

**ENEE381**

Networks

# N-port Linear Systems

Wave Systems are characterized by scattering (or S) parameters. Actually a matrix

These give the amplitudes of waves leaving the system as a linear function of the amplitudes of waves entering the system.

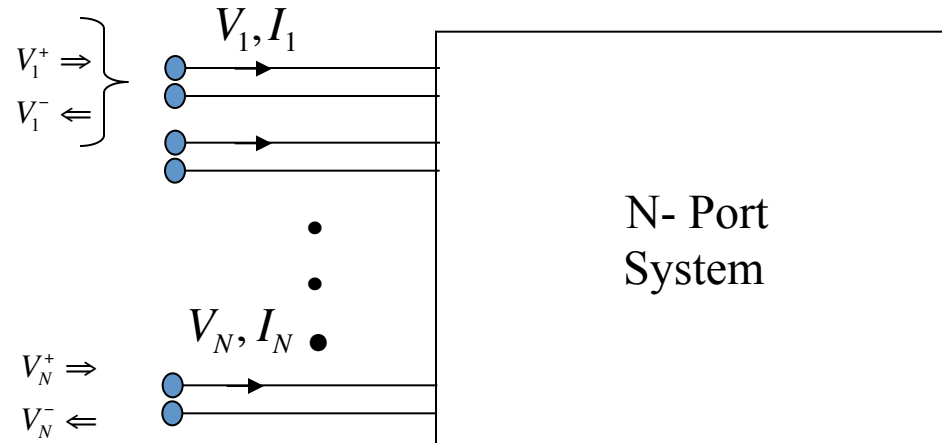
The simplest example is a reflection coefficient

$$V_{\text{refl}} / V_{\text{inc}} = \rho$$

# Impedance (Z) and Scattering (S) Matrices

N ports

- voltages and currents,
- incoming and outgoing waves



Z matrix

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \mathbf{Z} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}$$

voltage

current

S matrix

$$\begin{pmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_{N1}^- \end{pmatrix} = \mathbf{S} \begin{pmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_{N1}^+ \end{pmatrix}$$

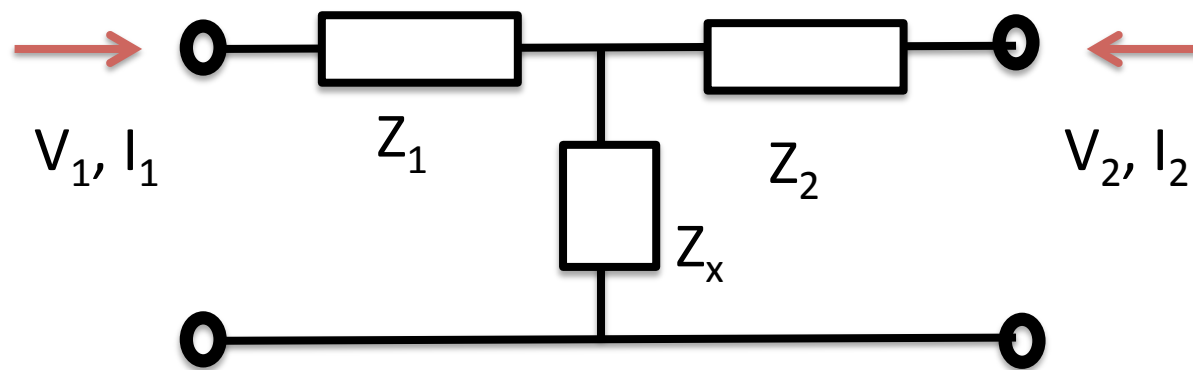
outgoing

incoming

$$\mathbf{S} = (\mathbf{Z} + \mathbf{Z}_0)^{-1} (\mathbf{Z} - \mathbf{Z}_0)$$

$$\mathbf{Z}(\omega), \mathbf{S}(\omega)$$

# Example – T-circuit



Impedance Matrix

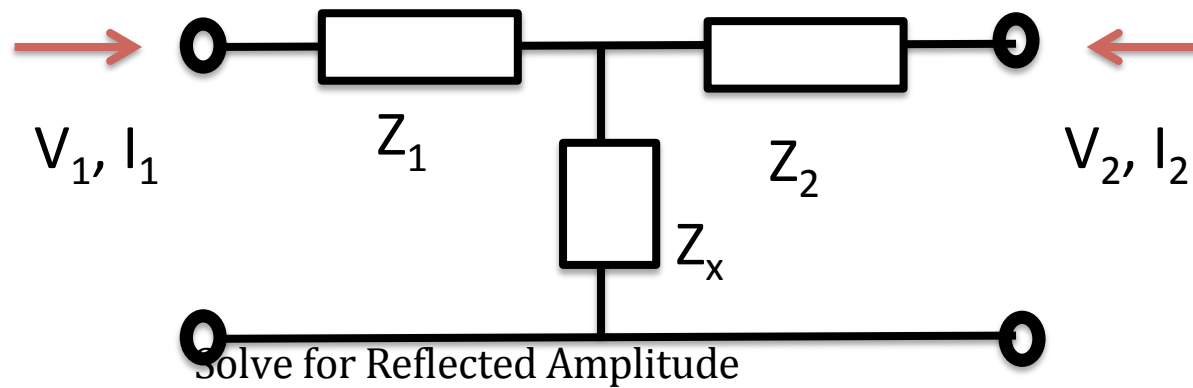
$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$Z_{11} = Z_1 + Z_x$$

$$Z_{22} = Z_2 + Z_x$$

$$Z_{12} = Z_{21} = Z_x$$

# Scattering Matrix



$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \underline{\underline{Z}} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} + \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \begin{pmatrix} \underline{\underline{Z}} & \underline{\underline{Z}}^{-1} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{pmatrix} \left\{ \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} - \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} \right\}$$

Incident and Reflected

Current Amplitudes

$$V_1 = (V_{1i} + V_{1r})$$

$$I_1 = Z_{01}^{-1} (V_{1i} - V_{1r})$$

$$V_2 = (V_{2i} + V_{2r})$$

$$I_2 = Z_{02}^{-1} (V_{2i} - V_{2r})$$

$$\begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} \begin{pmatrix} \underline{\underline{Z}} & \underline{\underline{Z}}^{-1} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{pmatrix} + \underline{\underline{1}} = \begin{pmatrix} \underline{\underline{Z}} & \underline{\underline{Z}}^{-1} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{pmatrix} \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} - \underline{\underline{1}}$$

$$\underline{\underline{Z}}_0 = \begin{bmatrix} Z_{01} & 0 \\ 0 & Z_{02} \end{bmatrix}, \quad \underline{\underline{Z}} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

Solve for Reflected Amplitude  
in terms of Incident

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} + \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left( \underline{\underline{Z}} \underline{\underline{Z}}_0^{-1} \right) \left\{ \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix} - \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} \right\}$$

$$\left( \underline{\underline{Z}} \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right) \begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left( \underline{\underline{Z}} \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix}$$

$$\begin{pmatrix} V_{1r} \\ V_{2r} \end{pmatrix} = \left( \underline{\underline{Z}} \underline{\underline{Z}}_0^{-1} + \underline{\underline{1}} \right)^{-1} \left( \underline{\underline{Z}} \underline{\underline{Z}}_0^{-1} - \underline{\underline{1}} \right) \begin{pmatrix} V_{1i} \\ V_{2i} \end{pmatrix}$$

# Define Normalized Wave Amplitudes

Power Entering the Network

$$P = \frac{1}{2} \text{Re} \{ \underline{I}^{*T} \cdot \underline{V} \} = \frac{1}{2} \text{Re} \left\{ (\underline{V}_i^* - \underline{V}_r^*)^T \cdot \underline{Z}_0^{-1} \cdot (\underline{V}_i + \underline{V}_r) \right\}$$

$$= \frac{1}{2} \text{Re} \left\{ \underbrace{\underline{V}_i^{*T}}_{\text{Incident}} \cdot \underline{Z}_0^{-1} \cdot \underline{V}_i - \underbrace{\underline{V}_r^{*T}}_{\text{Reflected}} \cdot \underline{Z}_0^{-1} \cdot \underline{V}_r \right\}$$

$$\underline{V}_r = (\underline{Z} \underline{Z}_0^{-1} + \underline{1})^{-1} (\underline{Z} \underline{Z}_0^{-1} - \underline{1}) \underline{V}_i$$

$$\underline{A}_r = \underline{Z}_0^{-1/2} (\underline{Z} \underline{Z}_0^{-1} + \underline{1})^{-1} (\underline{Z} \underline{Z}_0^{-1} - \underline{1}) \underline{Z}_0^{1/2} \underline{A}_i$$

$$(\underline{Z} \underline{Z}_0^{-1} + \underline{1})^{-1} = \left[ (\underline{Z} + \underline{Z}_0) \underline{Z}_0^{-1} \right]^{-1} = \underline{Z}_0 (\underline{Z} + \underline{Z}_0)^{-1}$$

$$(\underline{Z} \underline{Z}_0^{-1} - \underline{1}) = (\underline{Z} - \underline{Z}_0) \underline{Z}_0^{-1}$$

$$\underline{A}_r = \underline{Z}_0^{1/2} (\underline{Z} + \underline{Z}_0)^{-1} (\underline{Z} - \underline{Z}_0) \underline{Z}_0^{-1/2} \underline{A}_i$$

$$\underline{S} = \underline{Z}_0^{1/2} (\underline{Z} + \underline{Z}_0)^{-1} (\underline{Z} - \underline{Z}_0) \underline{Z}_0^{-1/2}$$

Normalized Amplitudes

$$\underline{A}_i = \underline{Z}_0^{-1/2} \underline{V}_i, \quad \underline{A}_r = \underline{Z}_0^{-1/2} \underline{V}_r$$

$$P = \frac{1}{2} \text{Re} \left\{ \underline{A}_i^{*T} \cdot \underline{A}_i - \underline{A}_r^{*T} \cdot \underline{A}_r \right\}$$

Scattering Matrix

$$\underline{A}_r = \underline{S} \cdot \underline{A}_i$$

# Scattering Matrix Properties

S and Z are often Symmetric – Reciprocity

Example: 
$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\begin{aligned} Z_{11} &= Z_1 + Z_x \\ Z_{22} &= Z_2 + Z_x \\ Z_{12} &= Z_{21} = Z_x \end{aligned}$$

Unitary – Consequence of absence of losses and Time Reversal Symmetry

$$\underline{\underline{S}}^* \cdot \underline{\underline{S}} = \underline{\underline{1}}$$

If Network is lossless elements of Z are imaginary.  $\underline{\underline{Z}}^* = -\underline{\underline{Z}}$

$$\underline{\underline{S}} = \underline{\underline{Z}}_0^{1/2} (\underline{\underline{Z}} + \underline{\underline{Z}}_0)^{-1} (\underline{\underline{Z}} - \underline{\underline{Z}}_0) \underline{\underline{Z}}_0^{-1/2}$$

$$\underline{\underline{S}}^* = \underline{\underline{Z}}_0^{1/2} (-\underline{\underline{Z}} + \underline{\underline{Z}}_0)^{-1} (-\underline{\underline{Z}} - \underline{\underline{Z}}_0) \underline{\underline{Z}}_0^{-1/2} = \underline{\underline{Z}}_0^{1/2} (\underline{\underline{Z}} - \underline{\underline{Z}}_0)^{-1} (\underline{\underline{Z}} + \underline{\underline{Z}}_0) \underline{\underline{Z}}_0^{-1/2}$$

$$\underline{\underline{S}}^* \underline{\underline{S}} = \underline{\underline{Z}}_0^{1/2} (\underline{\underline{Z}} - \underline{\underline{Z}}_0)^{-1} (\underline{\underline{Z}} + \underline{\underline{Z}}_0) (\underline{\underline{Z}} + \underline{\underline{Z}}_0)^{-1} (\underline{\underline{Z}} - \underline{\underline{Z}}_0) \underline{\underline{Z}}_0^{-1/2} = \underline{\underline{1}}$$



# 2 x 2 Unitary Scattering Matrix

$$\underline{\underline{S}} = \begin{bmatrix} R^{1/2} e^{i\theta_{11}} & T^{1/2} e^{i\theta_{12}} \\ T^{1/2} e^{i\theta_{12}} & R^{1/2} e^{i\theta_{22}} \end{bmatrix}$$

$$R + T = 1$$

$$\theta_{21} = \frac{\theta_{11} + \theta_{22}}{2} + \left( n + \frac{1}{2} \right) \pi$$

# 2 x 2 Unitary Scattering Matrix

$$\underline{\underline{S}}^* \cdot \underline{\underline{S}} = \underline{\underline{1}} = \begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}$$

$$\underline{\underline{1}} = \begin{bmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{21}^* S_{22} + S_{11}^* S_{21} \\ S_{21}^* S_{11} + S_{22}^* S_{21} & |S_{22}|^2 + |S_{21}|^2 \end{bmatrix}$$

$$|S_{11}|^2 + |S_{21}|^2 = 1 = |S_{22}|^2 + |S_{21}|^2 \quad S_{21}^* S_{11} + S_{22}^* S_{21} = 0 = S_{21}^* S_{22} + S_{11}^* S_{21}$$

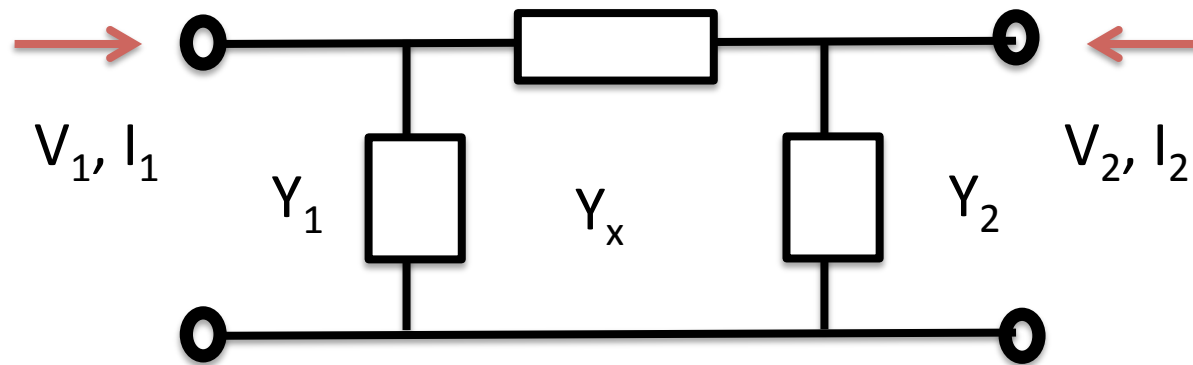
$$|S_{21}|^2 = T \quad (\text{RT})^{1/2} \left[ \exp(i(\theta_{11} - \theta_{21})) + \exp(i(\theta_{21} - \theta_{22})) \right] = 0$$

$$|S_{11}|^2 = |S_{22}|^2 = R \quad \exp(i(\theta_{11} + \theta_{22} - 2\theta_{21})) = -1$$

$$R + T = 1$$

$$\theta_{21} = \frac{\theta_{11} + \theta_{22}}{2} + \left( n + \frac{1}{2} \right) \pi$$

# Example – $\pi$ -circuit



Admittance Matrix

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$Y_{11} = Y_1 + Y_x$$

$$Y_{22} = Y_2 + Y_x$$

$$Y_{12} = Y_{21} = -Y_x$$

$$Y = Z^{-1}$$

# How to measure S-parameters

Network & Spectrum Analyzers > Network Analyzers > Vector Network Analyzers > Tektronix TTR506A USB Vector Network Analyzer



## Tektronix TTR506A USB Vector Network Analyzer

Manufacturer Part Number: TTR506A

TestEquity Part Number: 25859.1

Shipping Weight (Lbs): 1.000

Condition: New

Manufacturer: Tektronix

Your Price: **\$13,600.00**

Sold per: EA

In Stock

Qty on Hand: 1

QTY

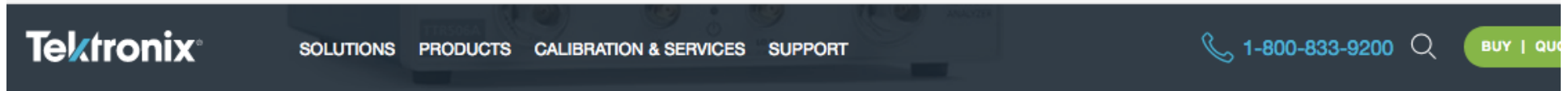
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### How does a Vector Network Analyzer (VNA) work?

A Vector Network Analyzer contains both a source, used to generate a known stimulus signal, and a set of receivers, used to determine changes to this stimulus caused by the device-under-test or DUT.

The stimulus signal is injected into the DUT and the Vector Network Analyzer measures both the signal that's reflected from the input side, as well as the signal that passes through to the output side of the DUT. The Vector Network Analyzer receivers measure the resulting signals and compare them to the known stimulus signal. The measured results are then processed by either an internal or external PC and sent to a display.

