

ENEE381

Lecture 17

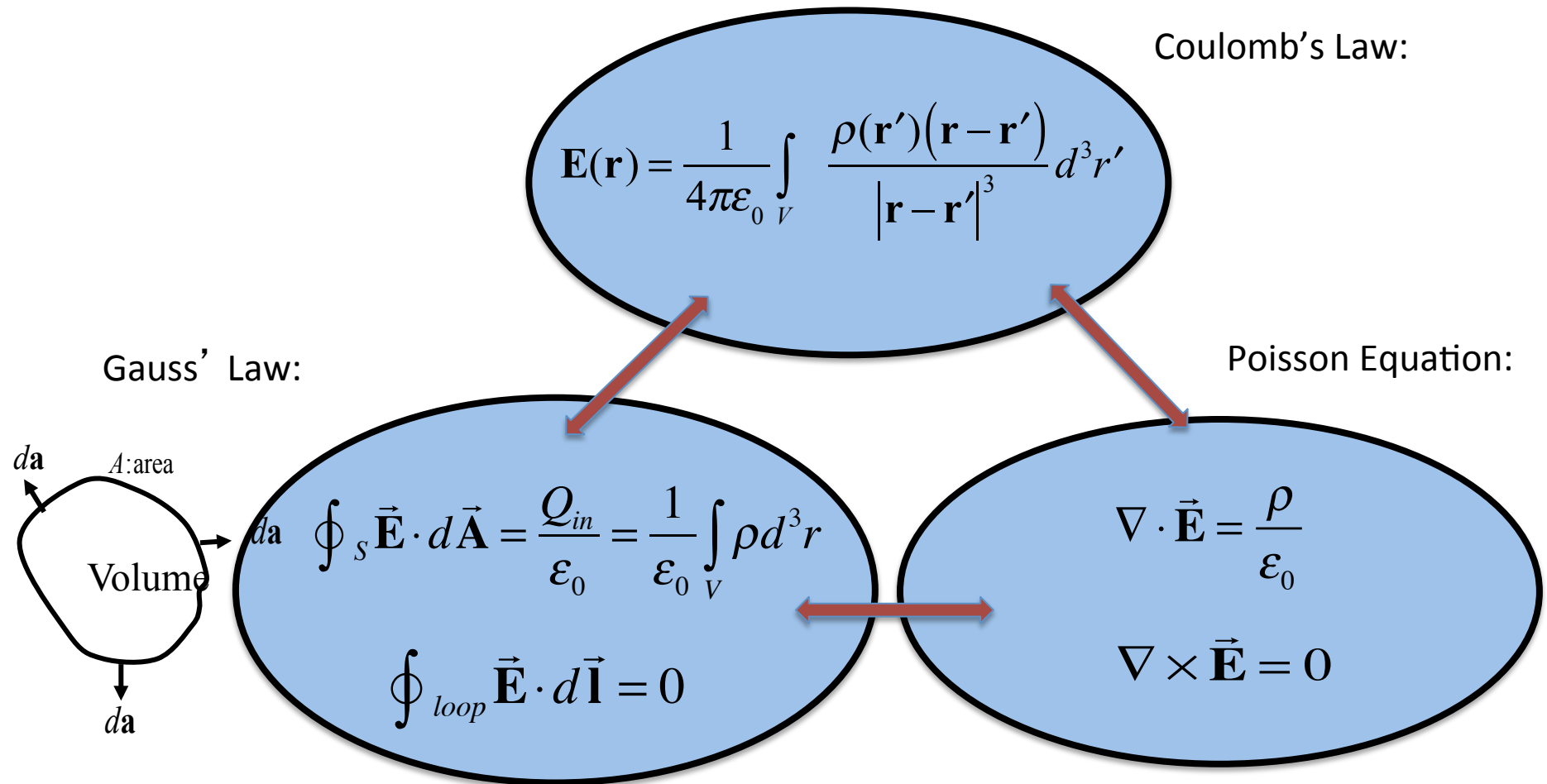
Radiation

Topics to be covered

Scalar and Vector Potentials

Green's functions for static and dynamic fields

Review: Static Electric Fields



Review: Magnetostatics

Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'$$

Ampere's Law:

$$\begin{aligned} \oint_{loop} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= \mu_0 I_{enclosed} \\ &= \mu_0 \oint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} \\ \oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} &= 0 \end{aligned}$$

Gauss' Law:

$$\begin{aligned} \nabla \times \vec{\mathbf{B}} &= \mu_0 \vec{\mathbf{J}} \\ \nabla \cdot \vec{\mathbf{B}} &= 0 \end{aligned}$$

Coulomb's Law for Electrostatic Potential

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'$$

This can also be written in terms of a scalar potential

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}), \quad \text{Automatically satisfies} \quad \nabla \times \mathbf{E}(\mathbf{r}) = -\nabla \times \nabla\phi(\mathbf{r}) = 0$$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r',$$

Show:

$$-\nabla\phi(\mathbf{r}) = \frac{-1}{4\pi\epsilon_0} \int_V \frac{\partial}{\partial \mathbf{r}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'$$

Coulomb's Law is Solution to the Following

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \quad \nabla \times \vec{\mathbf{E}} = 0$$

Introduce scalar potential

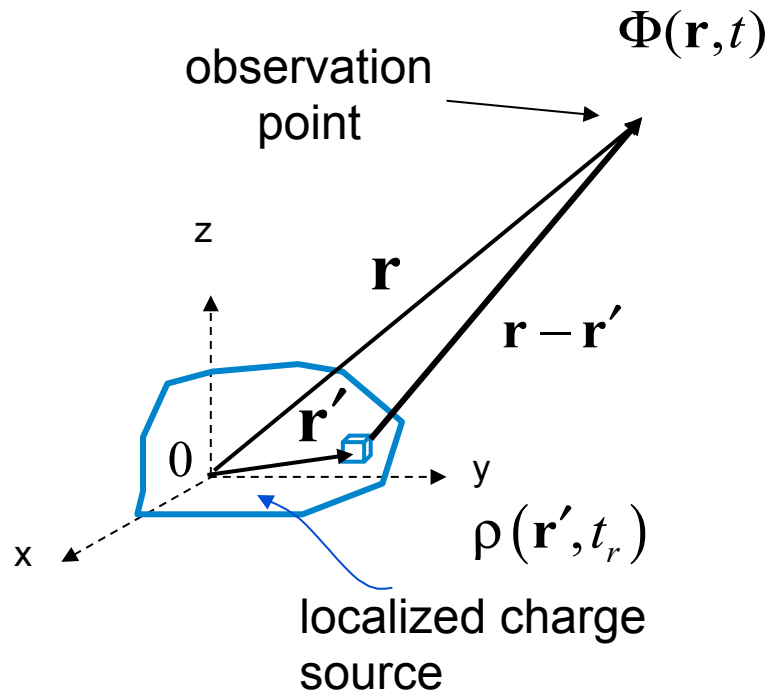
$$\nabla \cdot \vec{\mathbf{E}} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

Solution is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

Integral

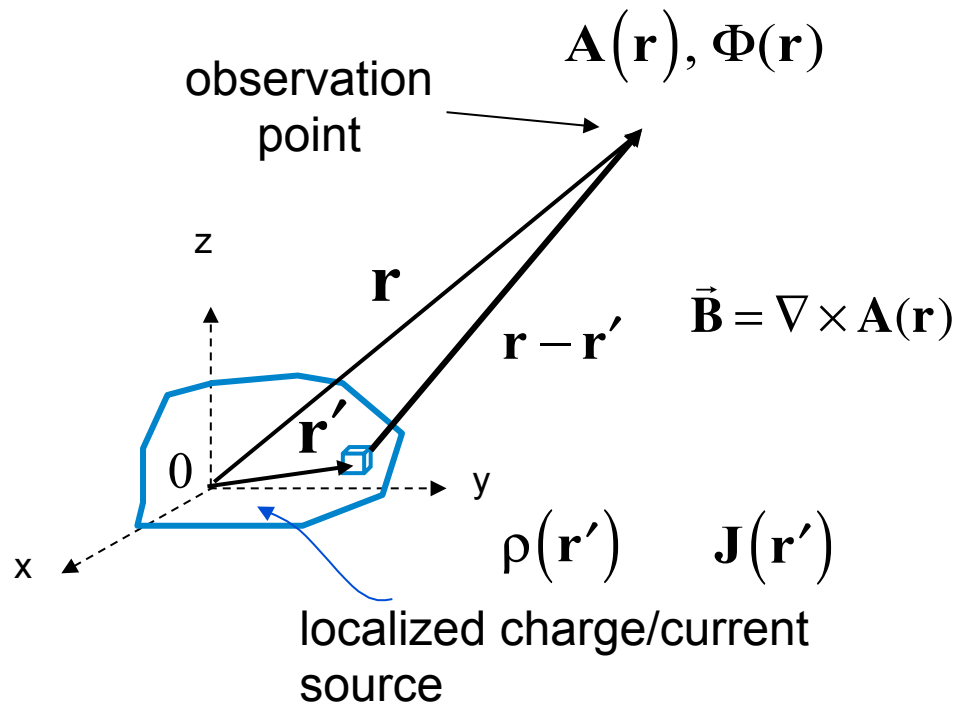


$$\nabla \cdot \vec{\mathbf{E}} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

Solution is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

Magnetic Field



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{Vol} d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{Vol} d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

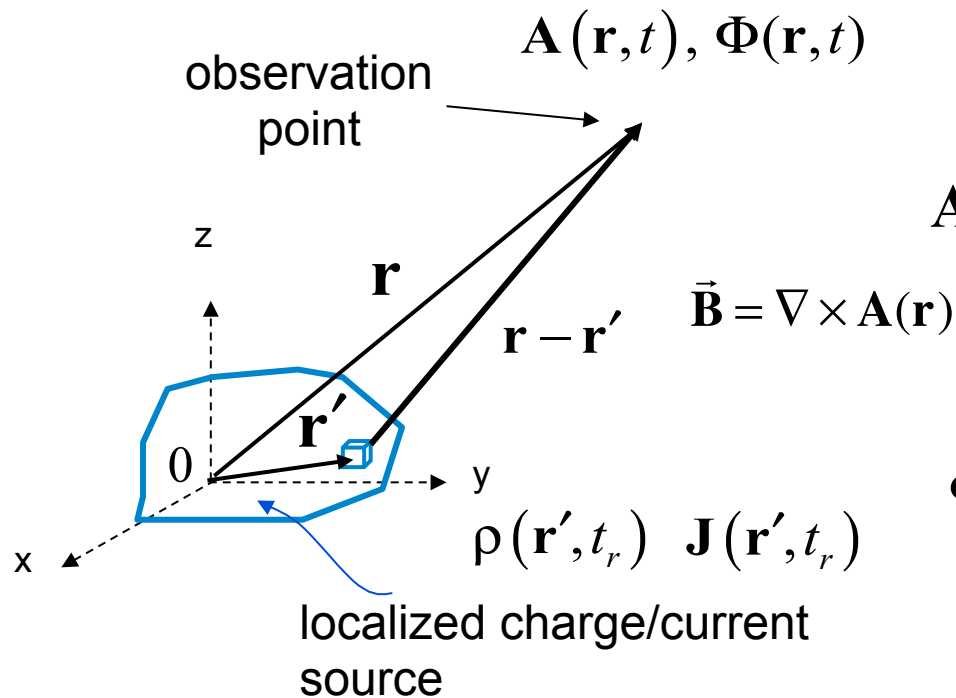
In static approximation time does not enter.

Changes in charge and current densities in time lead to changes in potentials.

No time delay!?

$$d\mathbf{r}' = dx' dy' dz'$$

Time-retarded potentials



$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{Vol} d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \Bigg|_{t_r = t - |\mathbf{r} - \mathbf{r}'|/c}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{Vol} d\mathbf{r}' \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \Bigg|_{t_r = t - |\mathbf{r} - \mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time (earlier time)

$$d\mathbf{r}' = dx' dy' dz'$$

What About Magnetic Field?

Basic Equations $\nabla \cdot \vec{\mathbf{B}} = 0$ $\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$

$\nabla \cdot \vec{\mathbf{B}} = 0$ Implies $\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$

$\nabla \cdot (\nabla \times \mathbf{A}(\mathbf{r})) = 0$ For any \mathbf{A}

\mathbf{A} is not unique.

If $\vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$ then $\vec{\mathbf{B}} = \nabla \times \mathbf{A}'(\mathbf{r})$,

where $\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi$ for any χ $\nabla \times \nabla \chi = 0$

How to understand

$\mathbf{A} = (\hat{\mathbf{x}}A_x(x,y,z) + \hat{\mathbf{y}}A_y(x,y,z) + \hat{\mathbf{z}}A_z(x,y,z))$ Has three independent functions

$\mathbf{B} = (\hat{\mathbf{x}}B_x(x,y,z) + \hat{\mathbf{y}}B_y(x,y,z) + \hat{\mathbf{z}}B_z(x,y,z))$ Has only two independent functions

How so?

$$\nabla \cdot \mathbf{B} = \left(\frac{\partial}{\partial x} B_x(x,y,z) + \frac{\partial}{\partial y} B_y(x,y,z) + \frac{\partial}{\partial z} B_z(x,y,z) \right) = 0$$

If I tell you B_x and B_y you can integrate over z to find B_z

Only two independent functions.

Gauges

We need to make up one scalar rule about \mathbf{A} to determine it uniquely.

Different rules are called different gauges.

Examples:

$$A_x = 0, \text{ No Name Gauge}$$

$$\nabla \cdot \mathbf{A} = 0, \text{ Coulomb Gauge}$$

$$\nabla \cdot \mathbf{A} + \epsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0, \text{ Lorenz Gauge}$$

What About Magnetic Field?

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \qquad \nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \qquad \text{Implies} \qquad \vec{\mathbf{B}} = \nabla \times \mathbf{A}(\mathbf{r})$$

Ampere's Law

$$\nabla \times \vec{\mathbf{B}} = \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

If Coulomb Gauge $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$,

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}, \qquad -\nabla^2 \phi = \rho / \epsilon_0 \qquad \text{Same equation!}$$

Static Scalar and Vector potential

$$-\nabla^2\phi = \frac{\rho}{\epsilon_0},$$

$$-\nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Solutions are:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r',$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

Time Dependent Fields

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Introduce Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Insert in Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} = -\nabla \times \frac{\partial}{\partial t} \mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} \right) = 0, \quad \mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} = -\nabla \phi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t} \mathbf{A}$$

Time Dependent Fields

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t} \mathbf{A}$$

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \rho / \epsilon_0$$

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\vec{\mathbf{J}} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \left(\vec{\mathbf{J}} - \epsilon_0 \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial}{\partial t} \mathbf{A} \right) \right)$$

$$\nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \phi \right) - \nabla^2 \mathbf{A} = \mu_0 \vec{\mathbf{J}} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{A}$$

Now Pick Lorenz Gauge

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \rho / \epsilon_0$$

$$\nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial}{\partial t}\phi\right) - \nabla^2\mathbf{A} = \mu_0\vec{\mathbf{J}} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}$$

Lorenz Gauge: $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial}{\partial t}\phi = 0$

$$-\left(\nabla^2\phi - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\phi\right) = \rho / \epsilon_0$$

$$-\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\mathbf{A}\right) = \mu_0\vec{\mathbf{J}}$$

Same Equation
Wave Equation

Maxwell's Equations for Vector and Scalar Potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

In the Lorentz gauge $\left(\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \right)$ the vector and scalar potentials obey wave equations

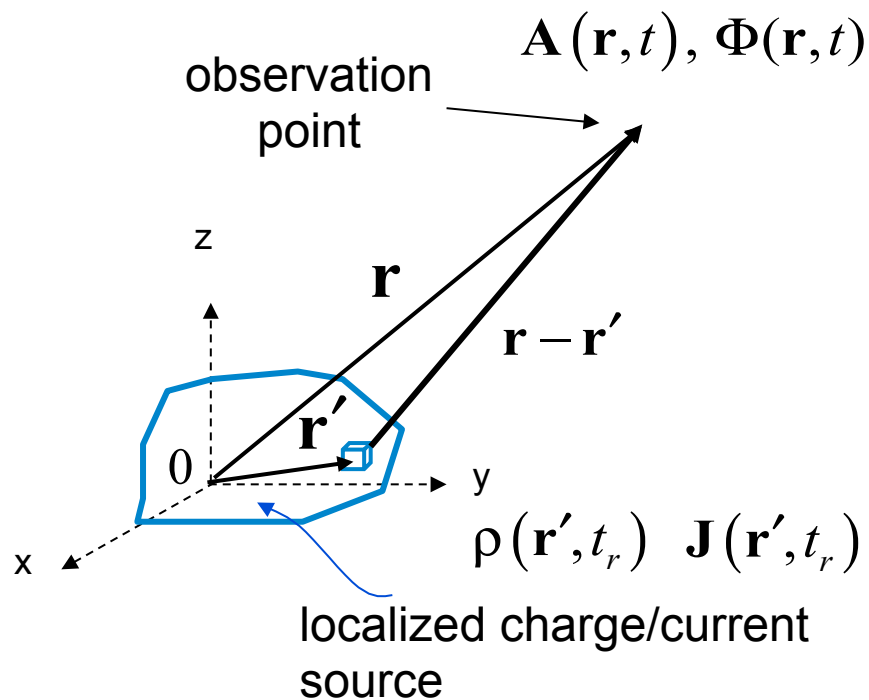
$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \quad \sqrt{\epsilon_0 \mu_0} = 1/c$$

$$\nabla^2 \Phi - \mu_0 \epsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

where \mathbf{J} and ρ are the current and charge densities

The solutions to the wave equations (in the absence of boundaries) are

Solution to Wave Equations



$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{Vol} d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \Bigg|_{t_r = t - |\mathbf{r} - \mathbf{r}'|/c}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{Vol} d\mathbf{r}' \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \Bigg|_{t_r = t - |\mathbf{r} - \mathbf{r}'|/c}$$

where $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time (earlier time)

$$d\tau' = dx' dy' dz'$$