ENEE381

Lecture-18

Radiation

Simple Antennas

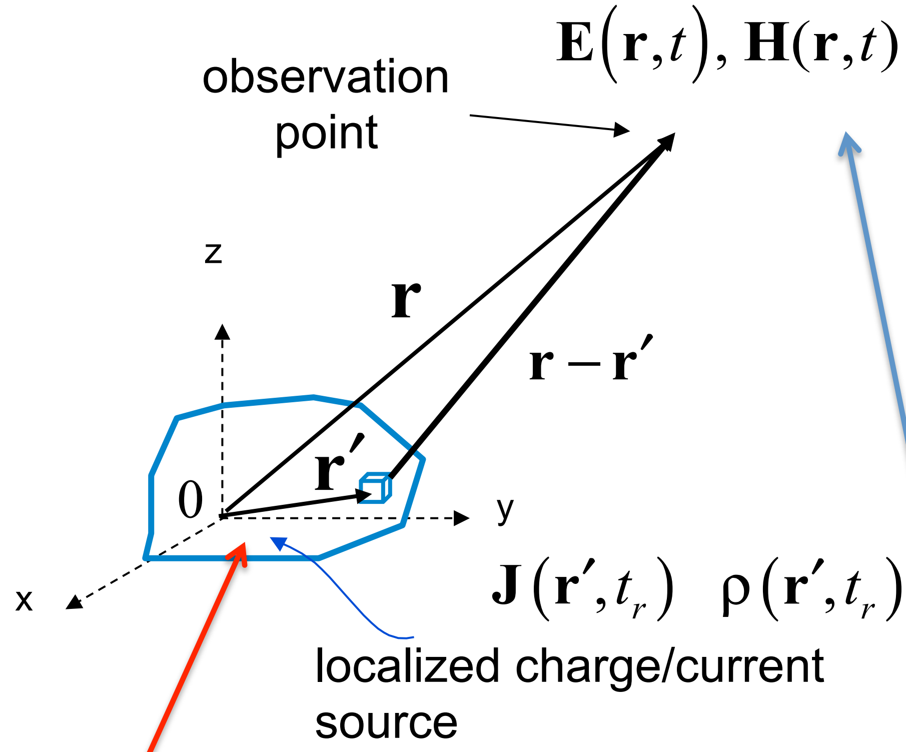
Goal for Today

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

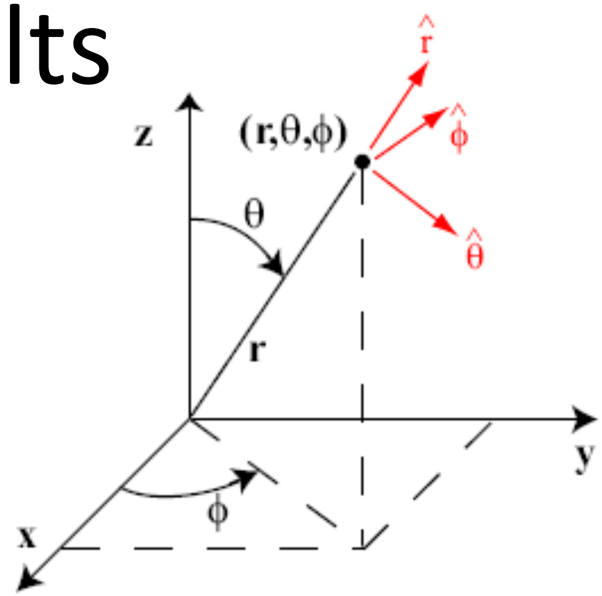
$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



Assume current and charge densities are sinusoidal and given. Calculate E and H far away. Find radiated power density S = E x H.

Preview of Results

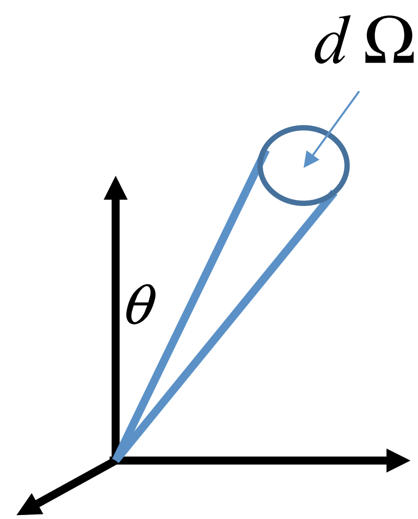


Total Radiated Power

In spherical coordinates $da = r^2\, d\,\Omega$

where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the solid angle

$$P_T = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2\, d\Omega = \oint_S \frac{dP_T}{d\Omega} d\Omega$$



Power radiated into the solid angle $d\Omega$

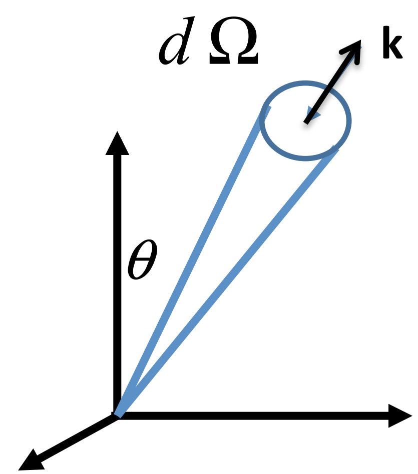
$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

Radiated Power Flux

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d^3r' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$P_T = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 \, d\Omega = \oint_S \frac{dP_T}{d\Omega} d\Omega$$



Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{c\,\varepsilon_0} = 377\,\Omega \text{ impedance of vacuum} \qquad c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}}$$

Time Dependent Fields

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Introduce Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Insert in Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

Time Dependent Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \ \mathbf{E} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\nabla \times \ \nabla \times \mathbf{A} = \mu_0 \left(\vec{\mathbf{J}} - \varepsilon_0 \frac{\partial}{\partial t} \left(\nabla\phi + \frac{\partial}{\partial t}\mathbf{A} \right) \right)$$

$$\nabla \ \left(\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\phi \right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} \text{ - } \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{A}$$

Now Pick Lorenz Gauge

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho/\varepsilon_0$$

$$\nabla\left(\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}$$

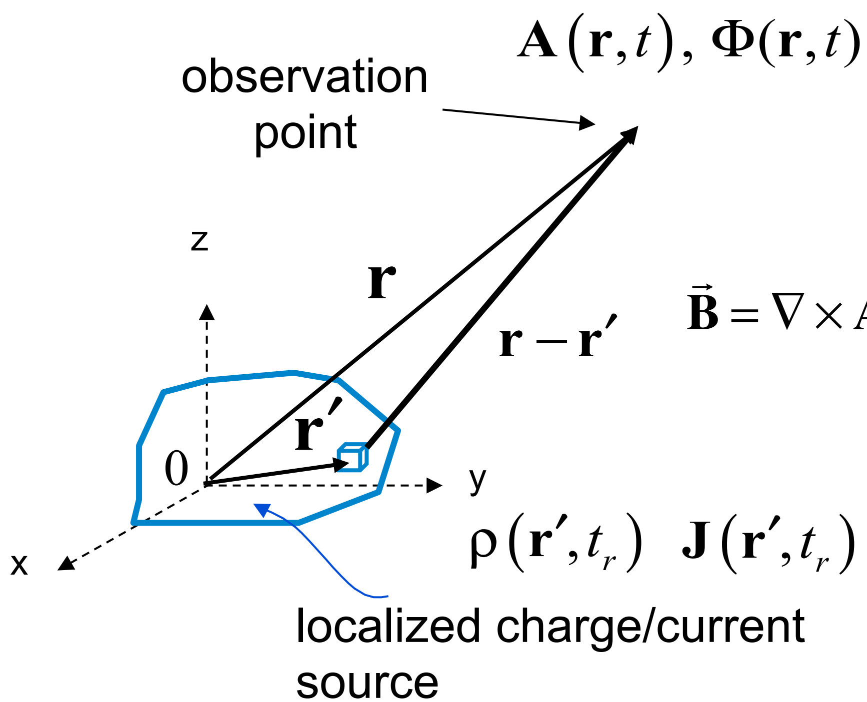
Lorenz Gauge: $\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi = 0$

$$-\left(\nabla^2\phi - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\phi\right) = \rho/\varepsilon_0$$

$$-\left(\nabla^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}\right) = \mu_0\vec{\mathbf{J}}$$

Same Equation
Wave Equation

Time-retarded potentials



$$\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}' \left.\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}' \left.\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\right|_{t_r = t - |\mathbf{r}-\mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the retarded time (earlier time)

$$dr' = dx'\,dy'\,dz'$$

Sinusoidal Dependence on Time

If we assume harmonic (sinusoid dependence on time) for all the fields and sources

$$\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\left[\hat{\mathbf{A}}(\mathbf{r})e^{-i\omega t}\right] \qquad \Phi(\mathbf{r},t) = \mathrm{Re}\left[\hat{\Phi}(\mathbf{r})e^{-i\omega t}\right]$$

$$\mathbf{J}(\mathbf{r}',t_r) = \mathrm{Re}\left[\hat{\mathbf{J}}(\mathbf{r}')e^{-i\omega t_r}\right] \qquad \rho(\mathbf{r}',t_r) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r}')e^{-i\omega t_r}\right]$$

$t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ where $k = \omega / c = 2\pi / \lambda$ is the wavenumber

$$e^{-i\omega t_r} = e^{-i\omega t}e^{ik|\mathbf{r}-\mathbf{r}'|}$$

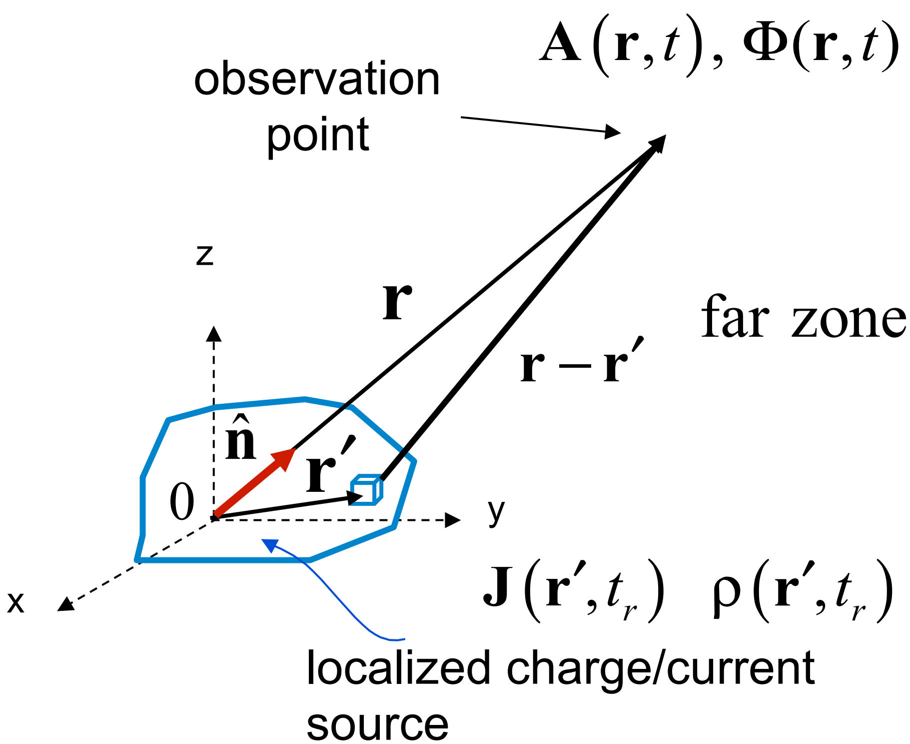
In phasor notation

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}' \frac{\hat{\mathbf{J}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} \qquad \hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}' \frac{\hat{\rho}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|}$$

Far Field Approximation

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \simeq r - \mathbf{r}\cdot\mathbf{r}'/r$$

Assume that the source is localized and the observation point is far away $(r >> r')$



$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}' \frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|} e^{ik\left|\mathbf{r}-\mathbf{r}'\right|}$$

$$\text{far zone } k\,r = 2\pi\frac{r}{\lambda} >> 1$$

$$\left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}' \quad \text{for } r >> r'$$

$$\hat{\mathbf{n}} = \frac{\mathbf{r}}{\left|\mathbf{r}\right|} \text{ unit vector}$$

Far Field Potentials

$$\text{Using } \left|\mathbf{r}-\mathbf{r}'\right| \simeq r - \hat{\mathbf{n}}\cdot\mathbf{r}'$$

$$\hat{\mathbf{A}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{Vol} d\mathbf{r}'\,\frac{\hat{\mathbf{J}}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{\mu_0}{4\pi r}e^{ikr}\int_{Vol} d\mathbf{r}'\,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\Phi}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\mathbf{r}'\,\frac{\hat{\rho}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}e^{ik\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \frac{e^{ikr}}{4\pi\varepsilon_0 r}\int_{Vol} d\mathbf{r}'\,\hat{\rho}(\mathbf{r}')\,e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

where $\mathbf{k} = k\,\hat{\mathbf{n}}$

for $r >> r'$, $\dfrac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} \simeq \dfrac{1}{r}$ and $k\left|\mathbf{r}-\mathbf{r}'\right| \simeq kr - k\,\hat{\mathbf{n}}\cdot\mathbf{r}'$ in exponent

Calculating Fields from Potentials

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\nabla = \hat{\mathbf{n}} \frac{\partial}{\partial r} \rightarrow i\,\mathbf{k} \qquad \hat{\mathbf{B}}(\mathbf{r}) = \nabla \times \hat{\mathbf{A}}(\mathbf{r}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = i\,\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{r})$$

$$\hat{\mathbf{B}}(\mathbf{r}) = i \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \mathbf{k} \times \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$\hat{\mathbf{B}}(\mathbf{r})$ is transvere to $\hat{\mathbf{J}}$, $\mathbf{k} = k\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}(\mathbf{r})$

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx' \, dy' \, dz'$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \hat{\mathbf{C}}(\mathbf{k}) \qquad \hat{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi r} e^{ikr} \, i\,\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k})$$

Electric Field

Ampere's law

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \partial \varepsilon_0 \mathbf{E} / \partial t \quad \rightarrow \quad i\mathbf{k} \times \hat{\mathbf{H}} = -i\omega\, \varepsilon_0 \hat{\mathbf{E}}$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi r} e^{ikr}\, i\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\frac{1}{\varepsilon_0\, \omega} \mathbf{k} \times \hat{\mathbf{H}}(\mathbf{r})$$

$$\mu_0 \hat{\mathbf{H}}(\mathbf{r}) = \hat{\mathbf{B}}(\mathbf{r})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i \frac{e^{ikr}}{4\pi r} \frac{1}{\varepsilon_0\, \omega} \mathbf{k} \times \left(\mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right)$$

Summary

In the far zone $k\,r = 2\pi\,r\,/\,\lambda >> 1$

Define the Fourier transform of the current density

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \,\hat{\mathbf{J}}(\mathbf{r}')\,e^{-i\,\mathbf{k}\cdot\mathbf{r}'} \qquad d\tau' = dx'\,dy'\,dz'$$

The fields in terms of the $F-T$ of the current density are

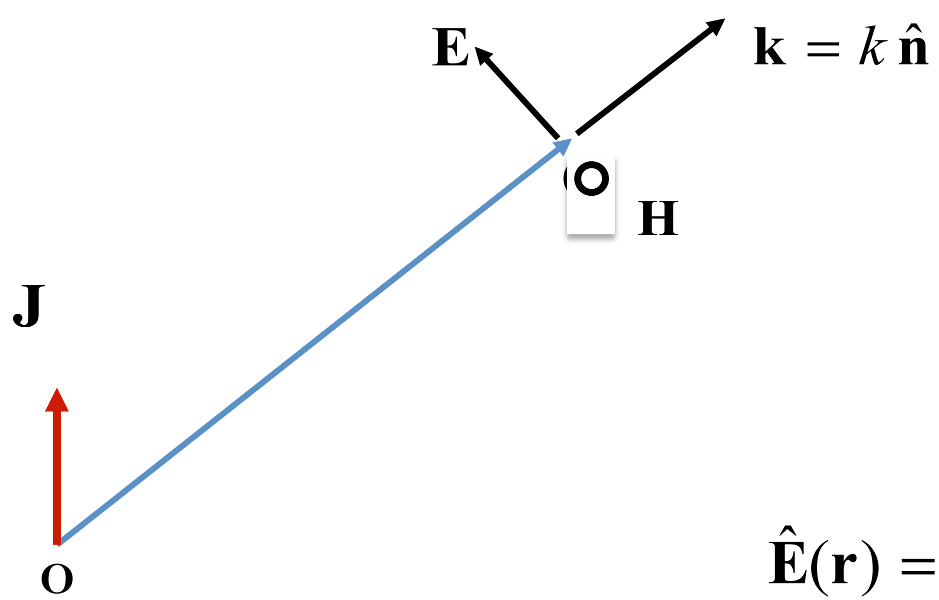
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi\,r} e^{i\,k\,r}\,\hat{\mathbf{C}}(\mathbf{k}) \qquad \hat{\mathbf{H}}(\mathbf{r}) = i\frac{1}{4\pi\,r} e^{i\,k\,r}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{i\,k\,r}}{4\pi\,r}\,\frac{1}{\varepsilon_0\,\omega}\,\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)$$

Radiation

In the far field zone

Direction of energy flow, Poynting's vector



$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr}\,\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{H}}(\mathbf{r}) = i\frac{1}{4\pi r} e^{ikr}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)$$

$\hat{\mathbf{H}}(\mathbf{r})$ is transverse to $\hat{\mathbf{J}}$, $\mathbf{k} = k\,\hat{\mathbf{n}}$ and $\hat{\mathbf{E}}$

Radiation

Radiation intensity [W/m^2]
(average over time of Poynting's vector)

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)$$

Radiation intensity is $\left|\langle \mathbf{S} \rangle\right|$

The average is over an optical period

$$\hat{\mathbf{E}}(\mathbf{r}) = -\,i\,\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right) \qquad \hat{\mathbf{H}}(\mathbf{r}) = i\,\frac{1}{4\pi r}e^{ikr}\,\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left(-\,i\,\frac{e^{ikr}}{4\pi r}\frac{1}{\varepsilon_0\,\omega}\left(\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)\right)\times\frac{-i}{4\pi r}e^{-ikr}\left(\mathbf{k}\times\hat{\mathbf{C}}^*(\mathbf{k})\right)\right)$$

$$\langle \mathbf{S} \rangle = -\,\frac{1}{2}\left(\frac{1}{4\pi r}\right)^2\frac{1}{\varepsilon_0\,\omega}\mathrm{Re}\left(\left(\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)\right)\times\mathbf{k}\times\hat{\mathbf{C}}^*(\mathbf{k})\right)$$

Radiated Power Flux

using the vector identity $(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) - \mathbf{b}(\mathbf{a}\cdot\mathbf{c})$

$$\left(\mathbf{k}\times\left(\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right)\right)\times\left(\mathbf{k}\times\hat{\mathbf{C}}^{*}(\mathbf{k})\right) = -\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^{2}\mathbf{k}$$

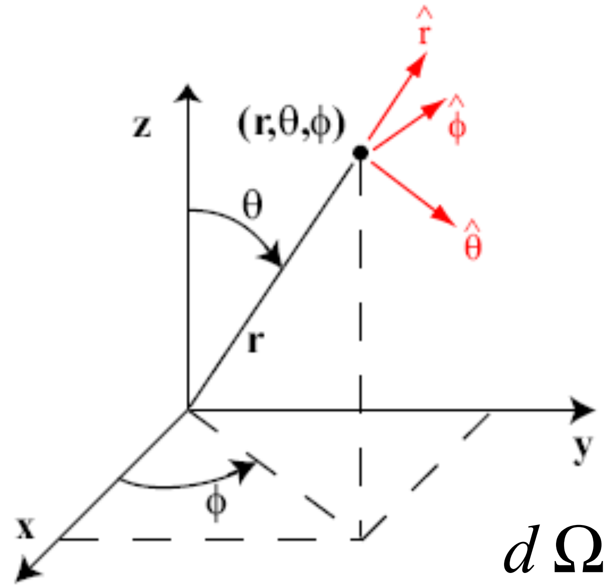
$$\langle\mathbf{S}\rangle = \frac{1}{32\,\pi^{2}\,\varepsilon_{0}\,\omega}\frac{\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^{2}}{r^{2}}\,\mathbf{k}$$

The power flux falls off like $1/r^{2}$ and is in the direction of $\mathbf{k}=k\,\hat{\mathbf{n}}$

$$\langle\mathbf{S}\rangle = \frac{Z_{0}}{32\,\pi^{2}}\frac{\left|\mathbf{k}\times\hat{\mathbf{C}}(\mathbf{k})\right|^{2}}{r^{2}}\,\hat{\mathbf{n}}$$

$Z_{0} = \sqrt{\dfrac{\mu_{0}}{\varepsilon_{0}}} = \dfrac{1}{c\,\varepsilon_{0}} = 377\,\Omega$ impedance of vacuum $\qquad c = \dfrac{1}{\sqrt{\varepsilon_{0}\,\mu_{0}}}$

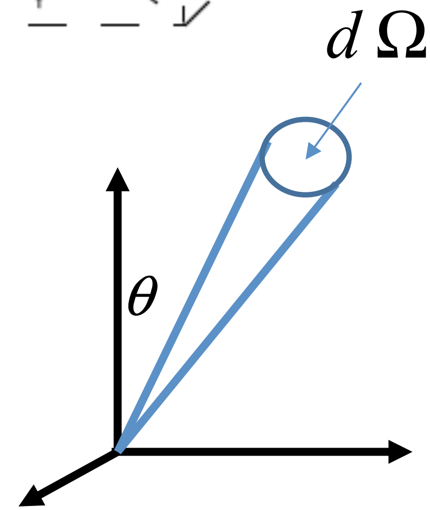
Total Radiated Power



$$P_T = \oint_S \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \, da$$

In spherical coordinates da = $r^2 \, d\Omega$

where $d\Omega = \sin\theta \, d\theta \, d\varphi$ is the solid angle



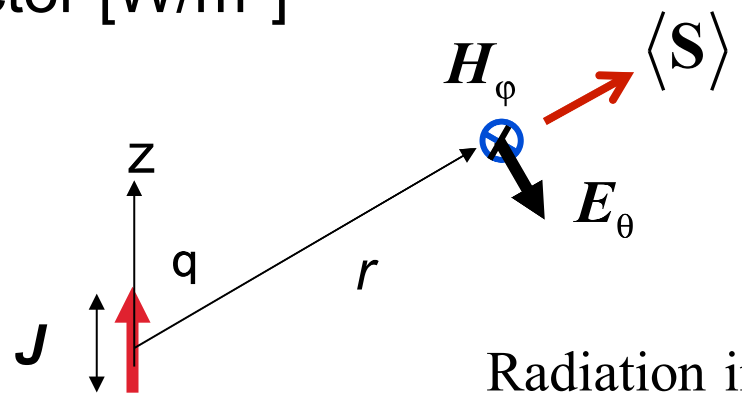
$$P_T = \oint_S \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 \, d\Omega = \oint_S \frac{dP_T}{d\Omega} d\Omega$$

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle r^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

Electric Dipole Radiation

Poynting vector [W/m²]



$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)$$

Radiation intensity is $\left|\langle \mathbf{S} \rangle\right|$

The dipole moment is $\hat{\mathbf{p}} = \int_{Vol} \mathbf{r}' \hat{\rho}(\mathbf{r}')\, d\tau'$

The charge density is $\rho(\mathbf{r}, t) = \mathrm{Re}\left[\hat{\rho}(\mathbf{r})\, e^{-i\omega t}\right]$

Conservation of charge, $\partial\rho / \partial t + \nabla \cdot \mathbf{J} = 0$, gives $\nabla \cdot \hat{\mathbf{J}}(\mathbf{r}) = i\,\omega \hat{\rho}(\mathbf{r})$

Electric Dipole Radiation

The dipole moment is

$$\hat{\mathbf{p}} = \int_{Vol} \mathbf{r}'\hat{\rho}(\mathbf{r}')\,d\tau' = -\frac{i}{\omega}\int_{Vol} \mathbf{r}'\nabla'\cdot\hat{\mathbf{J}}(\mathbf{r}')\,d\tau'$$

Integrating by parts gives $(\nabla'\cdot\mathbf{r}' = 1)$

$$\hat{\mathbf{p}} = \frac{i}{\omega}\int_{Vol} \hat{\mathbf{J}}(\mathbf{r}')\,d\tau'$$

Electric Dipole Radiation

Power radiated into the solid angle $d\Omega \quad (da = r^2 d\Omega)$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 \qquad P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$$

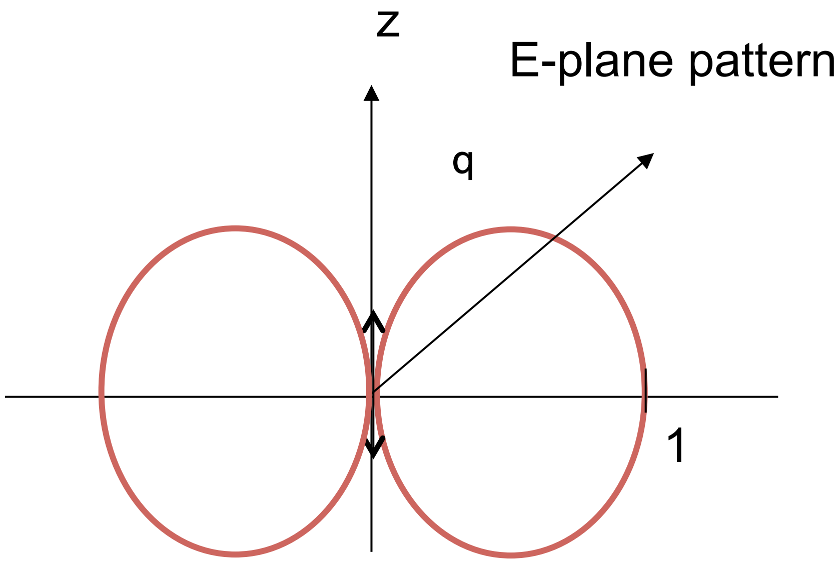
where $\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'}$ is F-T of the current density

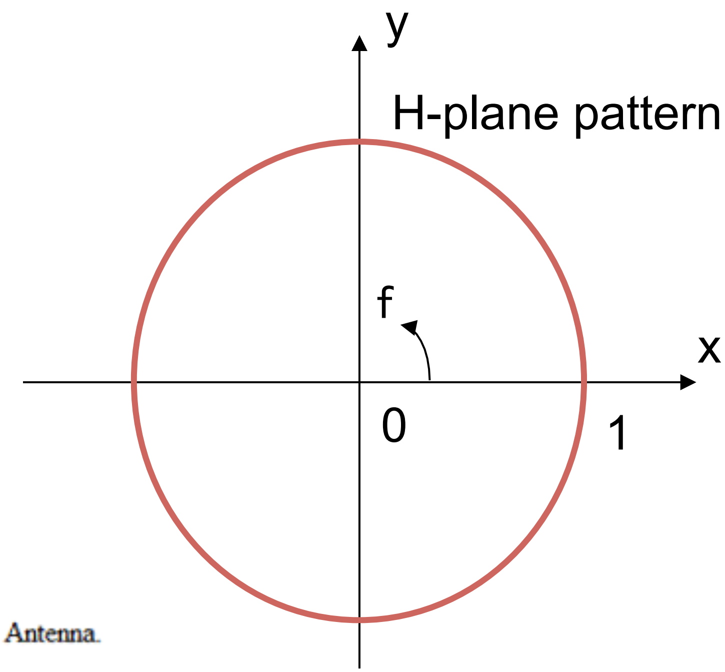
If the wavelength is large compared to the dimensions of the dipole

$|\mathbf{k} \cdot \mathbf{r}'| << 1, \quad k = 2\pi / \lambda \qquad \hat{\mathbf{C}}(\mathbf{k}) ; \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \qquad \hat{\mathbf{C}}(\mathbf{k}) ; -i\omega\hat{\mathbf{p}}$

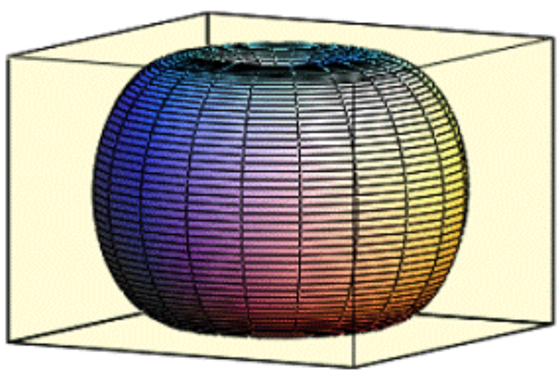
$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} |\hat{\mathbf{n}} \times \hat{\mathbf{p}}|^2 = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} |\hat{\mathbf{p}}|^2 \sin^2\theta \ \sim \ \omega^4$$

Antenna Pattern of a Hertzian Dipole





Radiation Pattern of the Dipole Antenna.

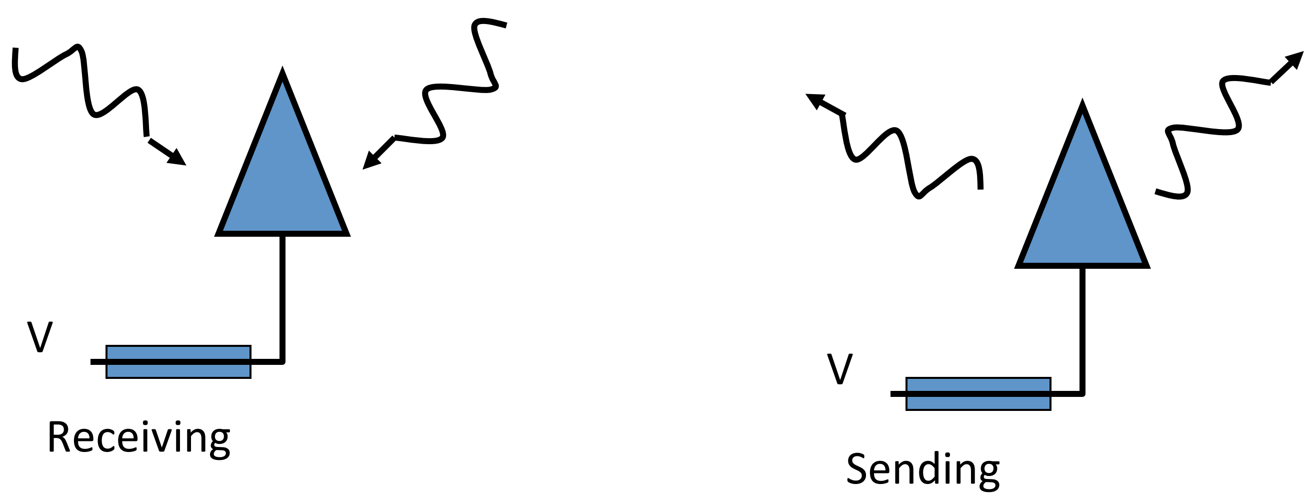


For a dipole of length
$L = 0.01 \ (\lambda_0)$.

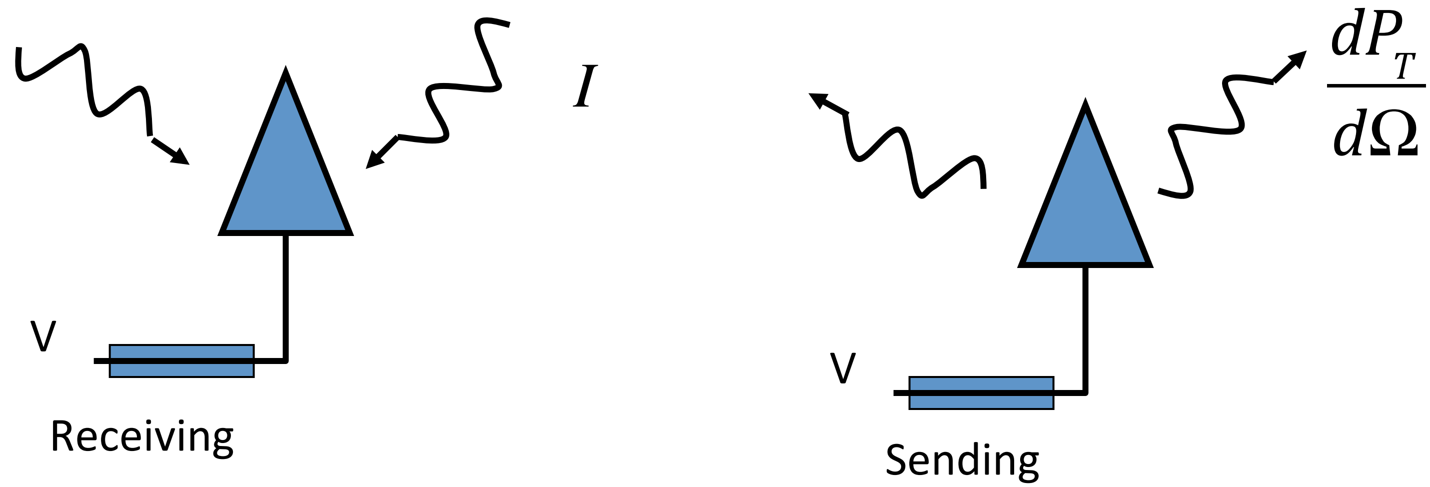
EM Reciprocity

Example:

- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity



Effective Area – Antenna Gain



Power received → $P_R = A_e(\Omega) I$ ← Incident intensity

Effective area → $A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi}$ ← gain →

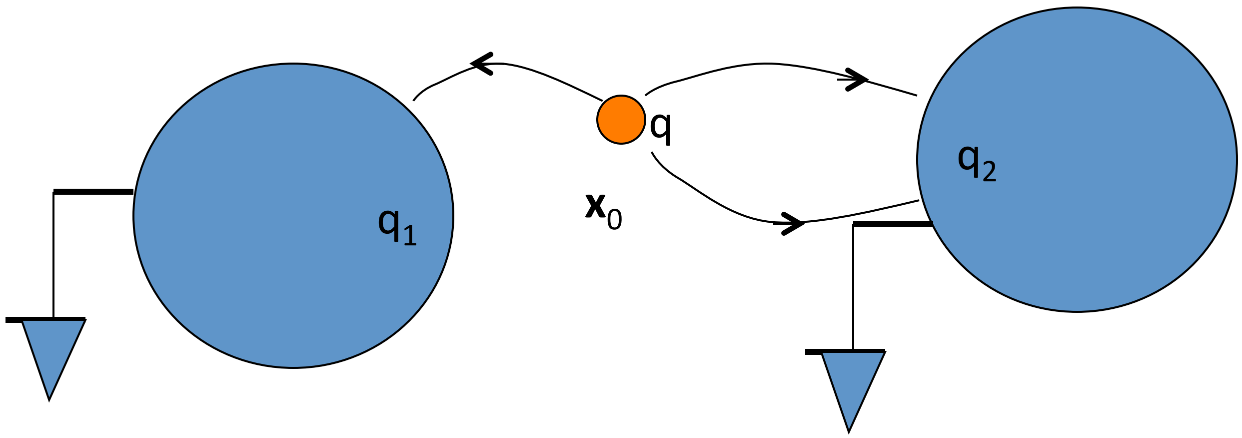
$$G(\Omega) = \frac{dP_T}{d\Omega} / \langle P_T \rangle$$

← Power per unit solid angle

$$\langle P_T \rangle = \int \frac{dP_T}{d\Omega} d\Omega$$

Example From "Classical Electrodynamics" by J. D. Jackson, Problems 1.12 and 1.13

A charge q is placed at an arbitrary point, $\mathbf{x}_o$, relative to two grounded, conducting electrodes.



What is the charge q_1 on the surface of electrode 1?

<u>Repeat for different $\mathbf{x}_0$</u>

Solution - Green's Reciprocation Theorem

<u>Prob #1</u>
Your Problem

$$\nabla^2 \phi = -q\delta(\mathbf{x} - \mathbf{x}_0)$$

BC: $\phi\big|_{B1} = \phi\big|_{B2} = \phi(x \to \infty) = 0$

$$q_1 = \int_{B1} d^2x\, \mathbf{n} \cdot \nabla \phi$$

<u>Prob #2</u>
Adjoint.

$$\nabla^2 \psi = 0$$

BC: $\psi\big|_{B1} = 1, \quad \psi\big|_{B2} = \psi(x \to \infty) = 0$

Green's Theorem

$$\int_V d^3x \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi\right) = \int_S da\, \mathbf{n} \cdot \left(\psi \nabla \phi - \phi \nabla \psi\right)$$

$$\left(\psi \frac{d^2}{dx^2}\phi - \phi \frac{d^2}{dx^2}\psi\right) = \frac{d}{dx}\left(\psi \frac{d}{dx}\phi - \phi \frac{d}{dx}\psi\right)$$

When the dust settles:

$$q_1 = -q\psi(\mathbf{x}_0)$$

George Green 1793-1841

The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard Physics Today Dec. 2003

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

- Born in Nottingham (Home of Robin Hood)
- Father was a baker
- At age 8 enrolled in Robert Goodacre's Academy
- Left after 18 months (extent of formal education pre 40)
- Worked in bakery for 5 years
- Sent by father to town mill to learn to be a miller

- Taught himself math
- Fell in love with Jane, the miller's daughter.
- Green's father forbade marriage.
- Had 7 children with Jane.
- Self published work in 1828
- With help, entered Cambridge 1833, graduated 1837.
- "Discovered" by Lord Kelvin in 1840.
- Theory of Elasticity, refraction, evanescence
- Died of influenza, 1841 (age 48)

Green's Mill: still functions

Features of Problems Suited to Adjoint Approach

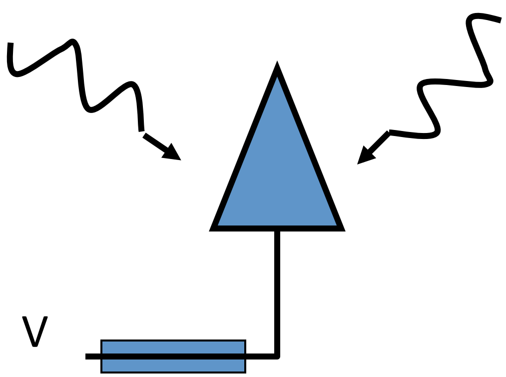
1. Many computations need to be repeated. (many different locations of charge, q)

2. Only a limited amount of information about the solution is required. (only want to know charge on electrode #1)

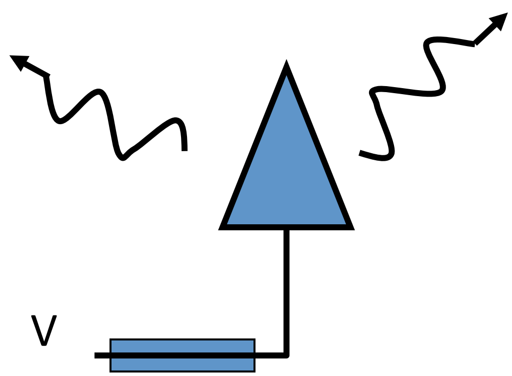
EM Reciprocity

Example:

- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity

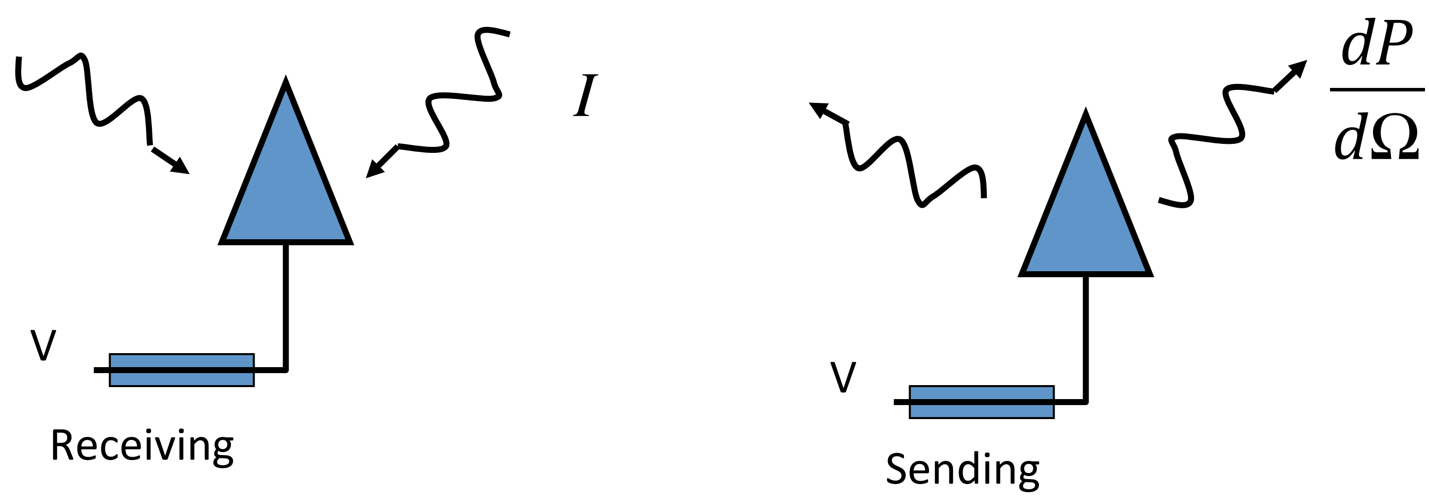


Receiving



Sending

Effective Area – Antenna Gain



Power received → $$P_R = A_e(\Omega) I$$ ← Incident intensity

Effective area → $$A_e(\Omega) = \frac{\lambda^2 G(\Omega)}{4\pi}$$ ↔ gain

$$G(\Omega) = \frac{dP}{d\Omega} / P_T$$ ← Power per unit solid angle

$$P_T = \int \frac{dP}{d\Omega} d\Omega$$

Higher Order Moments of the Fields

Include both electric and magnetic dipole contributions

In the far field zone

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \, e^{-i\mathbf{k}\cdot\mathbf{r}'}$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \, \hat{\mathbf{J}}(\mathbf{r}') \left(1 - ik\,\hat{\mathbf{n}}\cdot\mathbf{r}' - \frac{k^2}{2}(\mathbf{n}\cdot\mathbf{r}')^2 + \ldots \right)$$

terms fall off rapidly

Electric and Magnetic Dipole Radiation

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \,\hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq \frac{\mu_0}{4\pi r} e^{ikr} \int_{Vol} d\tau' \,\hat{\mathbf{J}}(\mathbf{r}')(1 - ik\,\hat{\mathbf{n}}\cdot\mathbf{r}')$$

electric dipole — magnetic dipole

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \left\{ -i\omega\hat{\mathbf{p}} - ik\,\hat{\mathbf{m}}\times\hat{\mathbf{n}} \right\}$$

The F-T of the current density including both the **electric** and **magnetic dipole** contributions is

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \,\hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega(\hat{\mathbf{p}} + \hat{\mathbf{m}}\times\hat{\mathbf{n}}/c)$$

Electric and Magnetic Dipole Radiation

Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2$$

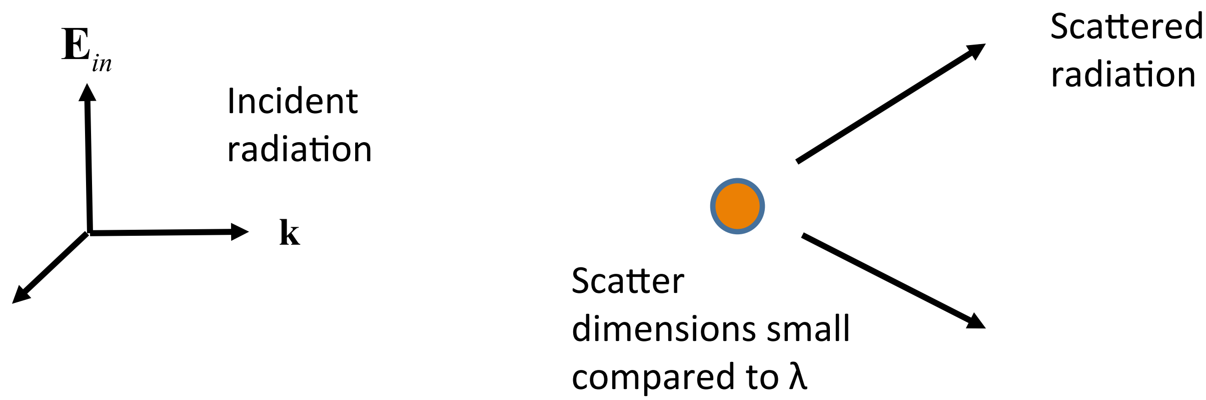
$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} \frac{\omega^4}{c^2} \left| \hat{\mathbf{n}} \times (\hat{\mathbf{p}} + \hat{\mathbf{m}} \times \hat{\mathbf{n}} / c) \right|^2$$

Far field zone

$\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$ are the electric and magnetic dipole moments

Shorter wavelengths scatter more (blue sky)

Scattering at Long Wavelengths



- The incident radiation induces an oscillating electric and magnetic dipole moment in the scatter

- The induced dipole moments radiate (scattered radiation)

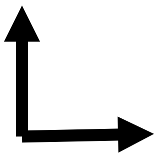
- The scattered radiation is a function of the polarization and direction of both incident and scattered radiation

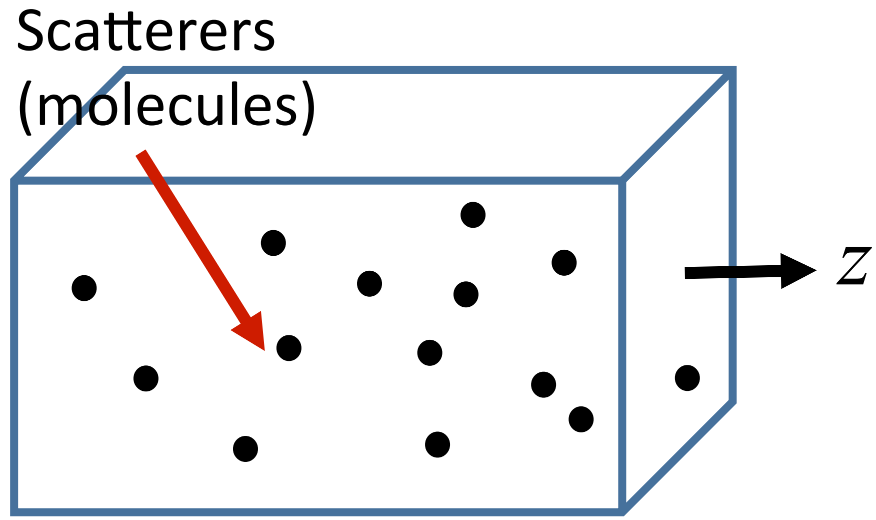
- If the wavelength is large compared to the size of the scatter the induced electric and magnetic dipole moments are sufficient to describe the scattered radiation   (opposite case is called Mie scattering)

Coherent vs Incoherent Scattering

Incident Plane Wave

$\hat{\mathbf{E}}\exp[ikz]$





Amplitude due to ensemble of spatially distributed scatterers

$$\hat{\mathbf{C}}(\mathbf{k}) = \int_{Vol} d\tau' \hat{\mathbf{J}}(\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} \simeq -i\omega \sum_i \hat{\mathbf{p}}_i e^{-i\mathbf{k}\cdot\mathbf{r}_i}$$

Dipole moment proportional to local electric field

$$\hat{\mathbf{p}}_i = \gamma \hat{\mathbf{E}} \exp[ikz_i]$$

Radiated Power

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

$$\frac{dP_T}{d\Omega} = \frac{dP_{T1}}{d\Omega} f(\mathbf{k}, N)$$

Radiation due to single dipole

$$\frac{dP_{T1}}{d\Omega} = \frac{Z_0}{32\pi^2}\frac{\omega^4}{c^2}\left|\hat{\mathbf{n}}\times\gamma\hat{\mathbf{E}}\right|^2$$

Form factor

$$f = \left|\sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i]\right|^2$$

Three cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k} \cdot \mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength

Dipoles distributed randomly in a volume larger than a wavelength

Dipoles in an ordered array

Cases

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

Dipoles localized to a volume smaller than a wavelength - coherent

$$f = \left| \sum_i \right|^2 = N^2$$

Dipoles distributed randomly in a volume larger than a wavelength - incoherent

$$f = \left| \sum_{i,j} \exp[ik\left(z_i - z_j\right) - i\mathbf{k}\cdot\left(\mathbf{r}_i - \mathbf{r}_j\right)] \right|$$

Terms with i an j different average to zero. Only j=i survive

$$f = \left| \sum_i \right| = N$$

Ordered array 1D

$$f = \left| \sum_i \exp[ikz_i - i\mathbf{k}\cdot\mathbf{r}_i] \right|^2$$

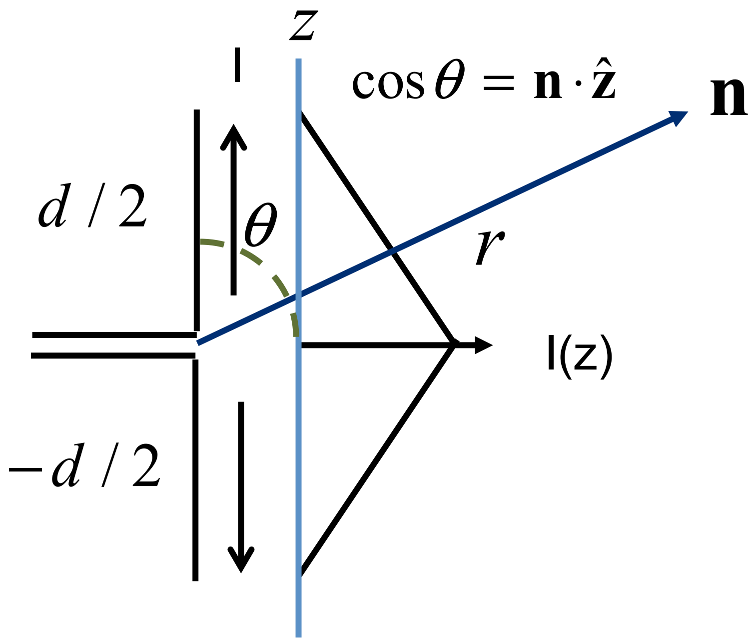
$$\mathbf{r}_i = \mathbf{d}i$$

$$f = \left| \sum_i \exp[i\left(kd\cos\theta - \mathbf{k}\cdot\mathbf{d}\right)i] \right|^2$$

f peaks when

$$\left(kd\cos\theta - \mathbf{k}\cdot\mathbf{d}\right) = 2\pi n$$

Center Fed Linear Antenna



In short antennas current varies ~ linearly with z

$$\text{Current density}\quad \hat{\mathbf{J}}(\mathbf{r}) = I_0\,\delta(x)\delta(y)\left(1-2\frac{|z|}{d}\right)\hat{\mathbf{z}} \qquad |z| \le \frac{d}{2}$$

for $r >> r'$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r}e^{ikr}\int_{Vol} d\tau'\,\hat{\mathbf{J}}(\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'} = \frac{\mu_0}{4\pi r}e^{ikr} I_0\,\hat{\mathbf{z}}\int_{-d/2}^{d/2} dz'\left(1-2\frac{|z'|}{d}\right)e^{-ikz'\cos\theta}$$

Center Fed Linear Antenna

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' (1-2\frac{|z'|}{d}) e^{-ikz'\cos\theta}$$

use Euler's equ.

even odd

$$= I_0\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} dz' \left(1-2\frac{|z'|}{d}\right)\left(\cos(kz'\cos\theta) - i\sin(kz'\cos\theta)\right)$$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2I_0\,\hat{\mathbf{z}} \int_{0}^{d/2} dz' \left(1-2\frac{z'}{d}\right)\cos(kz'\cos\theta)$$

Center Fed Linear Antenna

To carry out the integration, let $\rho' = k\,z'\cos\theta$ and $\rho_0 = \dfrac{k\,d\cos\theta}{2}$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq 2\,I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0}\int_0^{\rho_0} d\rho'\left(1-\frac{\rho'}{\rho_0}\right)\cos\rho'$$

using $\int x\cos x\,dx = \cos x + x\sin x$

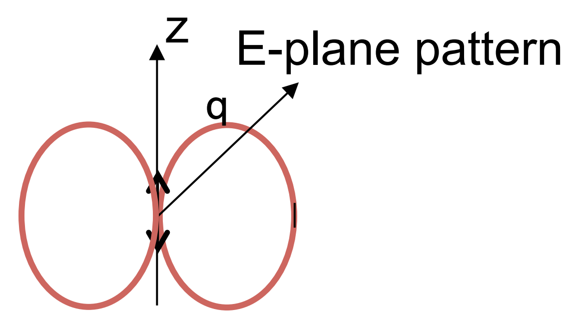
$$\hat{\mathbf{A}}(\mathbf{r}) \simeq I_0\,\hat{\mathbf{z}}\frac{d}{\rho_0^2}(1-\cos\rho_0) \quad \text{where } \rho_0 = \frac{k\,d\cos\theta}{2} = \frac{\pi\,d\cos\theta}{\lambda}$$

Antenna in the Dipoles Limit

In the dipole limit $\lambda >> d \quad (\rho_0 << 1)$

$$\hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} I_0 \frac{d}{2} \hat{\mathbf{z}} \qquad \hat{\mathbf{A}}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi r} e^{ikr} \hat{\mathbf{C}}(\mathbf{k})$$

hence, $\mathbf{C}(\mathbf{k}) = I_0 \frac{d}{2} \hat{\mathbf{z}}$



Power radiated into the solid angle $d\Omega$

$$\frac{dP_T}{d\Omega} = \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle \mathrm{r}^2 = \frac{Z_0}{32\pi^2} \left| \mathbf{k} \times \hat{\mathbf{C}}(\mathbf{k}) \right|^2 = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta$$

Total Power Radiated and Radiation Resistance

The total power radiated is $P_T = \oint_S \frac{dP_T}{d\Omega} d\Omega$

where $d\Omega = \sin\theta \, d\theta \, d\varphi$ is the solid angle

$$\frac{dP_T}{d\Omega} = \frac{Z_0}{32\pi^2} k^2 \frac{I_0^2 d^2}{4} \sin^2\theta \qquad \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta = 2\pi \frac{4}{3}$$

$$P_T = \frac{Z_0}{48\pi} k^2 d^2 I_0^2 = \frac{1}{2} Z_{rad} I_0^2$$

where the radiation resistance is $Z_{rad} = \frac{Z_0}{24\pi} k^2 d^2$ [Ω]

Time Dependent Fields

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Introduce Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Insert in Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \frac{\partial}{\partial t}\mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

Time Dependent Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

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$$\nabla \times \mathbf{B} = \mu_0\left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \ \mathbf{E} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0\left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \times \ \nabla \times \mathbf{A} = \mu_0\left(\vec{\mathbf{J}} - \varepsilon_0 \frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial}{\partial t}\mathbf{A}\right)\right)$$

$$\nabla\left(\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{A}$$

Now Pick Lorenz Gauge

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho / \varepsilon_0$$

$$\nabla\left(\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}$$

Lorenz Gauge: $\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi = 0$

$$-\left(\nabla^2\phi - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\phi\right) = \rho / \varepsilon_0$$

$$-\left(\nabla^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}\right) = \mu_0\vec{\mathbf{J}}$$

Same Equation
Wave Equation

Time Dependent Fields

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

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$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t}\mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

Time Dependent Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

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$$\nabla \cdot \ \mathbf{E} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \rho / \varepsilon_0$$

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$$\nabla \ \left(\nabla \cdot \mathbf{A} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{A}$$

Now Pick Lorenz Gauge

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho/\varepsilon_0$$

$$\nabla\left(\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}$$

Lorenz Gauge: $\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\phi = 0$

$$-\left(\nabla^2\phi - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\phi\right) = \rho/\varepsilon_0$$

$$-\left(\nabla^2\mathbf{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}\right) = \mu_0\vec{\mathbf{J}}$$

Same Equation
Wave Equation