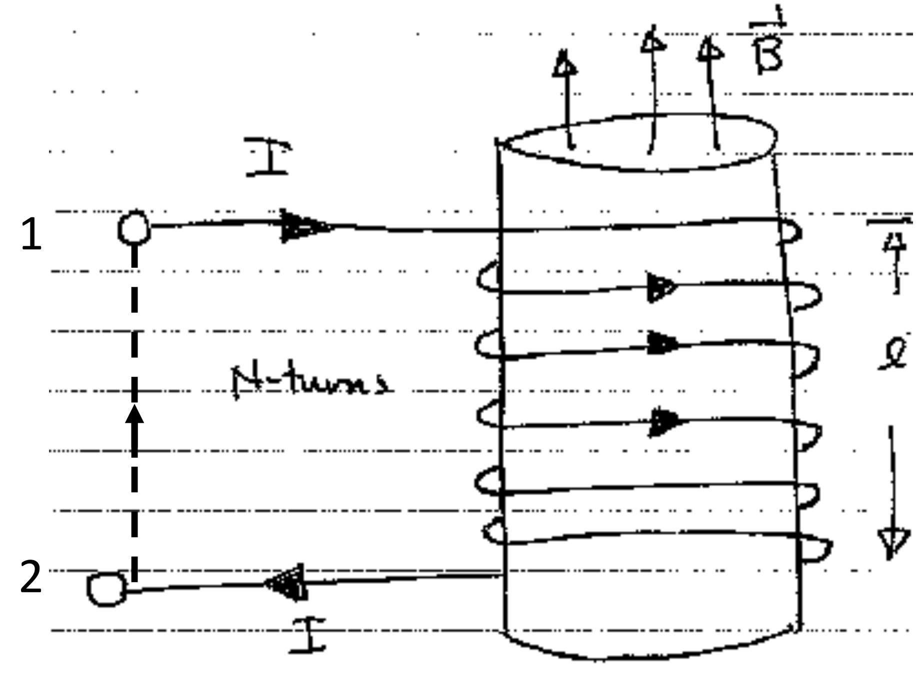


Top view

Inductance

$$\int_2^1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -N\pi a^2 \frac{\partial B_z}{\partial t}$$

$$B_z = \frac{\mu_o N I}{l}$$



$$V_1 - V_2 = -\int_2^1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \frac{\mu_0 N^2 \pi a^2}{l} \frac{dI}{dt} = L \frac{dI}{dt}$$

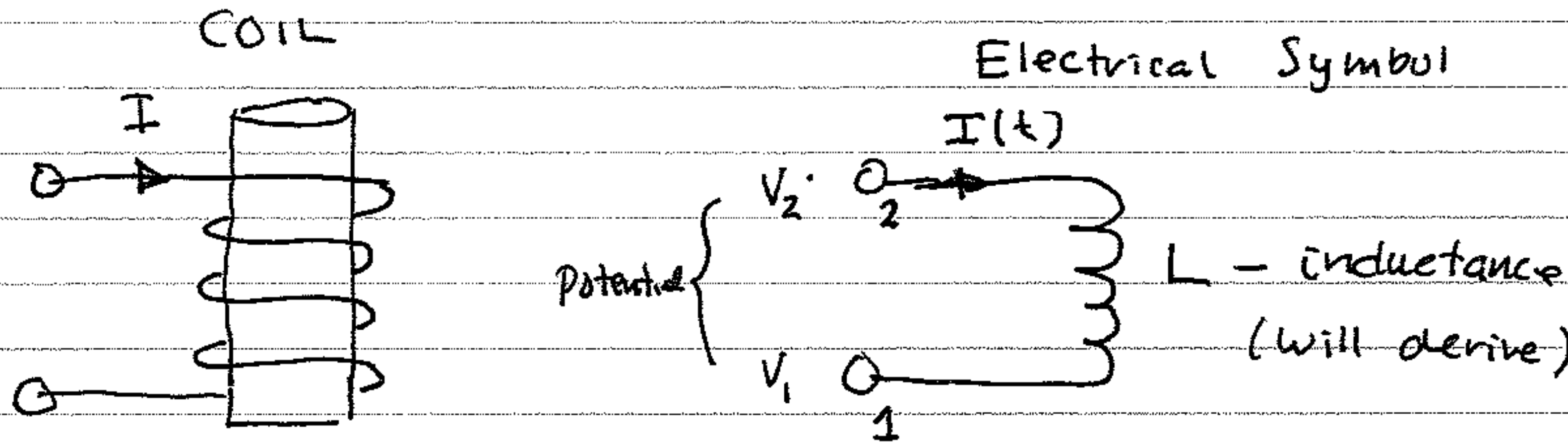
$$L = \frac{\mu_0 N^2 \pi a^2}{l}$$

Depends in geometry of coil, not I

Inductors

An inductor is a coil of wire

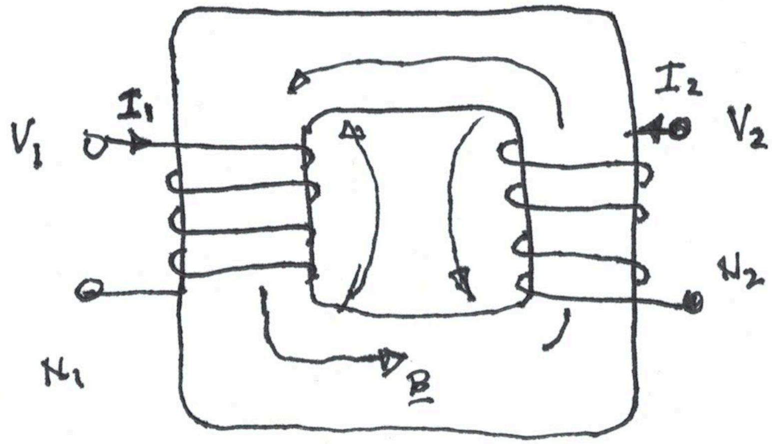
Any length of wire has inductance: but it's usually negligible



Engineering sign convention for labeling voltage and current

$$V_L = V(2) - V(1) = L\, dI/dt$$

Transformer



$$\det\left[\mathbf{L}\right] = L_{11}L_{22} - L_{12}L_{21} > 0$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Reciprocal

$$L_{12} = L_{21}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{pmatrix} dI_1 / dt \\ dI_2 / dt \end{pmatrix}$$

Coupling coefficient, $-1 < k < 1$

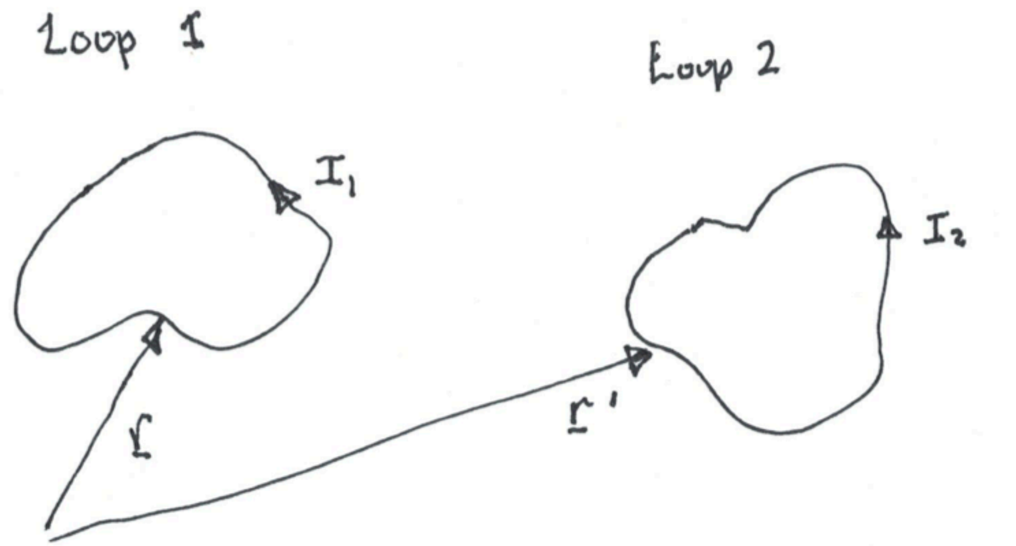
$$L_{12} = k\sqrt{L_{11}L_{22}}$$

Inductance Matrix

Symmetry of Inductance Matrix

$$\mathbf{B}_{12}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}_2(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3 r$$

$$= \frac{\mu_0 I_2}{4\pi}\int_{L2} \frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$



$$\psi_{12} = \int_{S1} da\mathbf{n}\cdot\mathbf{B}_{12}(\mathbf{r})$$

Step 1

$$\mathbf{B}_{12}(\mathbf{r}) = \nabla\times\mathbf{A}_{12}(\mathbf{r})$$

$$\mathbf{A}_{12}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi}\int_{L2} \frac{d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\nabla\times\mathbf{A}_{12}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi}\int_{L2}\nabla\times\frac{d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mu_0 I_2}{4\pi}\int_{L2} d\mathbf{l}'\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{\mu_0 I_2}{4\pi}\int_{L2} d\mathbf{l}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{B}_{12}(\mathbf{r})$$

$$\nabla \times \mathbf{A}_{12}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \int_{L2} \nabla \times \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} = -\frac{\mu_0 I_2}{4\pi} \int_{L2} d\mathbf{l}' \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

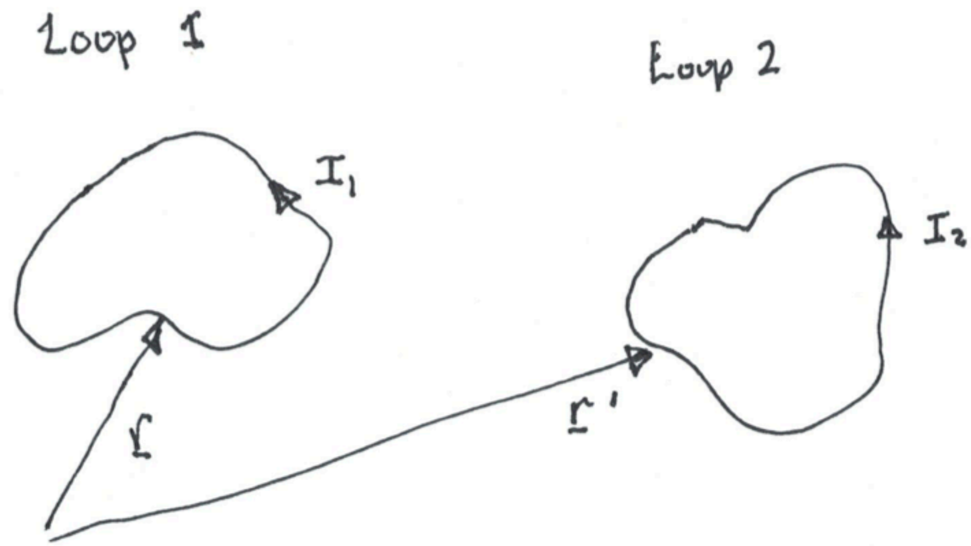
$$= \frac{\mu_0 I_2}{4\pi} \int_{L2} d\mathbf{l}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \mathbf{B}_{12}(\mathbf{r})$$

Step 2

$$\psi_{12} = \int_{S1} da\mathbf{n} \cdot \mathbf{B}_{12}(\mathbf{r})$$

$$= \int_{S1} da\mathbf{n} \cdot \nabla \times \mathbf{A}_{12}(\mathbf{r})$$

$$= \int_{L1} d\mathbf{l} \cdot \mathbf{A}_{12}(\mathbf{r})$$



$$\psi_{12} = \int_{L1} d\mathbf{l} \cdot \mathbf{A}_{12}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \int_{L1} \int_{L2} \frac{d\mathbf{l} \cdot d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} = L_{12} I_2$$

$$L_{12} = L_{21}$$

$$U = \int_{-\infty}^{t} dt' \mathbf{V}(t') \cdot \mathbf{I}(t') = \int_{-\infty}^{t} dt' \mathbf{I}(t') \cdot \mathbf{L} \cdot \frac{d}{dt} \mathbf{I}(t') = \frac{1}{2} \mathbf{I}(t) \cdot \mathbf{L} \cdot \mathbf{I}(t) \geq 0$$

Energy stored must be positive

All diagonal elements of $\mathbf{L}$ must be positive

Eigenvalues and Eigenfunctions of $\mathbf{L}$: $\quad \mathrm{L}_\mu \mathbf{I}_\mu = \mathbf{L} \cdot \mathbf{I}_\mu$

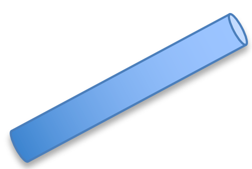
Suppose $\mathbf{I} = \mathbf{I}_\mu$, $\qquad U = \frac{1}{2} L_\mu \left| \mathbf{I}_\mu \right|^2 \rightarrow L_\mu \geq 0 \quad \text{all } \mu \rightarrow \det\left[\mathbf{L} \right] \geq 0$

What about self Inductance?

$$\psi_{11} = \int_{L1} d\mathbf{l} \cdot \mathbf{A}_{11}(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi} \int_{L1} \int_{L1} \frac{d\mathbf{l} \cdot d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} = L_{11} I_1$$

For wire of vanishing thickness $L_{11} \to \infty$

$$\oint B\, dl = 2\pi r B_\theta(r) = \mu_0 I \to B_\theta(r) = \frac{\mu_0 I}{2\pi r}$$



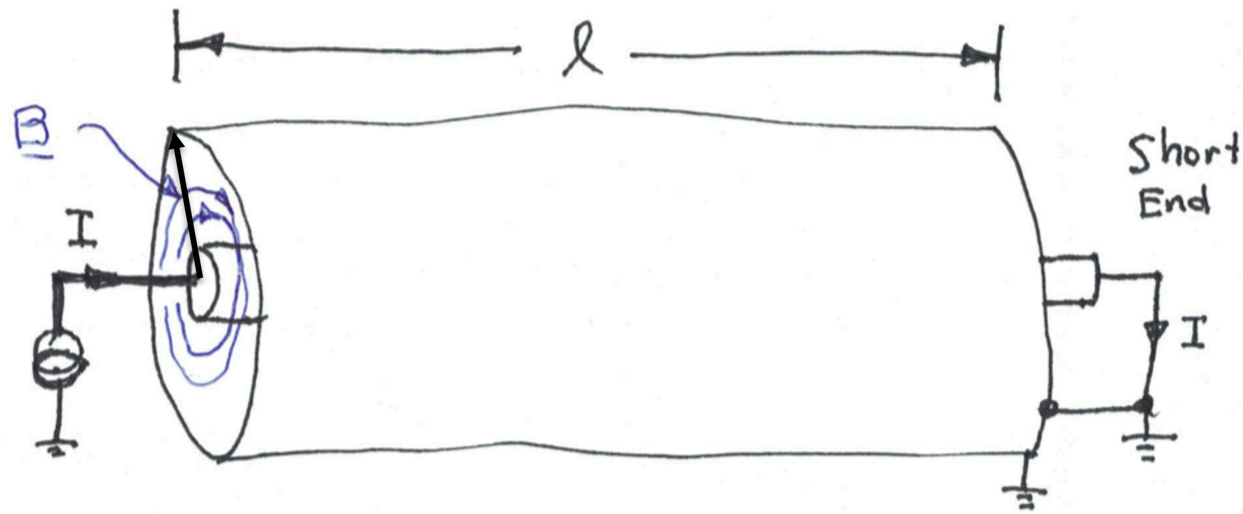
L' and C' for coaxial line

$$B_\theta(r) = \frac{\mu I}{2\pi r}$$

$$\psi = \int_a^b dr \int_0^l dz \frac{\mu I}{2\pi r} = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance/length

$$L' = \psi / Il = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

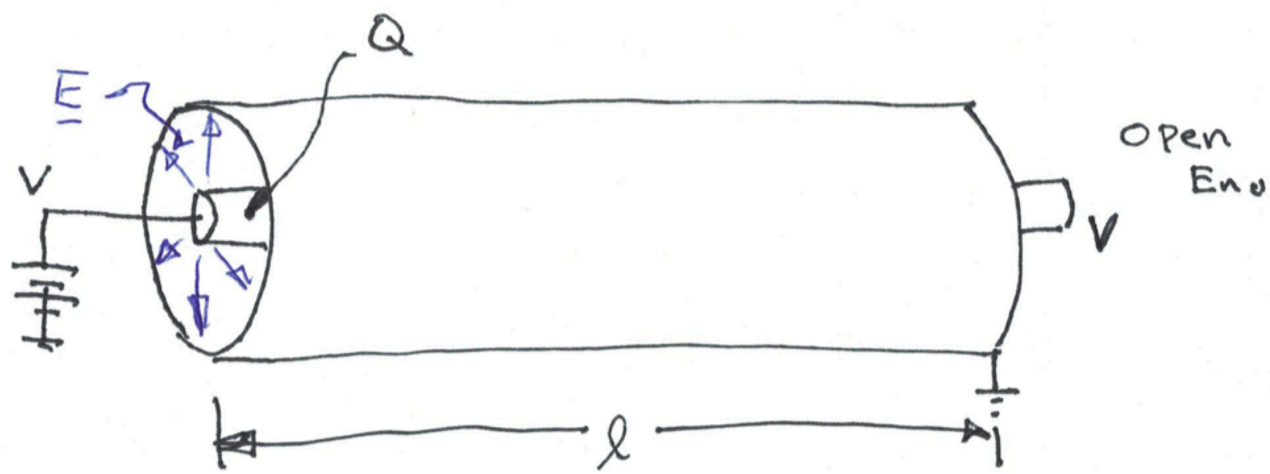


$$E_r(r) = \frac{Q/l}{2\pi\varepsilon r}$$

$$V = \int_a^b dr E_r = \frac{Q/l}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

Capacitance/length

$$C' = \frac{Q/l}{V} = 2\pi\varepsilon / \ln\left(\frac{b}{a}\right)$$



Coaxial Transmission Line

Inductance/length

$$L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

Capacitance/length

$$C' = 2\pi\varepsilon / \ln\left(\frac{b}{a}\right)$$

$$\left.\begin{array}{l} a \to 0 \\ b \to \infty \end{array}\right\} L \to \infty$$

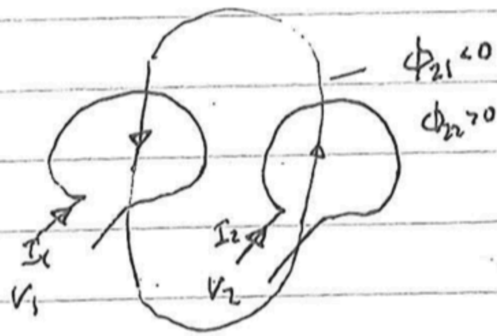
Propagation Speed

$$v^2 = 1/\left(L'C'\right) = 1/\left(\mu\varepsilon\right)$$

Characteristic Impedance

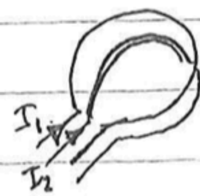
$$Z_0 = \sqrt{L'/C'} = \sqrt{\frac{\mu}{\varepsilon}}\left[\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)\right]$$

Two Loops



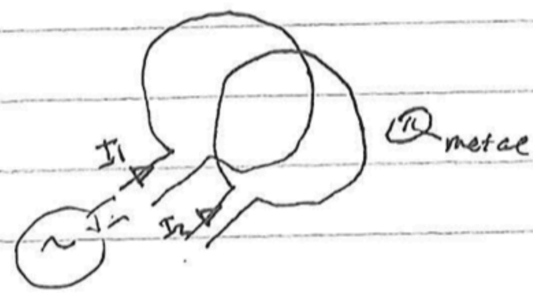
$L_{12} \gtrless 0$?

$L_{12} < 0$



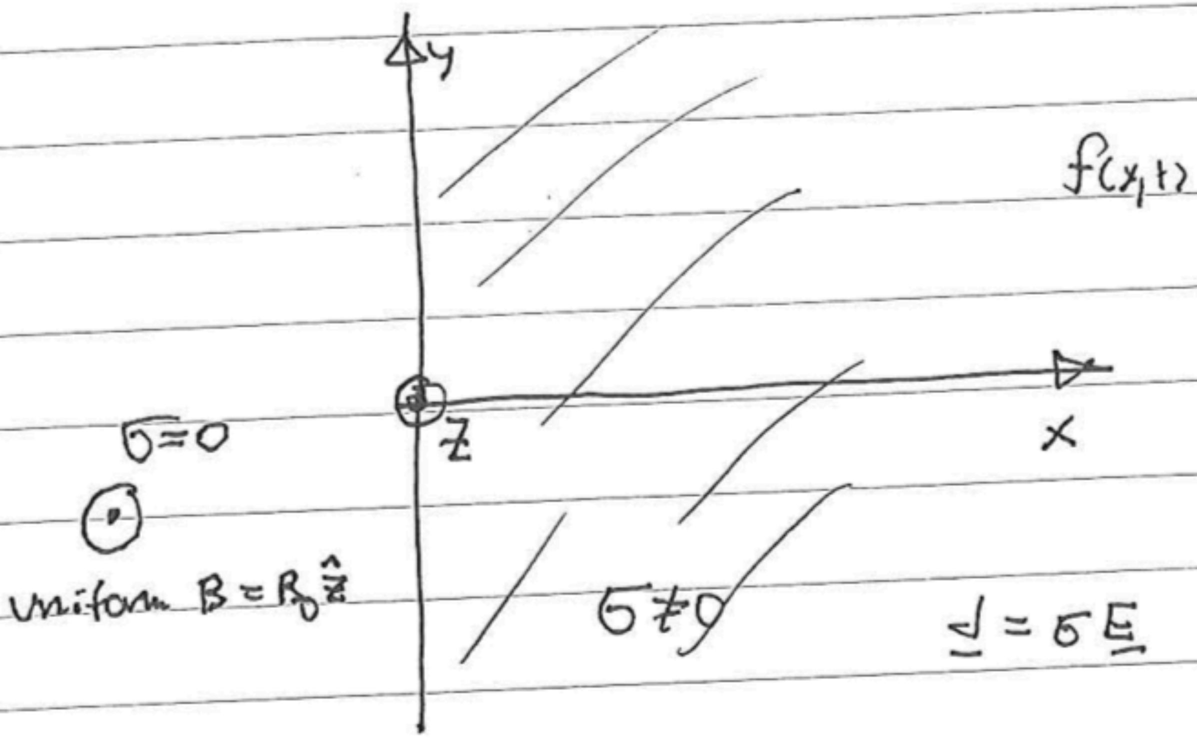
$L_{12} \gtrless 0$?

$L_{12} > 0$



$L_{12} \stackrel{?}{=} 0$

Skin Effect



$\underline{J} = \sigma \underline{E}$ Point form of Ohms law

For $x>0$ what components of fields

$$\underline{B} = (0, 0, B_z)$$

$$\underline{E} = (0, E_y, 0)$$

$$\underline{J} = (0, J_y, 0)$$

$$\frac{\partial}{\partial t} B_z(x,t) = -\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right)$$

$$-\frac{\partial}{\partial x} B_z(x,t) = \mu_0\left(J_y + \epsilon_0 \frac{\partial E_y}{\partial t}\right)$$

$$(\nabla \times B)_y$$

$$J_y = \sigma E_y$$

"good" conductor

$$\sigma E_y > \epsilon_0 \frac{\partial E_y}{\partial t}$$

$$E_y = -\frac{1}{\mu_0 \sigma} \frac{\partial}{\partial x} B_z$$

$$\frac{\partial B_z}{\partial t} = -\frac{\partial}{\partial x} E_y = \frac{1}{\mu_0 \sigma} \frac{\partial^2}{\partial x^2} B_z$$

D

diffusion Eqn

$$D = \frac{1}{\mu_0 \sigma} = \text{diffusion coefficient}$$

Diffusion Equation

$$\frac{\partial n}{\partial t} = D\frac{\partial^2}{\partial x^2}n \qquad \text{solution} \quad n(x,t) = \frac{n_0}{w(t)}\exp\left[-\frac{x^2}{w^2(t)}\right]$$

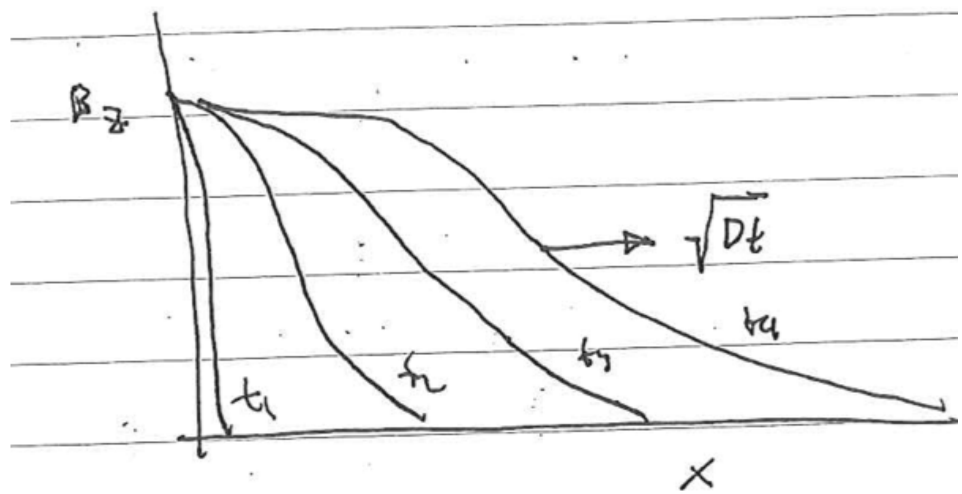
$$w^2(t) = w^2(0) + 4Dt$$

If $n(x,t)$ is a solution then so $B(x,t)$ where $n = \dfrac{\partial B}{\partial x}$

$$B(x,t) = \frac{2B_0}{\pi^{1/2}}\int_x^\infty dx'\frac{1}{w(t)}\exp\left[-\frac{x'^2}{w^2(t)}\right],$$

$$B(0,t) = B_0$$

$$B(x,t) \to 0 \text{ as } x \to \infty$$



Time Harmonic

$$\frac{\partial B_z(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}B_z(x,t) \qquad B_z(x=0,t) = \mathrm{Re}\left\{B_0 e^{-i\omega t}\right\}$$

$$B_z(x=0,t) = \mathrm{Re}\left\{\hat{B}(x)e^{-i\omega t}\right\}$$

$$-i\omega\hat{B}(x) = D\frac{\partial^2}{\partial x^2}\hat{B}(x) \qquad \text{Try } \hat{B}(x) = B_0\exp\left(-\kappa x\right)$$

$$-i\omega = D\kappa^2 \qquad \kappa = \pm\sqrt{\frac{-i\omega}{D}} = \pm\sqrt{-i}\sqrt{\frac{\omega}{D}} = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{D}} \qquad \text{Pick solution } \mathrm{Re}\left\{\kappa\right\} > 0$$

$$\kappa = \frac{1-i}{\delta} \qquad \delta = \sqrt{\frac{2D}{\omega}} = \sqrt{\frac{2}{\omega\sigma\mu_0}} \qquad \text{Skin Depth}$$

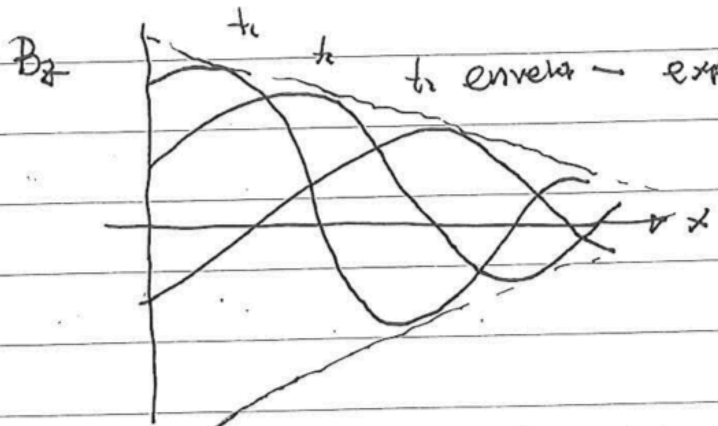
$$B_z(x,t) = \mathrm{Re}\{ \hat{B}(0) \exp(-i\omega t - (1-i)\, x/\delta) \}$$

($\hat{B}(0) = B_0$)

$$\delta = \sqrt{\frac{2D}{\omega}} = \sqrt{\frac{2\mu_0}{\sigma\omega}}$$

δ = skin depth

$$B_z(x,t) = B_0 \exp\left(-\frac{x}{\delta}\right) \cos(\omega t - x/\delta)$$



Surface Current Density

Surface Impedance

$$B_z(x,t) = \tfrac{1}{2} \mathrm{Re}\{ \hat{B}(x) \exp(-i\omega t) \} \qquad \hat{B}(x) = \hat{B}(0) \exp(-\mathbb{K} x)$$

$$\mathbb{K} = (1-i)\sqrt{\frac{\omega \mu_0 \sigma}{2}} \qquad \delta = \text{skin depth} = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

Ampere's law $\nabla \times B = \mu_0 (j + \epsilon_0 \partial E / \partial t)$

$$j_y = \frac{-1}{\mu_0} \frac{\partial}{\partial x} B_z \qquad j_y(x,t) = \tfrac{1}{2} \mathrm{Re}\left\{ \frac{\mathbb{K}}{\mu_0} \hat{B}(0) \exp(-i\omega t - \mathbb{K} x) \right\}$$

flows in narrow layer

$$E_y = \frac{j_y}{\sigma} \qquad E_y(x,t) = \tfrac{1}{2} \mathrm{Re}\left\{ \frac{\mathbb{K}}{\mu_0} \frac{\hat{B}(0)}{\sigma} \exp(-i\omega t - \mathbb{K} x) \right\}$$

Surface Current Density K_y Amps/meter $\underline{B} = \mu_0 \underline{H}$

H [A/m]

$$K_y = \int_0^\infty dx\, J_y(x,t) = \int_0^\infty dx \left\{ -\frac{1}{\mu_0} \frac{\partial}{\partial x} B_z \right\} = B_z(0)/\mu_0 = H_z(0)$$

$$E_y(0,t) = \mathrm{Re}\left\{ \frac{\mathbb{K}}{\sigma} \hat{H}_z(0) \right\} \qquad \frac{\mathbb{K}}{\sigma} = \frac{1-i}{\delta\sigma}$$

$$\hat{E}_y(0) = \frac{1-i}{\delta\sigma} \hat{H}_z(0)$$

V/m, A/m, ohms

$$Z_s = \frac{1-i}{\sigma\delta} = (1-i)\sqrt{\frac{\omega\mu_0}{2\sigma}}$$

R_s, X_s

TABLE 8–1
Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

† The ϵ of seawater is approximately $72\epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega\epsilon \cong 1$ (not $\gg 1$). Under these conditions, seawater is not a good conductor, and Eq. (8–57) is no longer applicable.

termine the attenuation constant, phase constant, intrinsic impedance, phase velocity