ENEE381

Lecture 8
Dispersion
Absorption
Skin Effect
Group and Phase Velocity

Topics

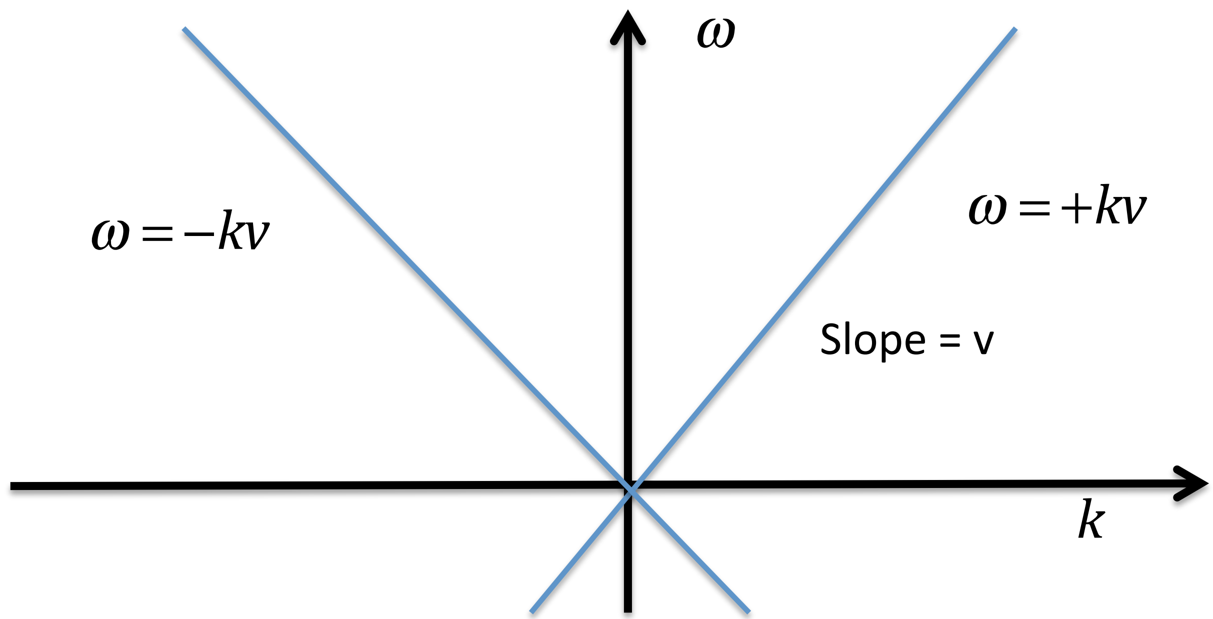
Dispersion
Absorption
Skin Effect
Group and phase velocity

Dispersion Relation

Plane waves in a nondispersive medium

$v = 1/\sqrt{\varepsilon\mu}$

ε, μ independent of frequency



$\omega = +kv$

$\lambda = v / f$

Slope of omega vs k is independent of k or omega

$\lambda = 2\pi / k$

$\omega = 2\pi f$

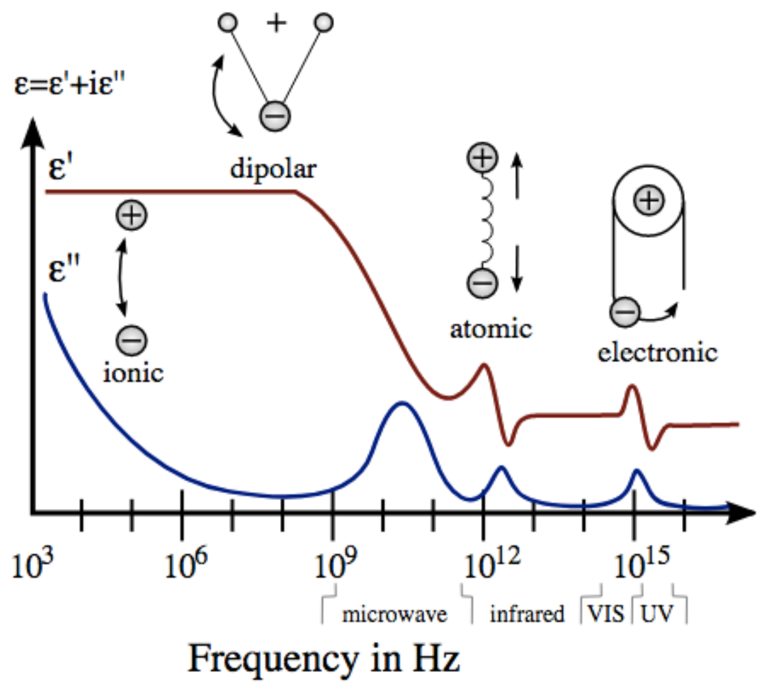
Pulses maintain their shape while propagating

Dispersion

$$\omega^2 = \frac{k^2}{\varepsilon\mu} = k^2 v^2$$

But in reality

$$\varepsilon = \varepsilon(\omega)$$
$$\mu = \mu(\omega)$$
$$v = v(\omega)$$



Different frequencies propagate with different speeds

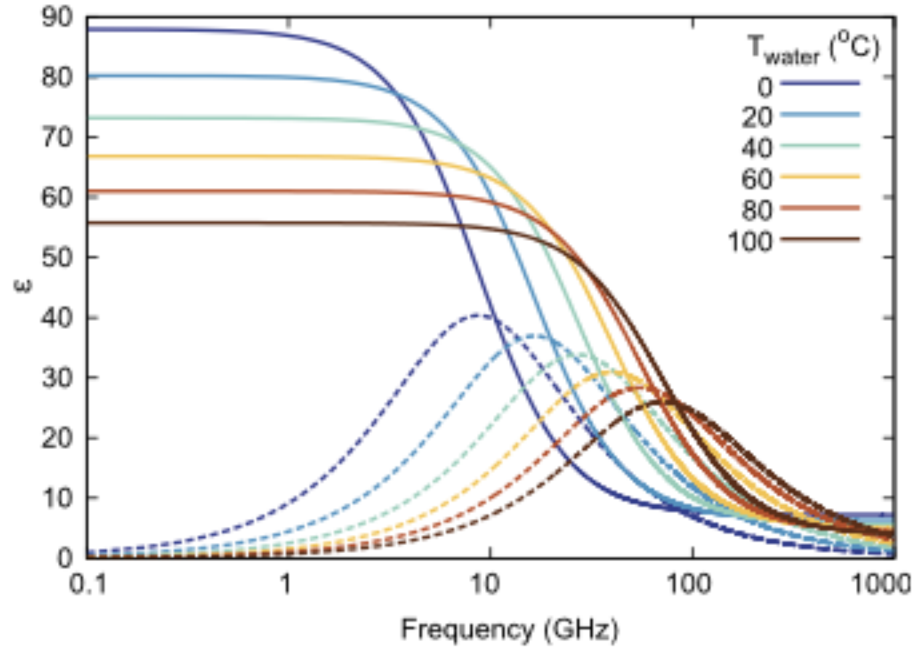
Group velocity dispersion - GVD

Dielectric function is complex

Absorption

https://en.wikipedia.org/wiki/Permittivity
"Dielectric Spectroscopy". Archived from the original on 2006-01-18. Retrieved 2018-11-20.

Dielectric Function for H_2O



See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/281552350

Water: Promising Opportunities For Tunable All-dielectric Electromagnetic Metamaterials

Article *in* Scientific Reports · August 2015

DOI: 10.1038/srep13535

Phase and Group Velocity

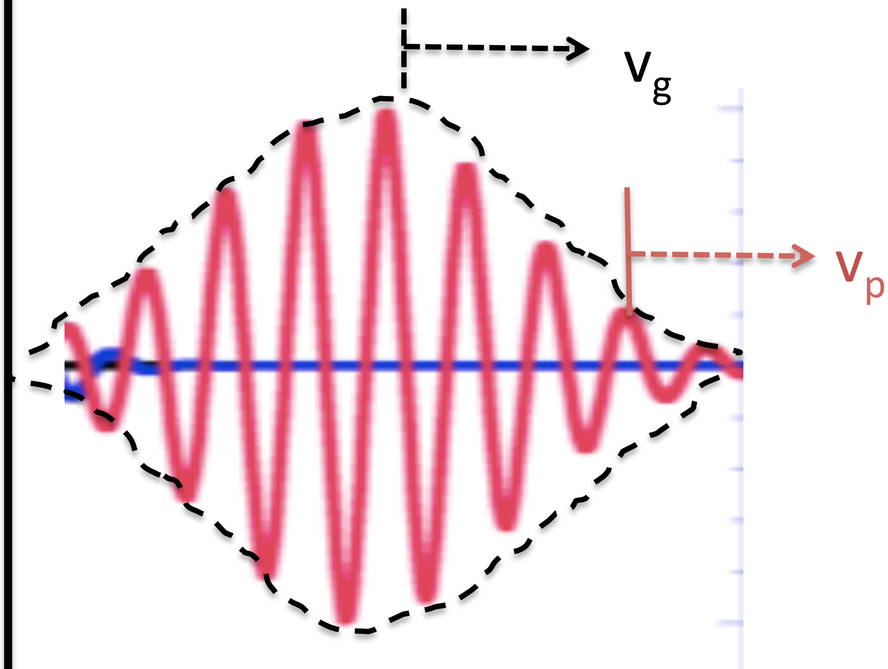
$$\omega\sqrt{\varepsilon(\omega)\mu_0} = k$$

The crests travel at the phase velocity

$$v_p = \frac{\omega_c}{k(\omega_c)}$$

The envelope travels at the group velocity

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega_c}$$



Group and Phase Velocity

Consider a superposition of two waves:

$$E = E_0 \cos\left(k_1 z - \omega_1 t\right) + E_0 \cos\left(k_2 z - \omega_2 t\right)$$

$$\omega_1 = \omega(k_1), \quad \omega_2 = \omega(k_2)$$

Trig identity::

$$\cos\left(A\right) + \cos\left(B\right) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$E = 2E_0 \cos\left(\bar{k}z - \bar{\omega}t\right)\cos\left(\Delta k z - \Delta\omega t\right)$$

$$\bar{k} = \frac{1}{2}\left(k_1 + k_2\right), \qquad \bar{\omega} = \frac{1}{2}\left(\omega_1 + \omega_2\right) = \frac{1}{2}\left(\omega(k_1) + \omega(k_2)\right)$$

$$\Delta k = \frac{1}{2}\left(k_1 - k_2\right), \qquad \Delta\bar{\omega} = \frac{1}{2}\left(\omega(k_1) - \omega(k_2)\right) \approx \Delta k \frac{d\omega}{d\bar{k}}$$

Carrier and Envelope

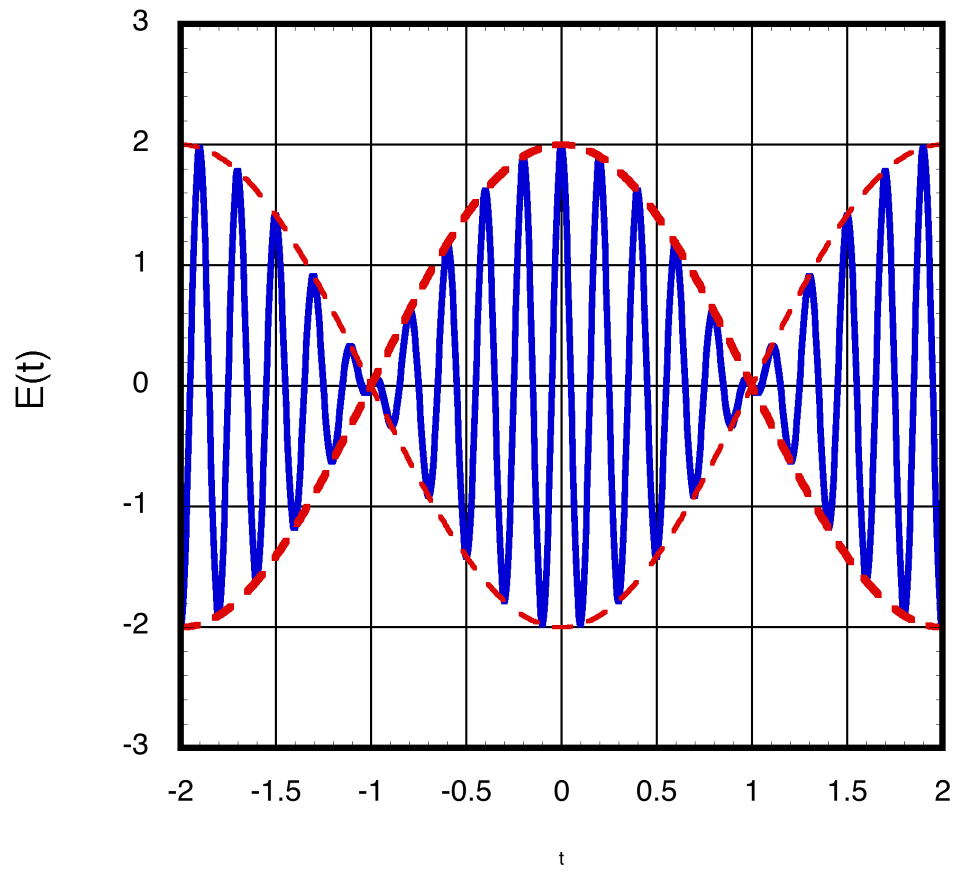
$$E = 2E_0 \cos\left(\bar{k}z - \bar{\omega}t\right)\cos\left(\Delta kz - \Delta\omega t\right)$$

Carrier envelope

Speed of carrier: $\bar{\omega} / \bar{k} = v_{phase}$

Speed of envelope: $\Delta\omega / \Delta \mathrm{k}$

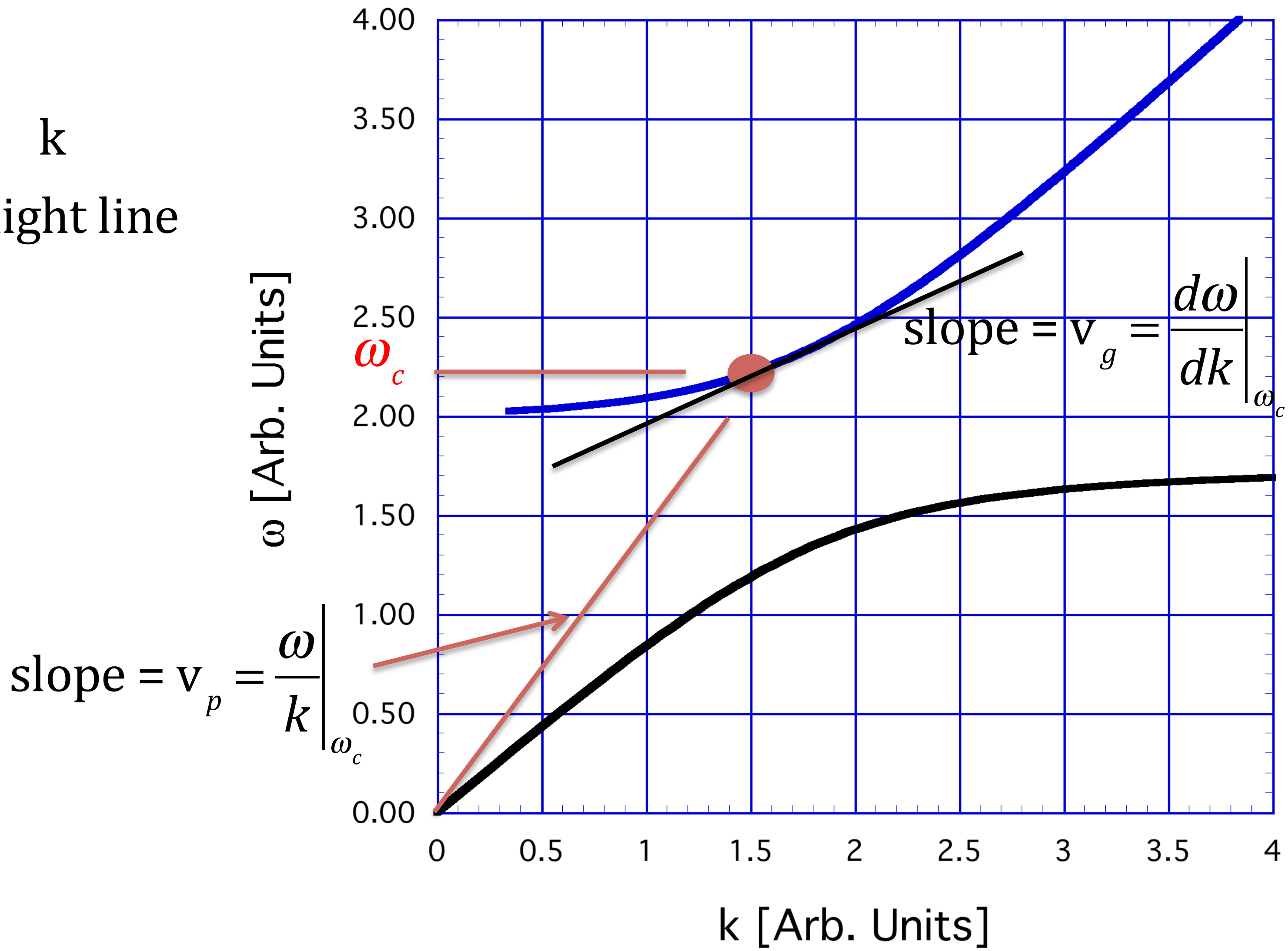
$\simeq \mathrm{d}\omega(\mathrm{k})/\mathrm{dk} = \mathrm{v}_{group}$



Dispersion Relation

$\omega(k)$ vs k

not a straight line



Problem

The dielectric function for a plasma is

$$\varepsilon(\omega)=\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right),\quad \text{where } \omega_p^2=\frac{ne^2}{m\varepsilon_0},\ n \text{ is the electron number density.}$$

ω_p is called the plasma frequency.

Find the dispersion relation $\omega(k)$, also find v_p, v_g

$$k^2=\omega^2\varepsilon\mu_0=\omega^2\varepsilon_0\mu_0\left(1-\frac{\omega_p^2}{\omega^2}\right)=\frac{1}{c^2}\left(\omega^2-\omega_p^2\right)$$

$$v_p=\frac{\omega}{k}=\frac{c\omega}{\left(\omega^2-\omega_p^2\right)^{1/2}}=\frac{c}{\left(1-\omega_p^2/\omega^2\right)^{1/2}}$$

rewriting the dispersion rellation

$$\omega=\left(k^2c^2+\omega_p^2\right)^{1/2}$$

$$v_g=\frac{d\omega}{dk}=\frac{kc^2}{\left(k^2c^2+\omega_p^2\right)^{1/2}}=c\left(1-\omega_p^2/\omega^2\right)^{1/2}$$

Simple Models

Suppose you have a dielectric material that also has a conductivity?

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_{free} + \frac{\partial \varepsilon \vec{\mathbf{E}}}{\partial t}$$

$$\vec{\mathbf{J}}_{free} = \sigma \vec{\mathbf{E}}$$ Ohm's Law

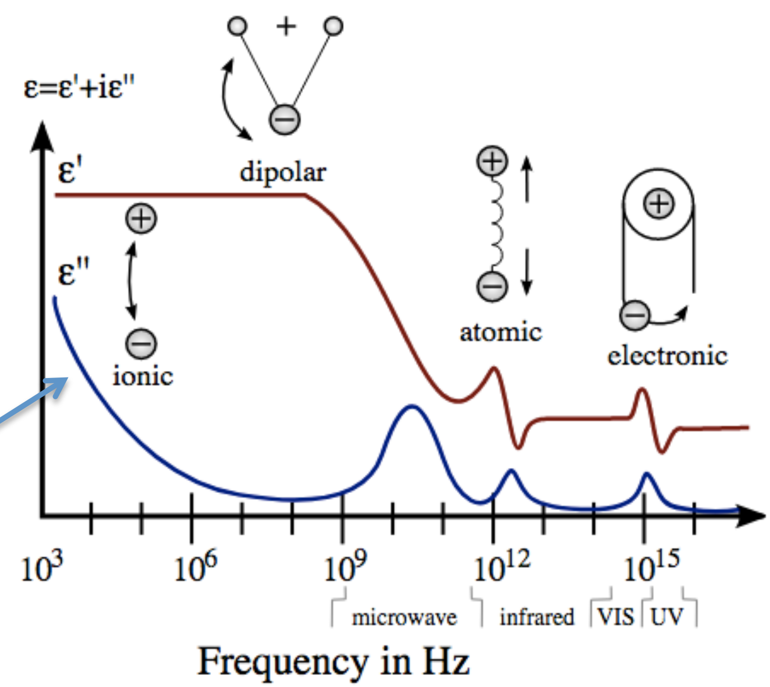
Pass to Phasor Representation

$$i\mathbf{k} \times \hat{\mathbf{H}} = (\sigma - i\omega\varepsilon)\hat{\mathbf{E}}$$

$$= i\omega\left(\varepsilon + i\frac{\sigma}{\omega}\right)\hat{\mathbf{E}}$$

Effective Dielectric function

$$\varepsilon' + i\varepsilon'' = \left(\varepsilon + i\frac{\sigma}{\omega}\right)$$



Modification to dispersion relation

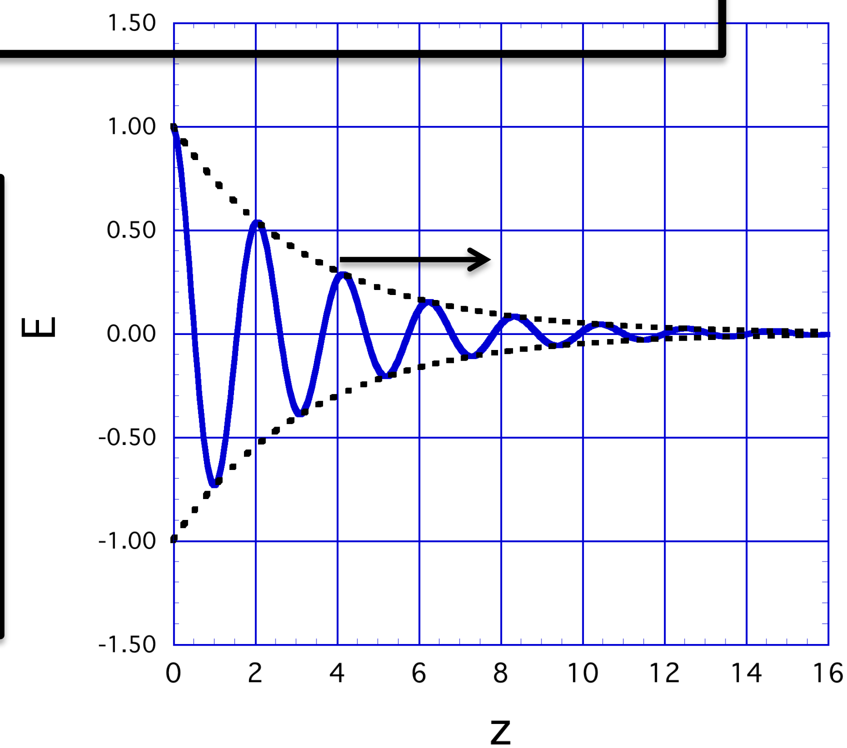
$$k^2 = \omega^2 \varepsilon \mu = \omega^2 \mu (\varepsilon' + i\varepsilon'')$$

If frequency is real, (controlled by source of waves), k must be complex.

$$k = k' + ik''$$

$$\mathbf{E} = \mathrm{Re}\left\{ \hat{\mathbf{E}} \exp\left[i(k' + ik'')z - i\omega t \right] \right\}$$

$$= \exp[-k''z]\,\mathrm{Re}\left\{ \hat{\mathbf{E}} \exp[ik'z - i\omega t] \right\}$$



Two limiting cases: 1. weak damping $\varepsilon' >> \varepsilon''$

2. good conductor $\varepsilon'' = \dfrac{\sigma}{\omega} >> \varepsilon$

Weak damping $k^2 = \omega^2 \mu \varepsilon' \left(1 + i \dfrac{\varepsilon''}{\varepsilon'} \right)$ $\quad 1 >> \dfrac{\varepsilon''}{\varepsilon'}$ loss tangent

$$k = k' + ik'' \qquad k'' << k'$$

$$k^2 = \left(k' + ik'' \right)^2 = k'^2 - k''^2 + 2ik'k'' \qquad \simeq k'^2 + 2ik'k''$$

$$k'^2 = \omega^2 \mu \varepsilon' \qquad 2k'k'' = \omega^2 \mu \varepsilon''$$

$$\mathbf{E} = \mathrm{Re}\left\{ \hat{\mathbf{E}} \exp\left[i\left(k' + ik'' \right) z - i\omega t \right] \right\}$$
$$= \exp\left[-k'' z \right] \mathrm{Re}\left\{ \hat{\mathbf{E}} \exp\left[ik' z - i\omega t \right] \right\}$$

Decay per wavelength determined by loss tangent

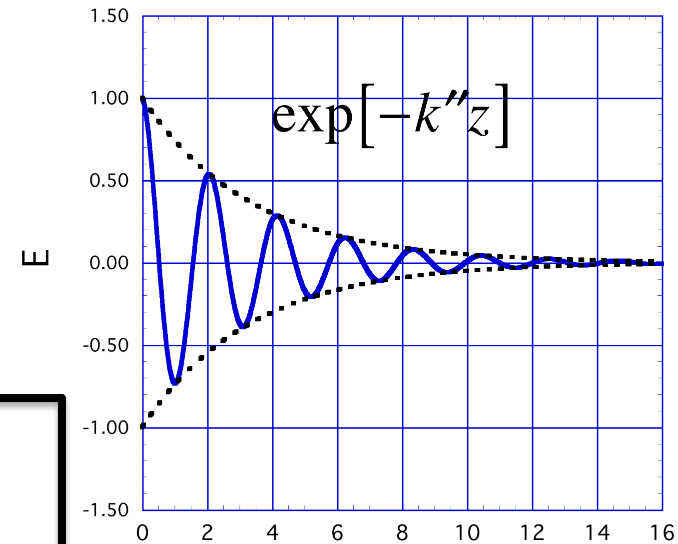
$$\frac{k''}{k'} = \frac{1}{2} \frac{\varepsilon''}{\varepsilon'}$$

Good Conductor

$$k^2 = \omega^2 \mu \left(\varepsilon + i \frac{\sigma}{\omega} \right) \simeq i\omega\mu\sigma$$

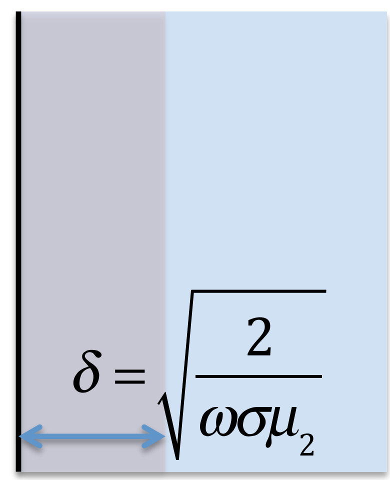
$$k = k' + ik'' \qquad k'' = k' = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$



Current flows in a layer

$$\delta = k''^{-1} = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$



Surface Impedance

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

Skin Depth

$$\delta = k''^{-1} = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

TABLE 8–1
Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	$f = 60$ (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

† The ϵ of seawater is approximately $72\epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega\epsilon \cong 1$ (not $\gg 1$). Under these conditions, seawater is not a good conductor, and Eq. (8–57) is no longer applicable.

Surface Impedance

Ratio of tangential E to tangential H at surface

$$\frac{E_{\tan}}{H_{\tan}} = \eta_2$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

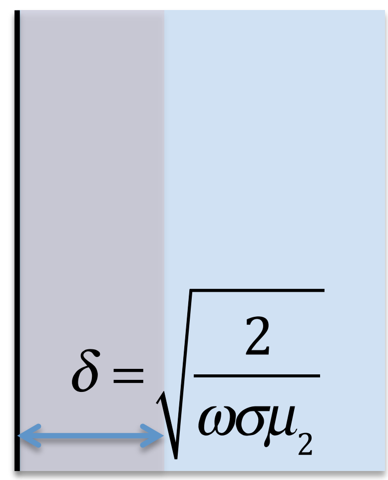
$$\eta_2 = Z_s = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

$$Z_s = (1-i)\sqrt{\frac{\omega\mu_2}{2\sigma}} = (1-i)R_s$$

Resistance + i Reactance

Poynting Flux

$$S_z = \frac{1}{2}\mathrm{Re}\left\{E_{\tan}^{*} H_{\tan}\right\} = \frac{1}{2}R_s\left|H_{\tan}\right|^2$$

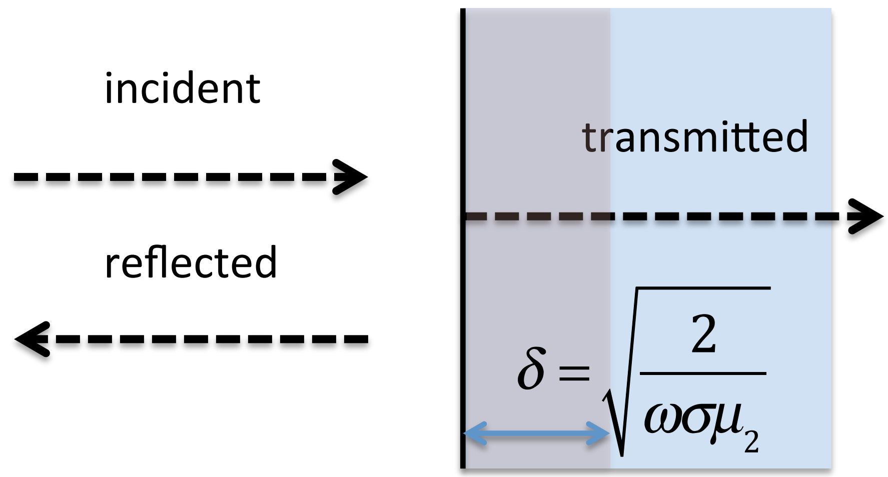


$$\delta = \sqrt{\frac{2}{\omega\sigma\mu_2}}$$

Reflection from conductor

Formally the same as reflection from dielectric. Just use the surface impedance

$$\rho = \frac{Z_s - \eta_1}{Z_s + \eta_1}$$



$$\eta_2 = Z_s = \sqrt{\frac{\omega\mu_2}{i\sigma}}$$

$$Z_s = (1-i)\sqrt{\frac{\omega\mu_2}{2\sigma}} \equiv (1-i)R_s$$

Calculate power transmission coefficient

$$T = 1 - R = 1 - |\rho|^2$$

Assume $|Z_s| << \eta_1$

$$S_z = \frac{1}{2}\mathrm{Re}\left\{E^*_{\tan} H_{\tan}\right\} = \frac{1}{2} R_s \left|H_{\tan}\right|^2$$

$$H_{\tan} = 2\hat{H}_{inc}$$

$$\boxed{S_z = \frac{1}{2}\frac{R_s}{\eta_1} 4\eta_1 \left|\hat{H}_{inc}\right|^2 = 4\frac{R_s}{\eta_1} P_{inc}}$$

Simple model of polarizable material

Atom/molecule

Displacement

$\mathbf{x}_d(t)$  q, m

Spring constant- $m\omega_0^2$

Restoring force — friction force — driving force

Newton's law

$$m\frac{d^2}{dt^2}\mathbf{x}_d(t) = -m\omega_0^2\mathbf{x}_d(t) - m\nu\frac{d}{dt}\mathbf{x}_d(t) + q\mathbf{E}(\mathbf{x},t)$$

Phasors

$$\left[\omega_0^2 - i\omega\nu - \omega^2\right]\hat{\mathbf{x}}_d = \frac{q}{m}\hat{\mathbf{E}}(\mathbf{x})$$

Current density

$$\mathbf{J} = qN\frac{d}{dt}\mathbf{x}_d(t) \qquad \hat{\mathbf{J}} = -i\omega qN\hat{\mathbf{x}}_d \qquad \hat{\mathbf{J}} - i\omega\varepsilon_0\hat{\mathbf{E}} = -i\omega\varepsilon_0\varepsilon_r\hat{\mathbf{E}}$$

Number density, m^{-3}

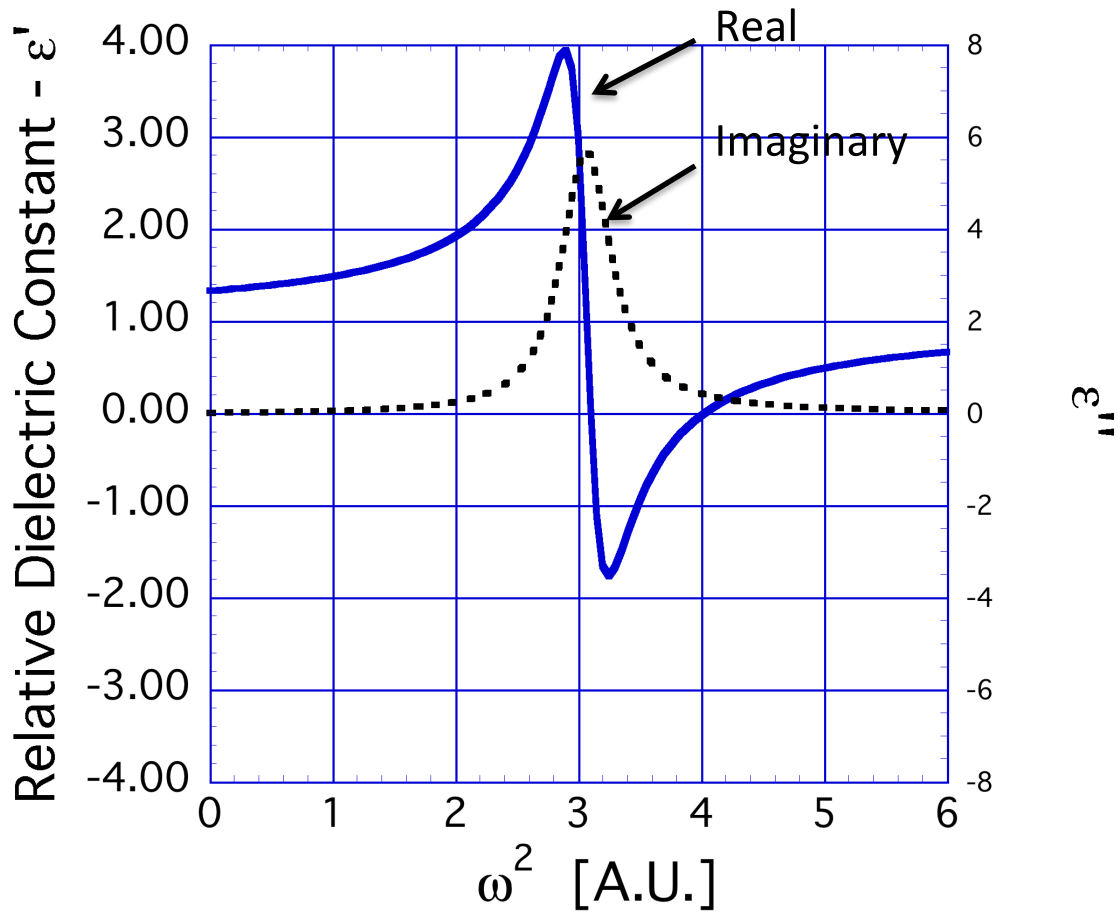
Complex relative dielectric function

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\nu - \omega^2}\right]$$

$$\omega_p^2 = \frac{Nq^2}{\varepsilon_0 m}$$

Model relative dielectric function

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\nu - \omega^2} \right]$$

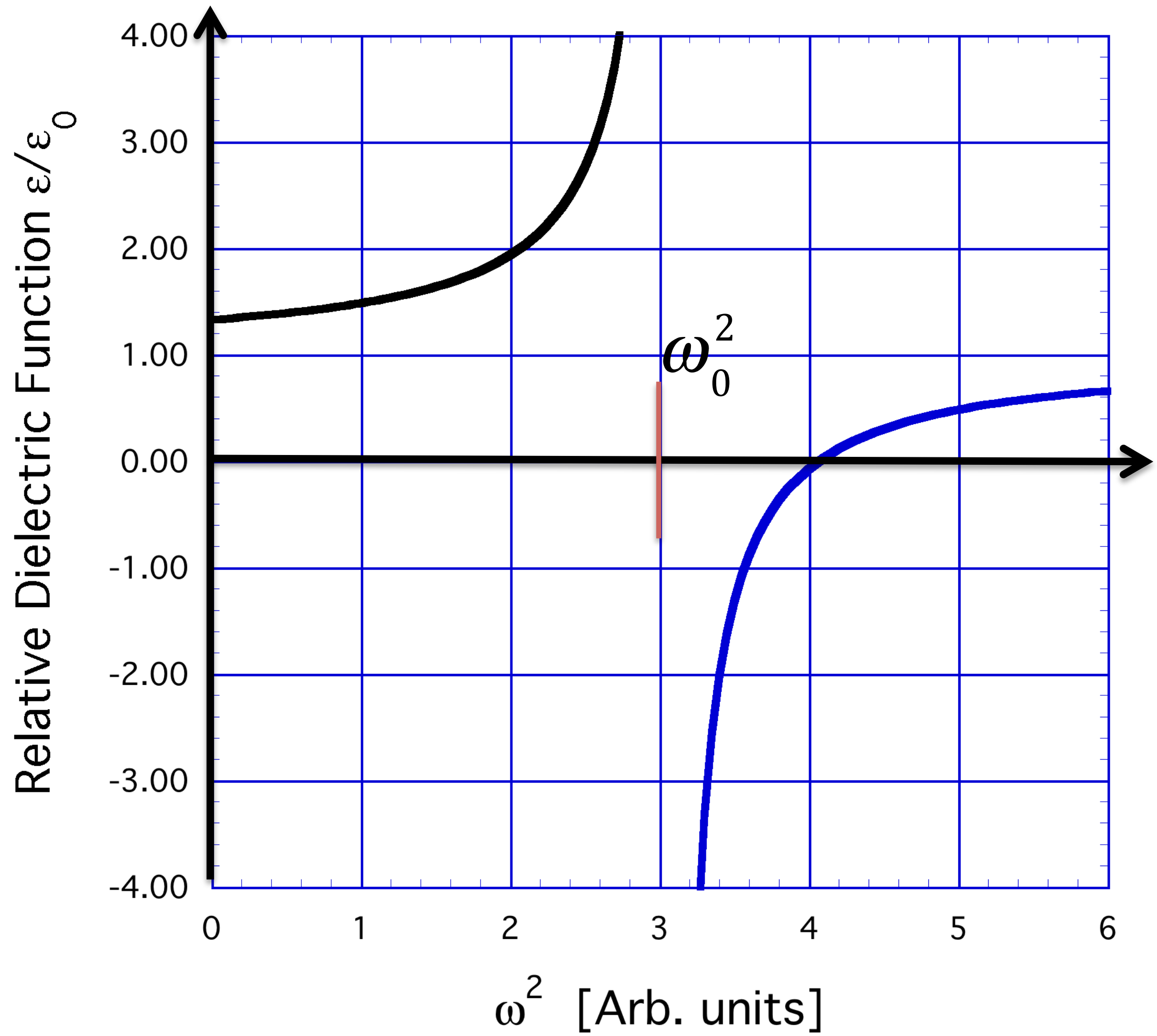


Simple Model for Dielectric Function

Loss free case

$$\varepsilon_r = \left[1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} \right]$$

Restoring force



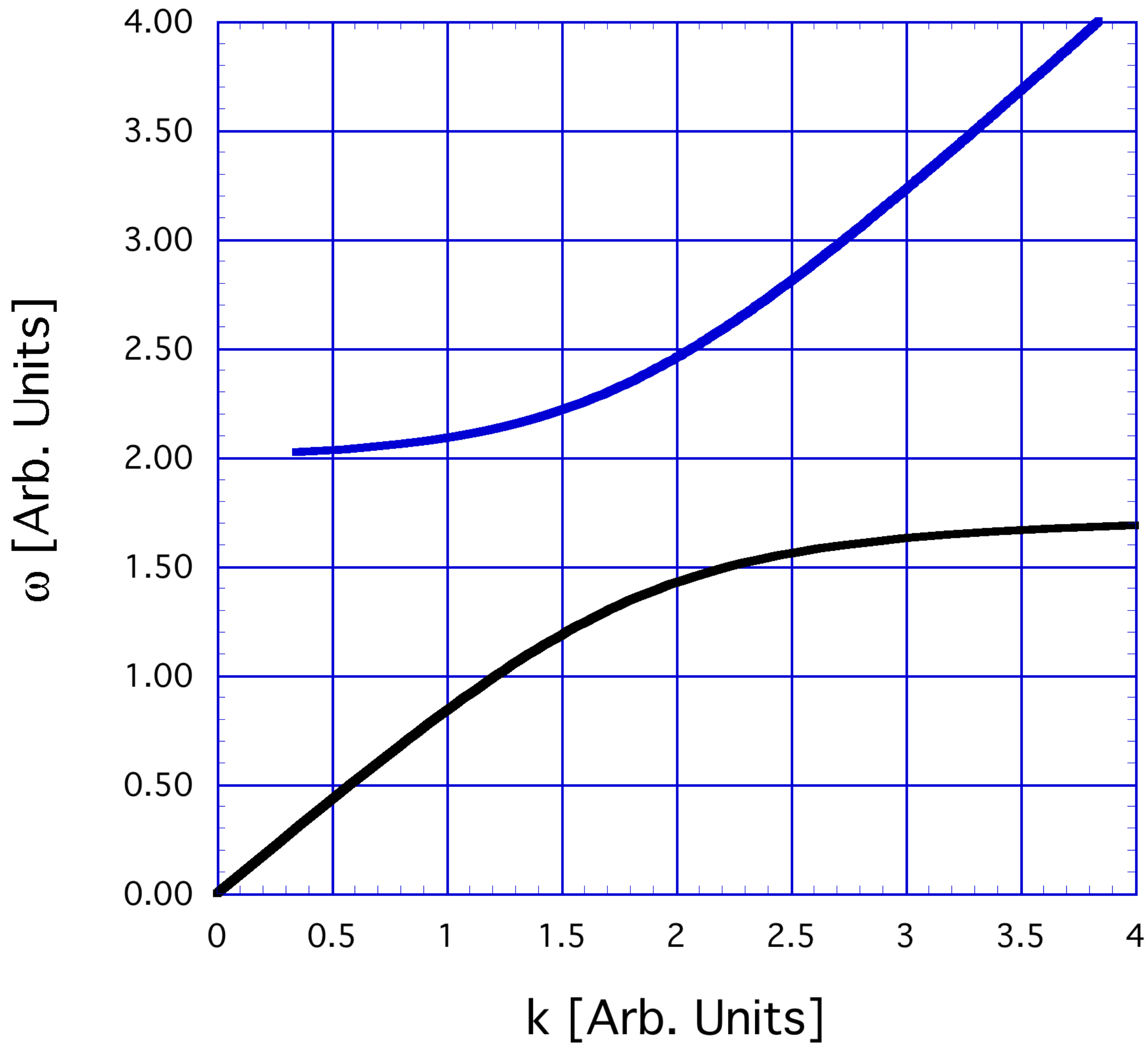
Dispersion Relation

$$\omega^2\left[1+\frac{\omega_p^2}{\omega_0^2-\omega^2}\right]=k^2c^2$$

Two modes

$\omega(k)$ vs k

not a straight line



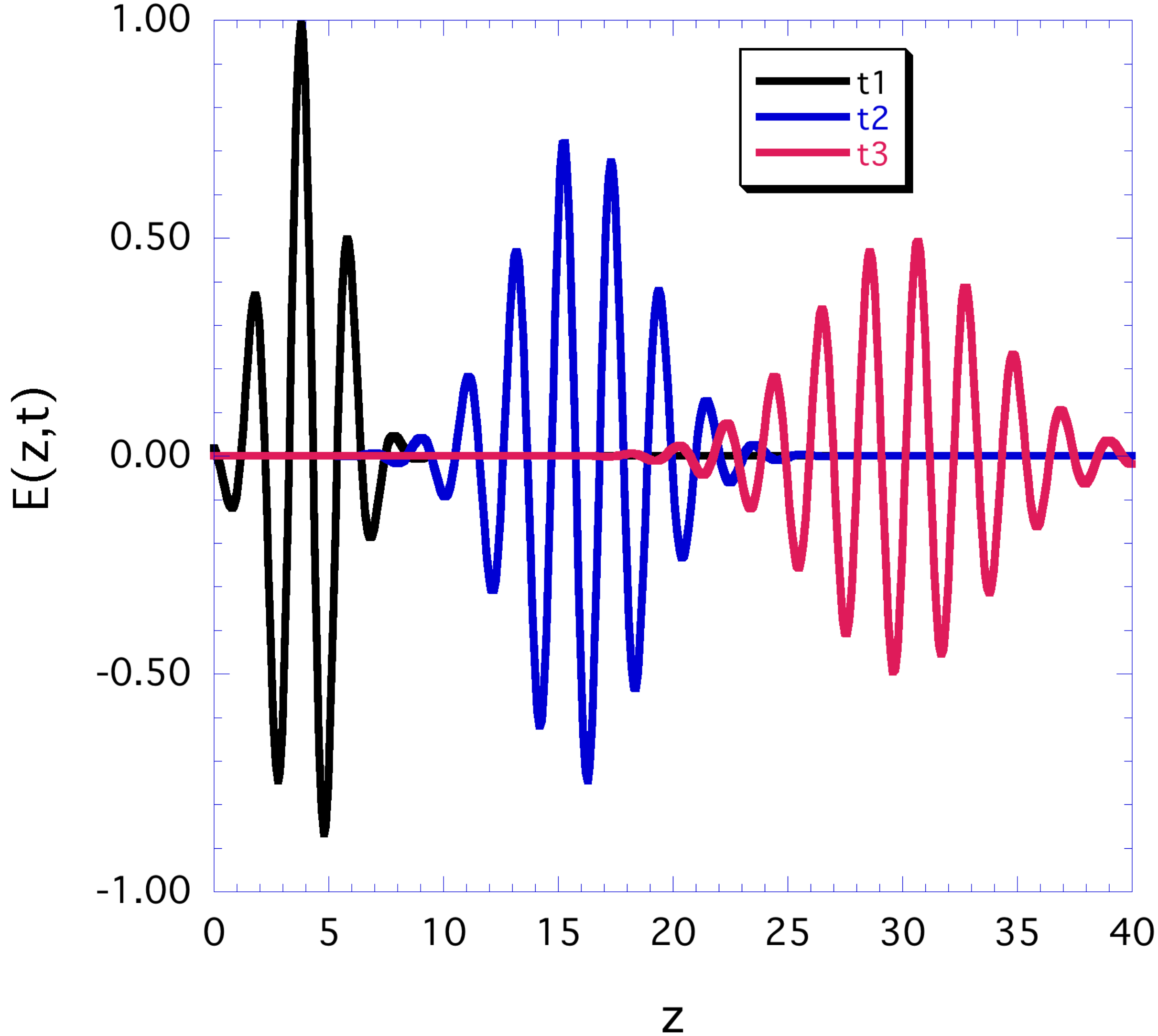
Dispersion and Attenuation

Pulses contain a spectrum of frequencies.

In dispersive media different frequency components propagate with different speeds.

Pulses spread out.

Losses lead to attenuation



How to represent a pulse

At z=0,

$$E(z=0,t)=\mathrm{Re}\left\{\hat{E}_e(z=0,t)\exp\left[-i\omega_c t\right]\right\}$$

$\hat{E}_e(z=0,t)$ - time varying envelope

ω_c - carrier frequency

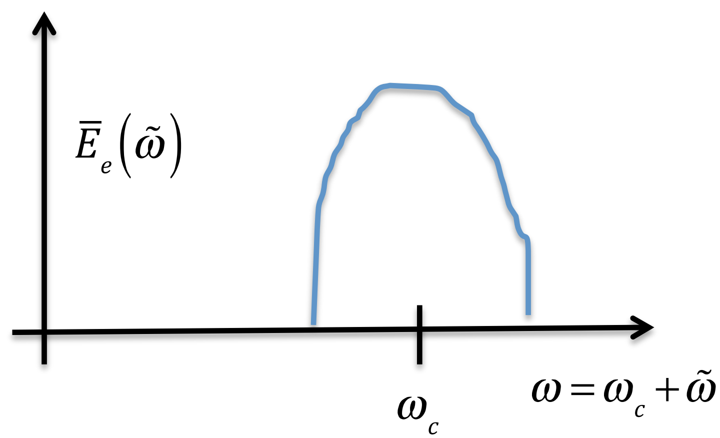
The same waveform can be represented as a Fourier integral

$$\hat{E}_e(z=0,t)\exp\left[-i\omega_c t\right]=\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Fourier transform of time varying envelope

$\omega_c+\tilde{\omega}$ - frequency

Spectrum is narrow and peaked at the carrier frequency



After Propagation

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right]=\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[ik(\omega_c+\tilde{\omega})z-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Peaked at $\tilde{\omega}=0$

Taylor expand $k(\omega_c+\tilde{\omega})=k(\omega_c)+\left.\frac{dk}{d\omega}\right|_{\omega_c}\tilde{\omega}+....$

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right]=\exp\left[i\left(k(\omega_c)z-\omega_c t\right)\right]\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[-i\tilde{\omega}\left(t-\left.\frac{dk}{d\omega}\right|_{\omega_c}z\right)\right]$$

Carrier Wave

Envelope

$\hat{E}_e(z=0,t-z/v_g)$,

$v_g=d\omega/dk|_{\omega_c}$ the group velocity

The pulse envelope retains its shape, but travels at a different speed than the crests of the carrier.

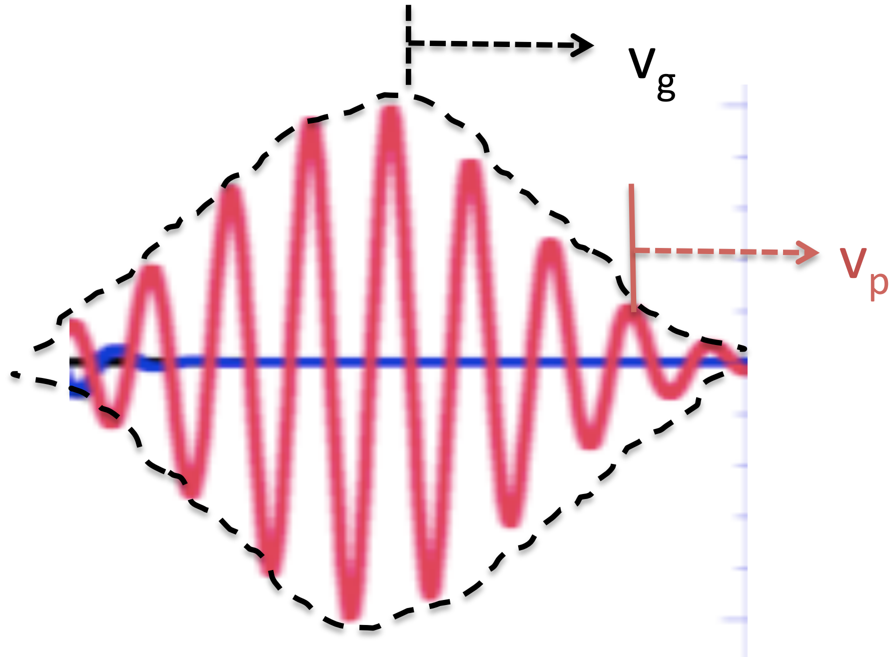
Phase and Group Velocity

The crests travel at the phase velocity

$$v_p = \frac{\omega_c}{k(\omega_c)}$$

The envelope travels at the group velocity

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega_c}$$



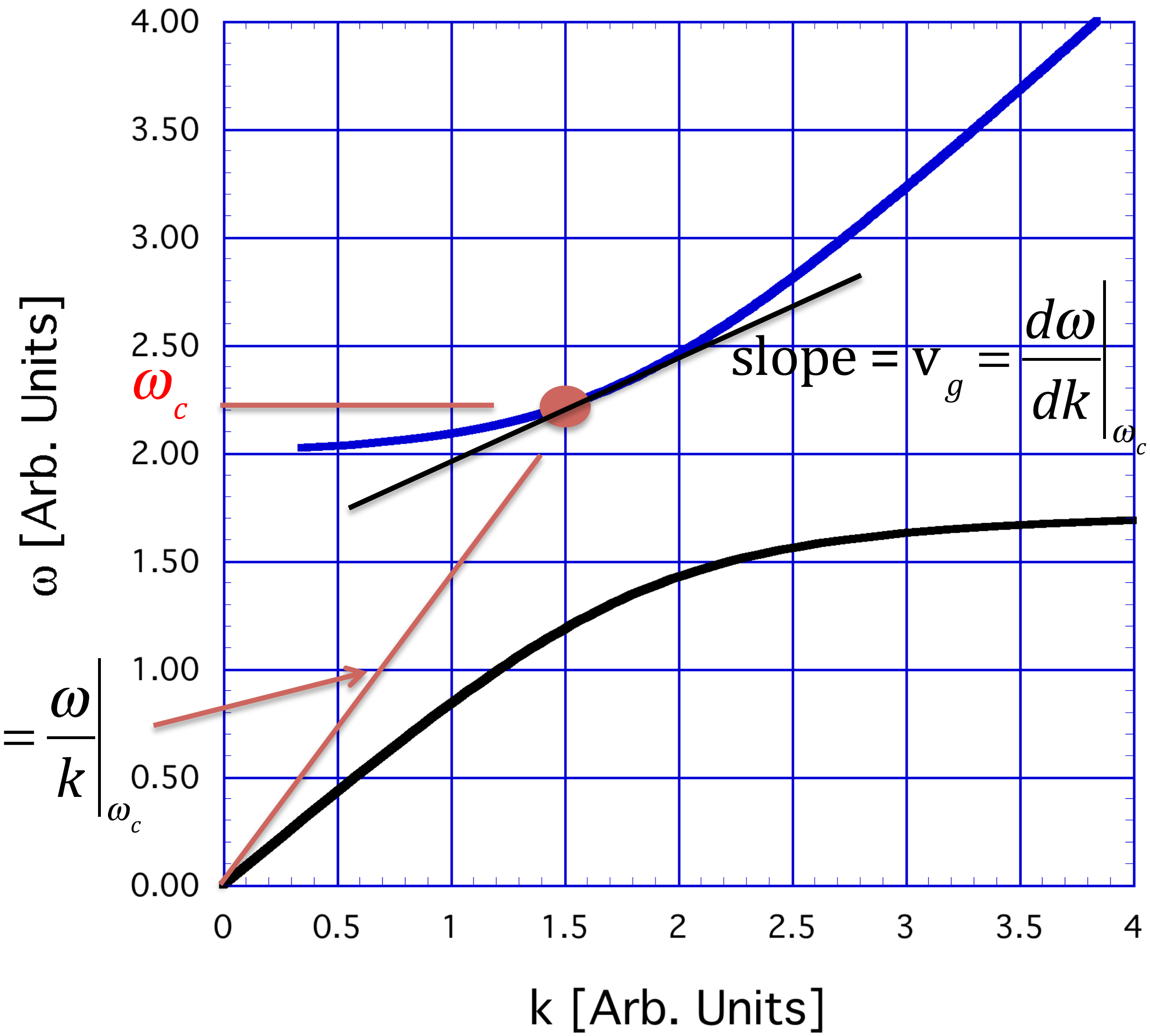
Dispersion Relation

Two modes

$\omega(k)$ vs k

not a straight line

$$\text{slope} = v_p = \left.\frac{\omega}{k}\right|_{\omega_c}$$



Energy and Momentum of Light

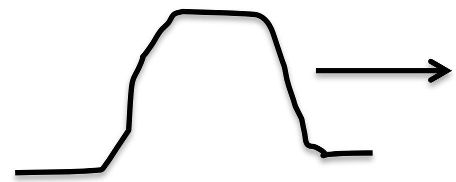
$$\left(\frac{\varepsilon_0 \left|\vec{\mathbf{E}}\right|^2}{2} + \frac{\left|\vec{\mathbf{B}}\right|^2}{2\mu_0} \right)$$

Energy density

$$\vec{\mathbf{S}} = \left(\vec{\mathbf{E}} \times \vec{\mathbf{H}} \right)$$

Power Flux

Units: Joules/m^3 Watts/m^2



Pulse
Moves at c

Power Flux = c Energy Density

EM linear momentum density: $\varepsilon_0 \mathbf{E} \times \mathbf{B} = \mathbf{S} / c^2$

$$\frac{\text{Energy Density}}{\text{Momentum Density}} = \frac{S / c}{S / c^2} = c$$

A pulse of light carries energy and momentum: ratio = c

Energy and Momentum

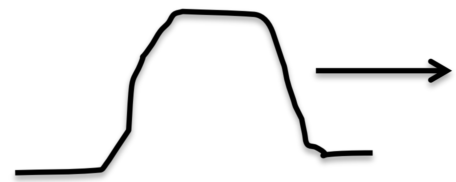
$$\left(\frac{\varepsilon\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\mu\left|\vec{\mathbf{H}}\right|^2}{2}\right)$$

Energy density

$$\vec{\mathbf{S}}=\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}\right)$$

Power Flux

Units: Joules/m^3 Watts/m^2



Pulse
Moves at v_g

Power Flux = v_g Energy Density

Energy Density

$$U=\int d^3x\left(\frac{\partial(\omega\varepsilon(\omega))}{\partial\omega}\frac{\left|\vec{\mathbf{E}}\right|^2}{2}+\frac{\partial(\omega\mu(\omega))}{\partial\omega}\frac{\left|\mathbf{H}\right|^2}{2}\right)$$

Group Velocity Dispersion (GVD)

Different frequencies propagate with different group velocities
Evolution of pulse envelope

$$\hat{E}_e(z,t)\exp\left[-i\omega_c t\right]=\int\frac{d\tilde{\omega}}{2\pi}\bar{E}_e(\tilde{\omega})\exp\left[ik(\omega_c+\tilde{\omega})z-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$\bar{E}_e(\tilde{\omega})$ - Peaked at $\tilde{\omega}=0$

Taylor expand $k(\omega_c+\tilde{\omega})=k(\omega_c)+\left.\frac{dk}{d\omega}\right|_{\omega_c}\tilde{\omega}+....$

Remember, we stopped at first order

Let's go one order higher

$$k(\omega_c+\tilde{\omega})=k(\omega_c)+\left.\frac{dk}{d\omega}\right|_{\omega_c}\tilde{\omega}+\frac{1}{2}\tilde{\omega}^2\left.\frac{d^2k}{d\omega^2}\right|_{\omega_c}$$

$$\boxed{\frac{d^2k}{d\omega^2}=-\frac{1}{v_g^2}\frac{d}{d\omega}v_g}$$

Gaussian Pulse Envelope

Pulse width - τ

Chirp- Ω'

$$\hat{E}_e = E_0 \exp\left[-\frac{t^2}{\tau^2} - i\Omega'\frac{t^2}{2}\right]$$

Instantaneous frequency - $\frac{d}{dt}\Omega'\frac{t^2}{2} = \Omega' t$

If $\Omega' > 0$ Low frequencies come before high frequencies

If $\Omega' < 0$ High frequencies come before low frequencies

Fourrier Transform

$$\overline{E}(\tilde{\omega}) = \int_{-\infty}^{\infty} dt \hat{E}_e(t) \exp\left[i\tilde{\omega}t\right] = \pi^{1/2}\tau_c E_0 \exp\left[-\frac{\tilde{\omega}^2\tau_c^2}{4}\right]$$

$$\tau_c^2 = \frac{\tau^2}{1 + i\Omega'\tau^2/2}$$

Inverse Transform

$$\hat{E}_e(z,t)\exp\left[ik(\omega_c)z-i\omega_c t\right]=$$

$$\int\frac{d\tilde{\omega}}{2\pi}\overline{E}_e(\tilde{\omega})\exp\left[iz\left(\frac{dk}{d\omega}\tilde{\omega}+\frac{d^2k}{d\omega^2}\frac{\tilde{\omega}^2}{2}\right)-i\left(\omega_c+\tilde{\omega}\right)t\right]$$

$$\hat{E}_e(z,t)=\frac{\tau_c}{\tau_s}E_0\exp\left[-\frac{\left(t-z/v_g\right)^2}{\tau_s^2}\right]$$

$$\tau_c^2=\frac{\tau^2}{1+i\Omega'\tau^2/2}$$

$$\tau_s^2=\tau_c^2-2iz\frac{d^2k}{d\omega^2}$$

You will be asked to make plots for HW

Simple Case: no initial chirp

$$\hat{E}_e(z,t) = \frac{\tau_c}{\tau_s} E_0 \exp\left[-\frac{\left(t - z/v_g\right)^2}{\tau_s^2} \right]$$

$$\tau_c^2 = \frac{\tau^2}{1 + i\Omega'\tau^2/2} = \tau^2$$

$$\tau_s^2 = \tau_c^2 - 2iz\frac{d^2k}{d\omega^2} = \tau^2 - i(2zk'')$$

$$\hat{E}_e(z,t) = \frac{\tau_c}{\tau_s} E_0 \exp\left[-\frac{\left(t - z/v_g\right)^2}{\tau^4 + (2zk'')^2}\left(\tau^2 + i(2zk'')\right) \right]$$

Pulse Width: $\tau_w(z) = \sqrt{\tau^2 + (2zk'')^2/\tau^2}$

Chirp: $\Omega'(\mathrm{z}) = 4(zk'')/\left(\tau^4 + (2zk'')^2\right)$

CPA – 2018 Nobel Prize in Physics

Gérard Mourou and Donna Strickland "for their method of generating high-intensity, ultra-short optical pulses"

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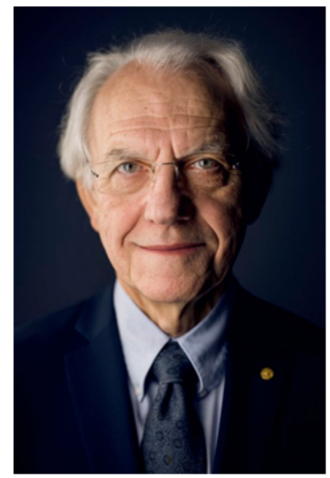
https://www.nobelprize.org/prizes/physics/2018/strickland/facts/



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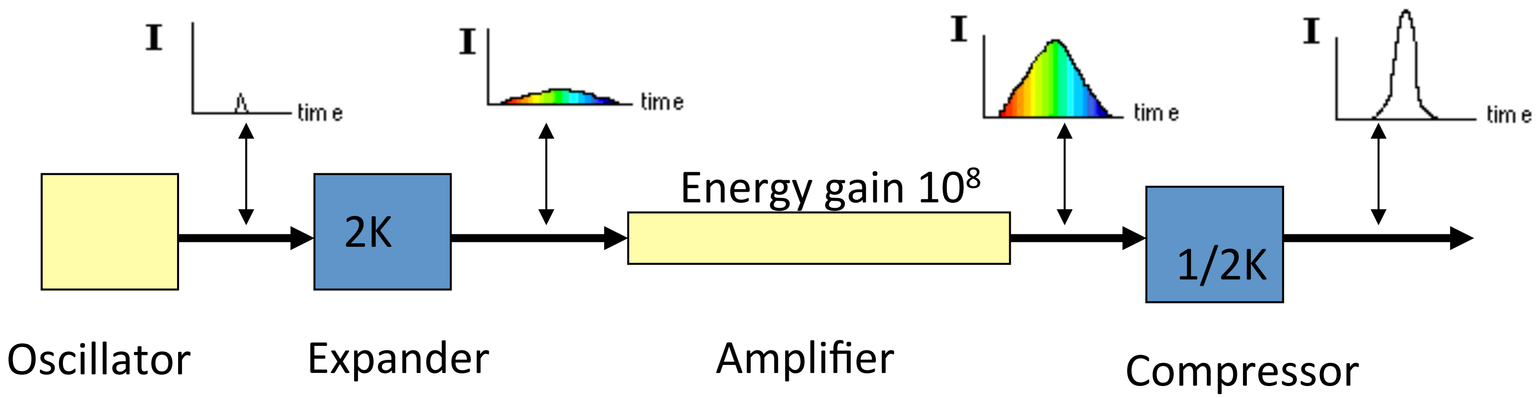
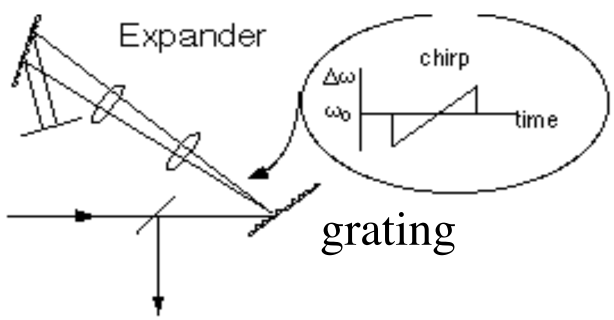
https://www.nobelprize.org/prizes/physics/2018/mourou/facts/

T^3 Lasers - (Table Top Terawatt) Ultra- High Intensity -CPA

CPA: Chirped Pulse Amplification,

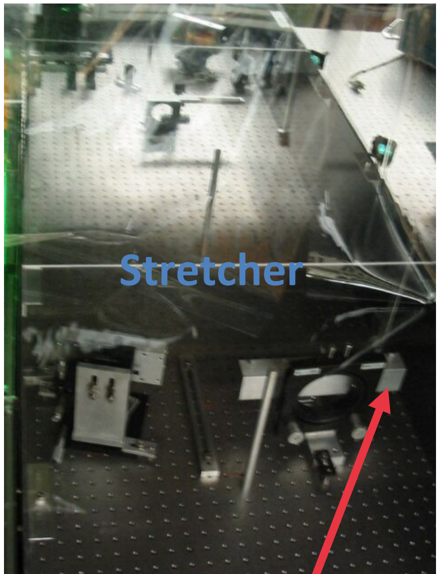




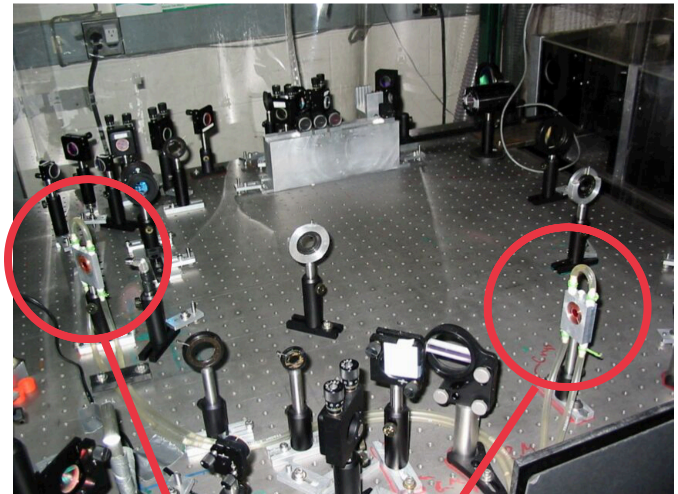
Sample parameters:

Pulse energy	1 Joule	
Pulse Duration	100 fsec	= 1 × 10^{-13} sec
Power	10 TW	= 1 × 10^{+13}W
Wave Length	1 mm	= 1 × 10^{-4} cm
Spot Size	10 mm	= 1 × 10^{-3} cm
Power Density	6.4 × 10^{+18} watts/cm^2	

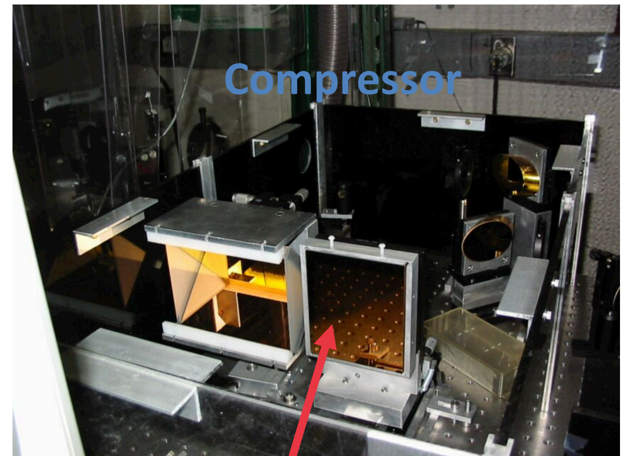
Realization at UMD – H. Milchberg



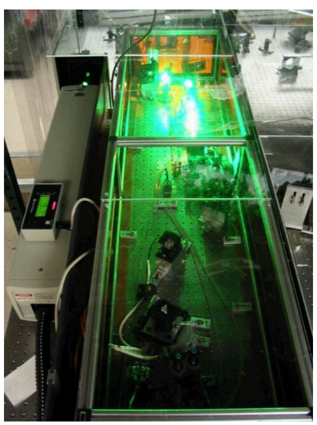
Grating



Power Amplifiers



Grating



Oscillator

Experimental chamber

Applications of Short Pulse Lasers

Review: G. A. Mourou, C. P. J. Barty, and M. D. Perry, Physics Today **51**, 22 (1998).

- **Particle Acceleration:** Use laser as a driver of plasma waves. Ultra High gradient 50 Gev/m (50 Mev/mm)

- **Ultra-short wavelength radiation:** generate high harmonics in nonlinear media
  soft X-rays in cluster gasses (Ditmire, Milchberg)

- **ICF fast Igniter:** Inertial fusion approach, use short pulse to burn a hole for subsequent long pulse to heat and compress target

- **X ray lasers:** create population inversions in partially ionized gasses