ENEE381

Plane Waves in 3D,

Polarization, Reflection,

Topics

Phasor Representation of Fields
Solving Maxwell's Equations
Polarization

Where We Stand

$$\nabla \cdot \vec{\mathbf{D}} = \rho_f$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

Assume the following:

Linear, isotropic, instantaneous, media

Propagation in free space, no free charge or current.

$$\rho_f = 0, \quad \vec{\mathbf{J}}_f = 0$$

$$\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}$$

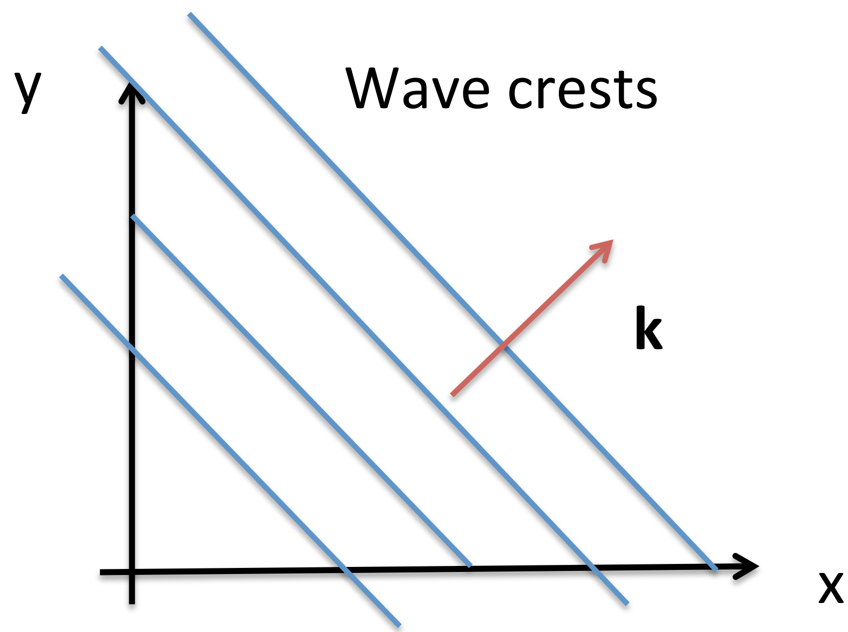
$$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$$

Introduce Phasor Notation

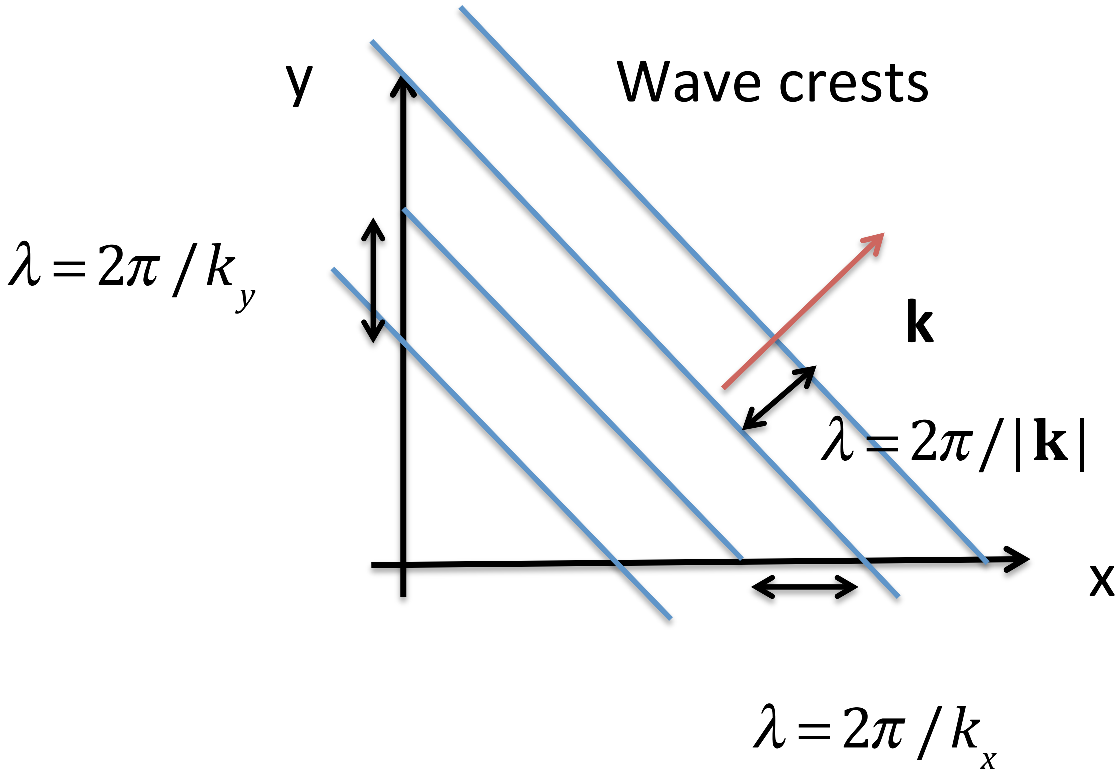
$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right\} \qquad \mathbf{H}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right\}$$

Note: two new elements

1. Phasor amplitudes are vectors. Will be independent of space and time.
2. Space and time dependence contained in complex exponential
3. Wave number k is now wave vector **k**.



$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]\right\}$$



$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(k_x x + k_y y - \omega t\right)\right]\right\}$$

When does this work?

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\} \quad \mathbf{H}(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

Works when ε and μ are independent of space and time.

$$\nabla\times\vec{\mathbf{E}} = -\mu\frac{\partial\vec{\mathbf{H}}}{\partial t}$$

$$\nabla\times\mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$= \mathrm{Re}\left\{\nabla\times\left(\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

$$= \mathrm{Re}\left\{i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$-\mu\frac{\partial}{\partial t}\mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}$$

$$= \mathrm{Re}\left\{-\mu\frac{\partial}{\partial t}\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

$$= \mathrm{Re}\left\{-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

Relating phasor amplitudes

$$\mathrm{Re}\left\{i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right\}=\mathrm{Re}\left\{-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)\right\}$$

If the real parts of two complex variables are equal, and there is no restriction on the imaginary parts, then I can make the complex variables equal.

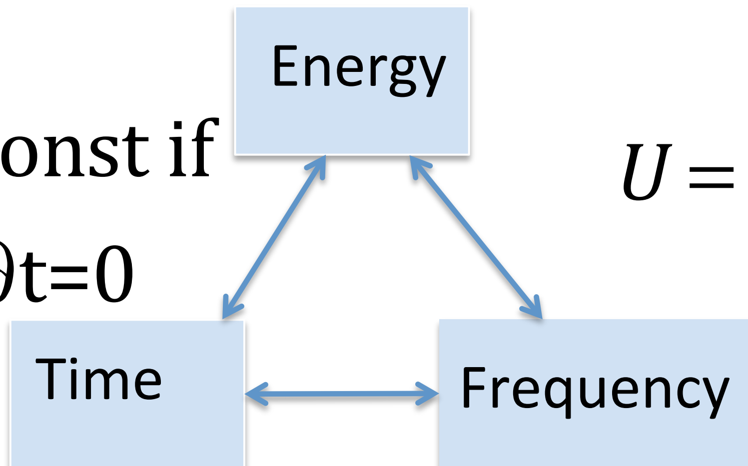
$$i\mathbf{k}\times\hat{\mathbf{E}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]=-\mu(-i\omega)\left(\hat{\mathbf{H}}\exp\left[i\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\right]\right)$$

Now cancel the exponential factor from both sides. Result must still hold for all x and t.

$$\boxed{i\mathbf{k}\times\hat{\mathbf{E}}=-\mu(-i\omega)\hat{\mathbf{H}}}$$

Linked Quantities

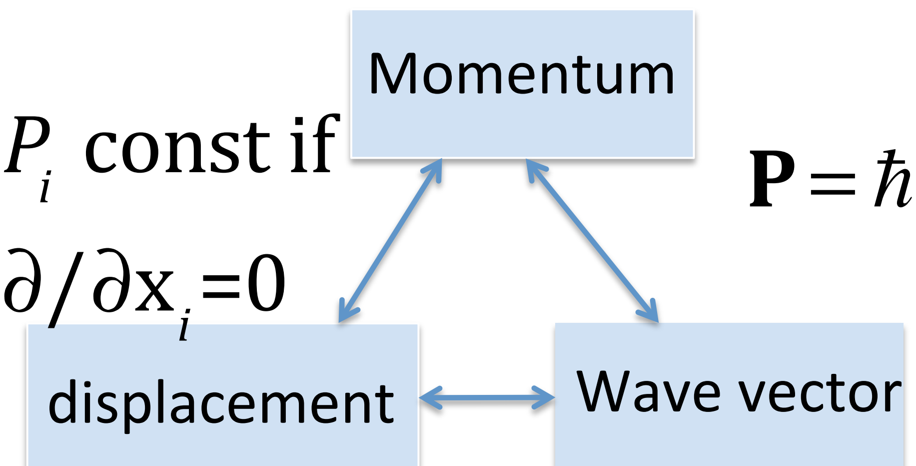
U const if $\partial/\partial t = 0$



$$U = \hbar\omega$$

$$1 = \Delta t \Delta\omega$$

P_i const if $\partial/\partial x_i = 0$



$$\mathbf{P} = \hbar\mathbf{k}$$

Sinusoidal waves

$$\exp\left(i\mathbf{k}\cdot\mathbf{x} - i\omega t\right)$$

$$1 = \Delta x \Delta k$$

Maxwell Eqs. Phasor Amplitudes

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \cdot \vec{\mathbf{H}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad \nabla \times \vec{\mathbf{H}} = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

To get equations for phasor amplitudes $\quad \vec{\mathbf{E}}, \vec{\mathbf{H}} \Rightarrow \hat{\mathbf{E}}, \hat{\mathbf{H}} \quad \frac{\partial}{\partial t}, \nabla \Rightarrow -i\omega, i\mathbf{k}$

$$i\mathbf{k} \cdot \hat{\mathbf{E}} = 0 \qquad i\mathbf{k} \cdot \hat{\mathbf{H}} = 0 \qquad i\mathbf{k} \times \hat{\mathbf{E}} = i\omega\mu\hat{\mathbf{H}} \qquad i\mathbf{k} \times \hat{\mathbf{H}} = -i\omega\varepsilon\hat{\mathbf{E}}$$

Combine

$$i\mathbf{k}\cdot\hat{\mathbf{E}}=0 \qquad i\mathbf{k}\cdot\hat{\mathbf{H}}=0 \qquad i\mathbf{k}\times\hat{\mathbf{E}}=i\omega\mu\hat{\mathbf{H}} \qquad i\mathbf{k}\times\hat{\mathbf{H}}=-i\omega\varepsilon\hat{\mathbf{E}}$$

combine

$$i\mathbf{k}\times\left(i\mathbf{k}\times\hat{\mathbf{E}}\right)=i\omega\mu\left(i\mathbf{k}\times\hat{\mathbf{H}}\right)=i\omega\mu\left(-i\omega\varepsilon\hat{\mathbf{E}}\right)=\omega^2\varepsilon\mu\hat{\mathbf{E}}$$

Use “BAC CAB”

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

$$\left(\mathbf{k}\cdot\mathbf{k}\right)\hat{\mathbf{E}}-\mathbf{k}\left(\mathbf{k}\cdot\hat{\mathbf{E}}\right)=k^2\hat{\mathbf{E}}=\omega^2\varepsilon\mu\hat{\mathbf{E}}$$

Plane waves in 3D

$$\left(k^2-\omega^2\varepsilon\mu\right)\hat{\mathbf{E}}=0,\quad \mathbf{k}\cdot\hat{\mathbf{E}}=0$$

E can be in any direction perpendicular to k,

Dispersion relation

$$\omega^2=\frac{k^2}{\varepsilon\mu}=k^2v^2 \qquad k^2=\left|\mathbf{k}\right|^2=k_x^2+k_y^2+k_z^2$$

Remember 1D

$$\omega=\pm kv$$

$$\lambda=v/f$$

$$\lambda=2\pi/k$$

$$\omega=2\pi f$$

Faraday's Law $\quad \mathbf{k}\times\hat{\mathbf{E}}=\omega\mu\hat{\mathbf{H}}$

$$\frac{\mathbf{k}}{\left|\mathbf{k}\right|}\times\hat{\mathbf{E}}=\frac{\omega\mu}{\left|\mathbf{k}\right|}\hat{\mathbf{H}}=\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{H}}$$

Superposition of Solutions

Maxwell's Eqs. are linear in E & H. Thus, separate solutions can be added together. Consider 2 solutions:

$$\vec{\mathbf{E}}_1(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}_1 \exp\left[i\left(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t\right)\right]\right\} \quad k_1^2 = \omega_1^2 \varepsilon\mu$$

$$\vec{\mathbf{E}}_2(\mathbf{x},t) = \mathrm{Re}\left\{\hat{\mathbf{E}}_2 \exp\left[i\left(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t\right)\right]\right\} \quad k_2^2 = \omega_2^2 \varepsilon\mu$$

Then $\mathbf{E}_1 + \mathbf{E}_2$ is also a solution of Maxwell's Equations

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\sum_j \hat{\mathbf{E}}_j \exp\left[i\left(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t\right)\right]\right\} \quad k_j^2 = \varepsilon\mu\omega^2(\mathbf{k}_j)$$

Turn the sum into an integral

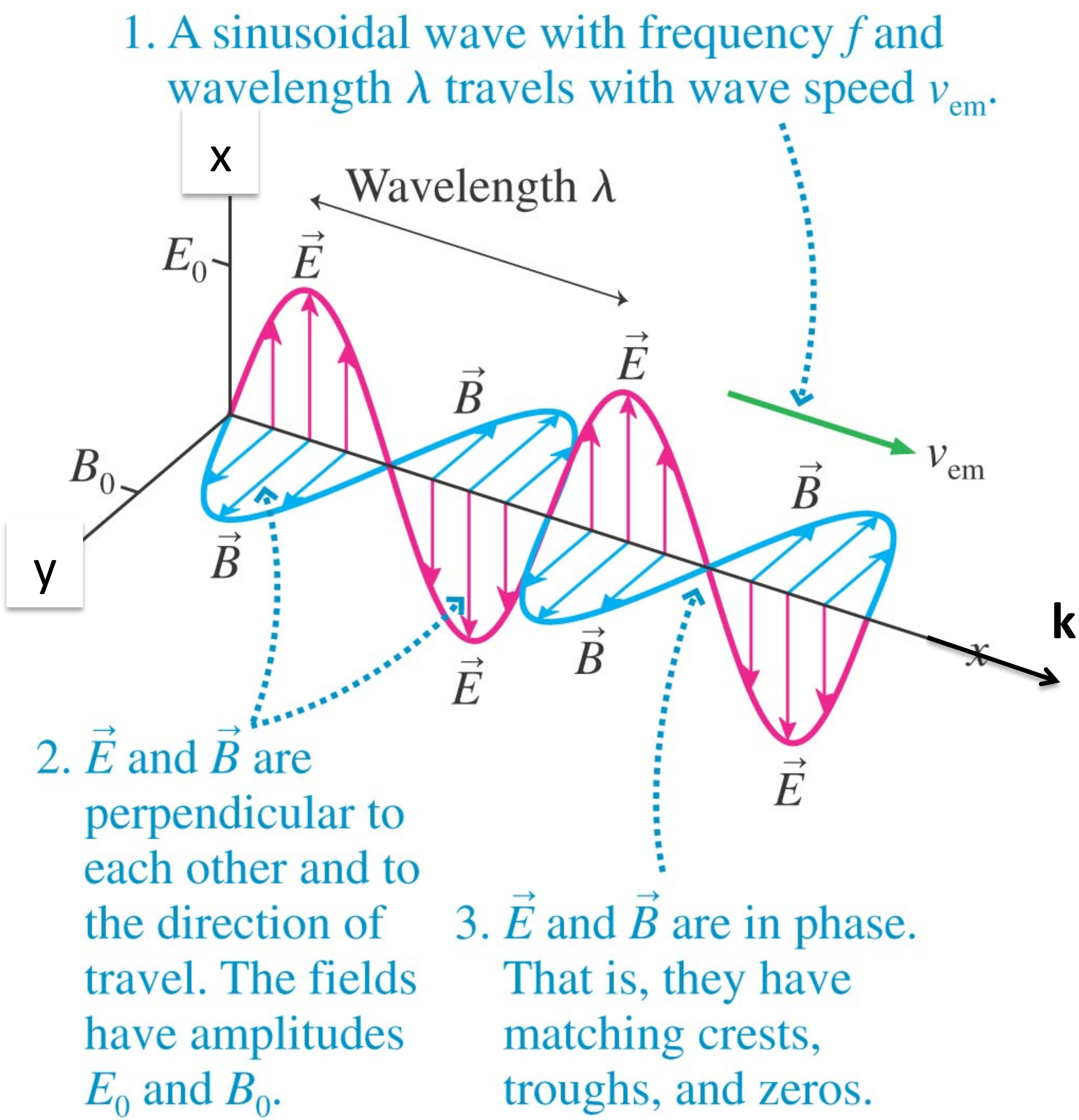
$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\sum_j \hat{\mathbf{E}}_j \exp\left[i\left(\mathbf{k}_j \cdot \mathbf{x} - \omega(\mathbf{k}_j)t\right)\right]\right\} \quad k_j^2 = \varepsilon\mu\omega^2(\mathbf{k}_j)$$

Fourier Integral - used to solve diffraction and dispersion

$$\vec{\mathbf{E}}(\mathbf{x},t) = \mathrm{Re}\left\{\int d^3k\, \bar{\hat{\mathbf{E}}}(\mathbf{k}) \exp\left[i\left(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t\right)\right]\right\} \quad k^2 = \varepsilon\mu\omega^2(\mathbf{k})$$

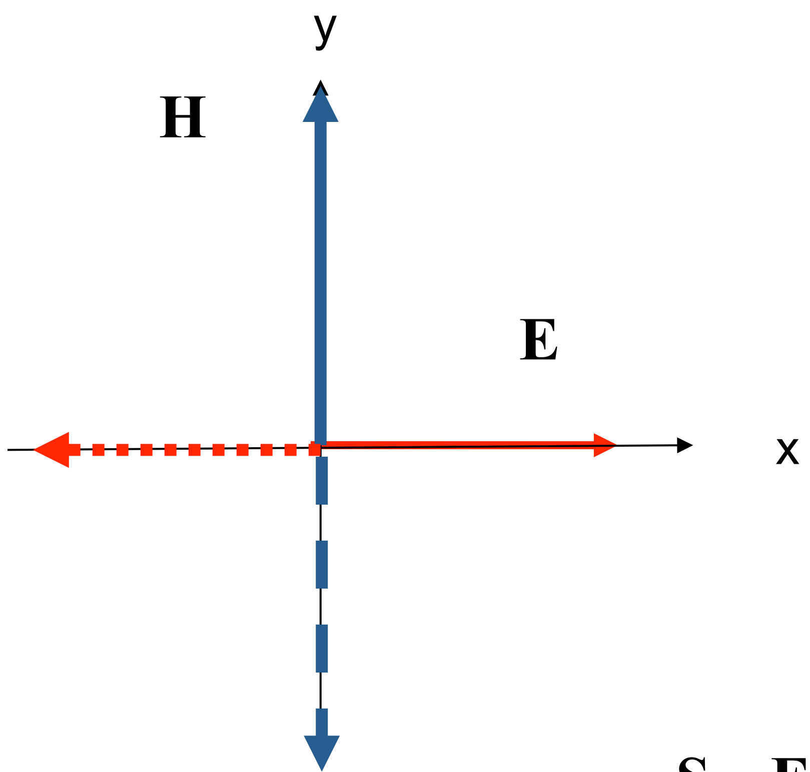
Linearly Polarized Waves

Electric field vector lies in one plane

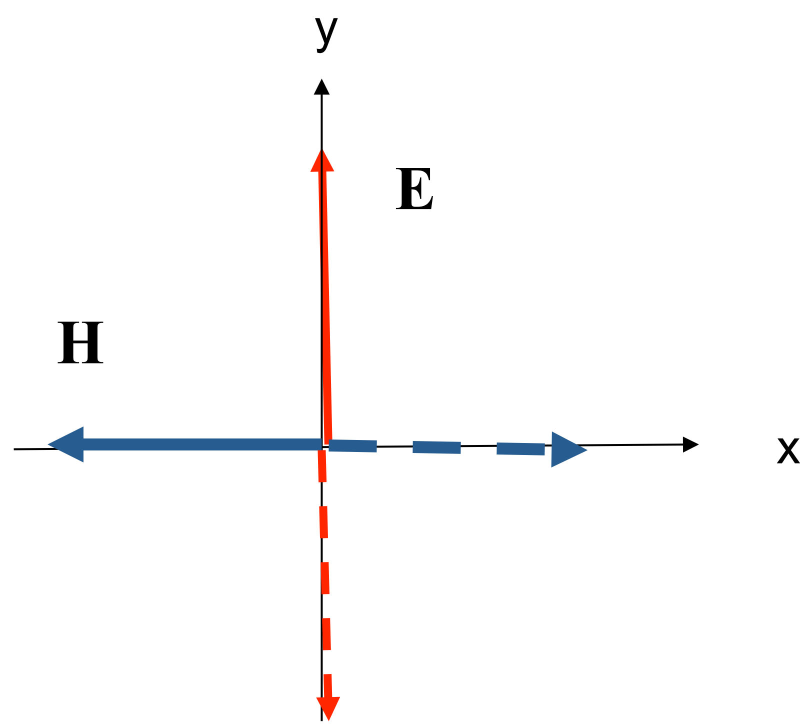


Linear Polarizations

Linearly Polarized in X direction



Linearly Polarized in y direction

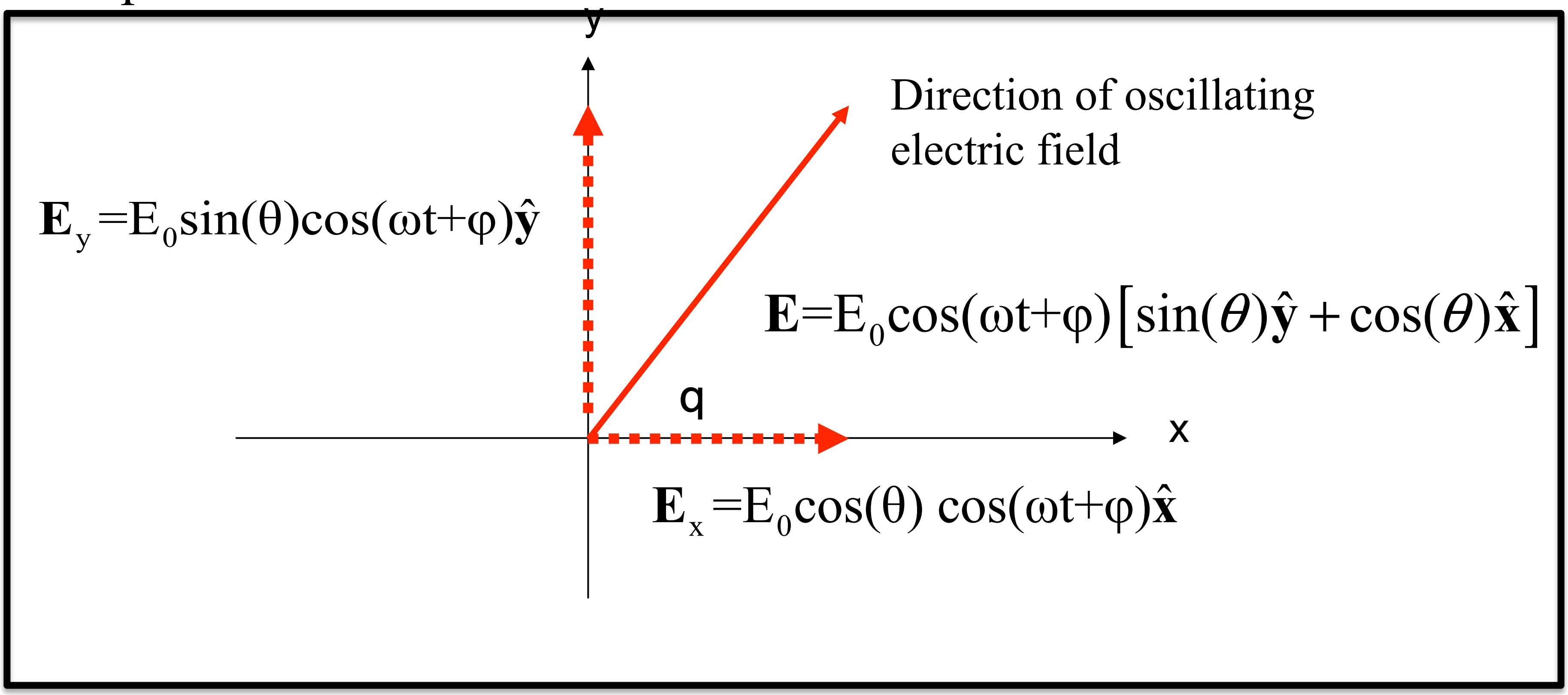


$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = S\hat{\mathbf{z}}$$

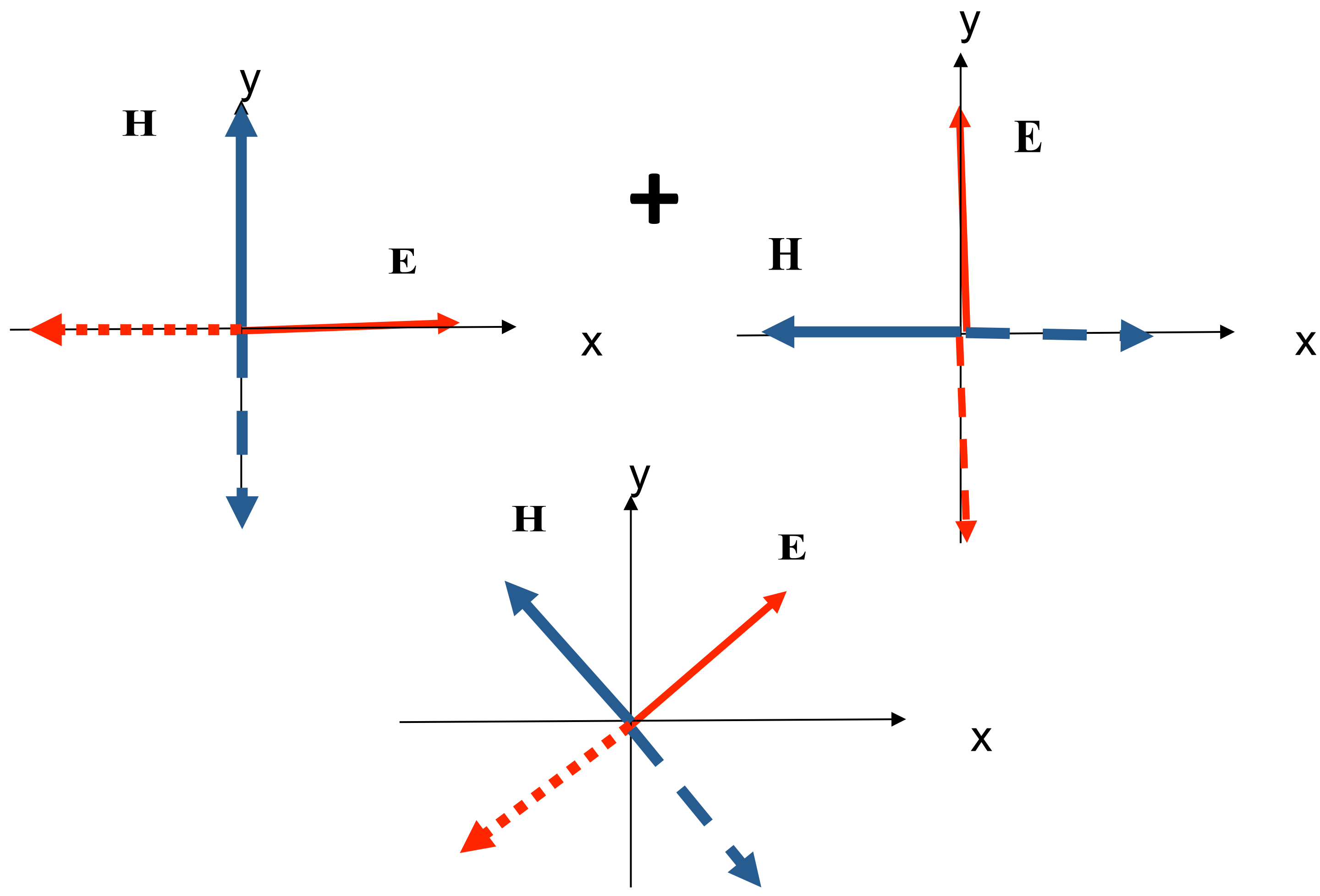
Linear Polarization

For a wave propagating in z direction, a linearly polarized wave has x and y components oscillating in **phase**

z - plane



In phase superposition of two linearly polarized waves

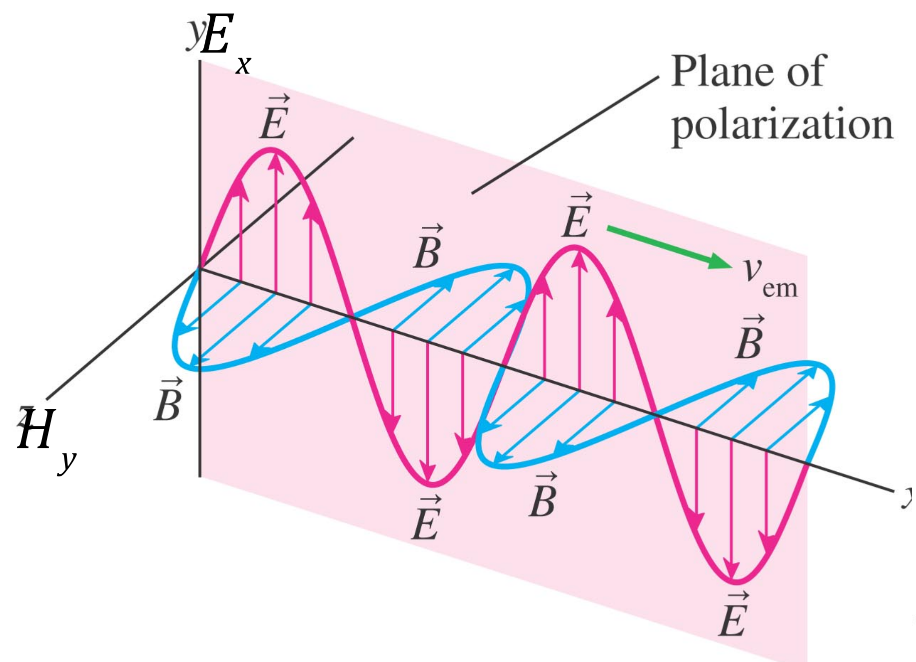


Polarizations

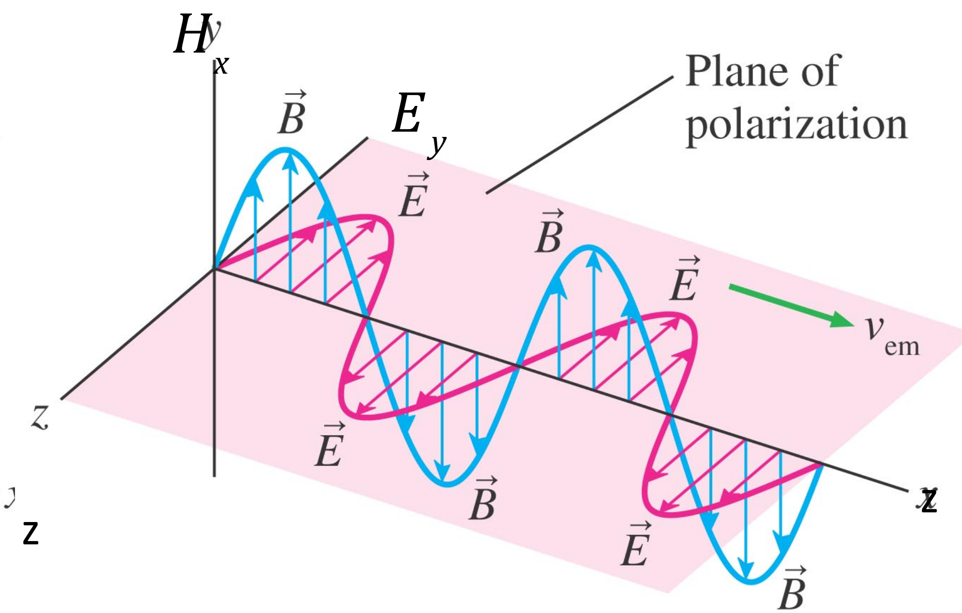
We picked this combination of fields: E_x - H_z

Could have picked this combination of fields: E_y - B_x

(a) Vertical polarization



(b) Horizontal polarization

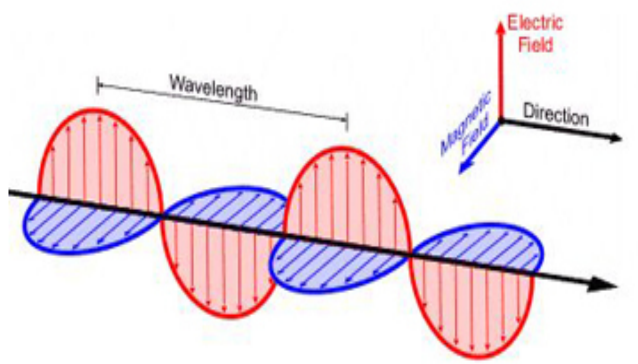


These are called plane polarized. Fields lie in plane

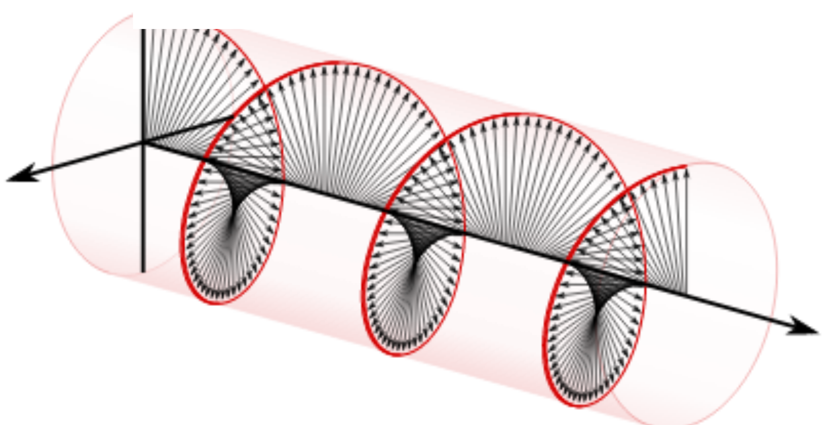
Polarization

Polariation is determined by the direction of **E** field

The wave is **linearly** polarized if the **electric field** oscillates in **one plane**



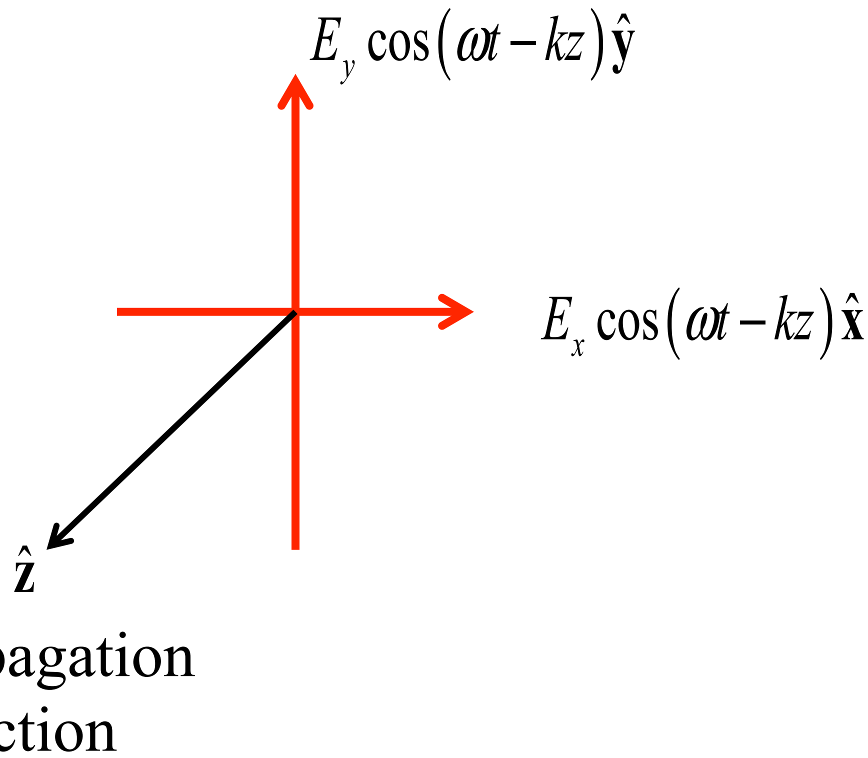
Linear polarization



Circular polarization

Polarization of Fields

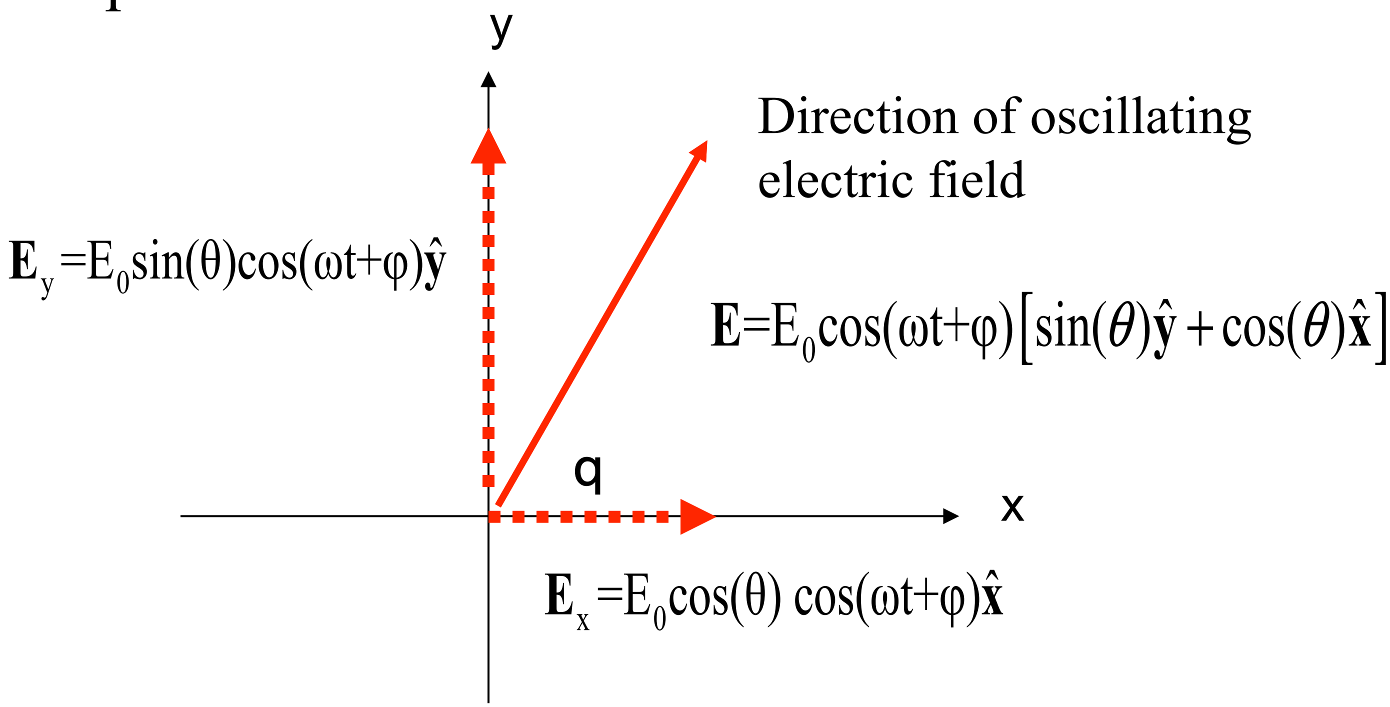
Consider two waves having the same frequency, same directions of propagation, but different orthogonal linear polarizations



Linear Polarization

For a wave propagating in z direction, a linearly polarized wave has x and y components oscillating in **phase**

z - plane



Polarization of Electric Fields

Three parameters determine the state of polarization

Consider a wave propagating in z direction

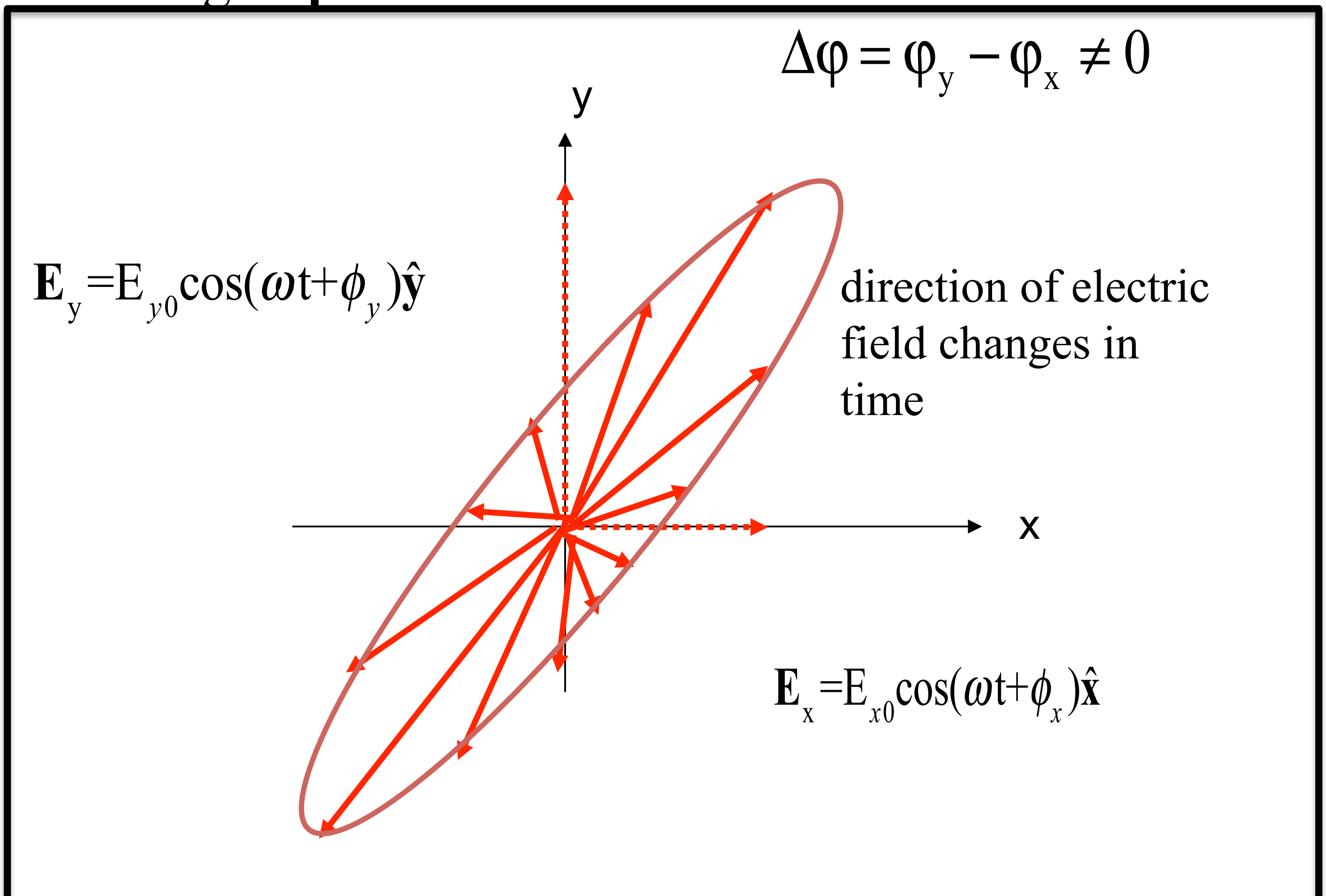
1. Field strength along x direction

2. Field strength along y direction

3. Relative phase shift between them

Elliptical Polarization

Elliptically polarized light has x and y field components **not** oscillating **in phase**



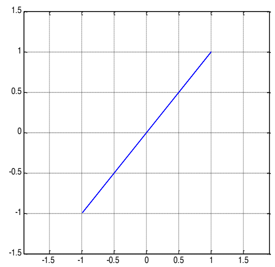
Different States of Polarization

$$\Delta\varphi = \varphi_y - \varphi_x$$

$$|E_x| = |E_y|$$

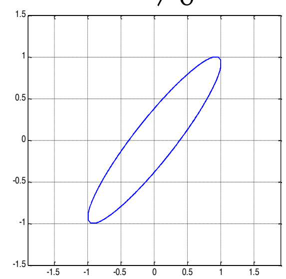
$$E_{x0} = E_{y0}$$

$\Delta\varphi = 0$



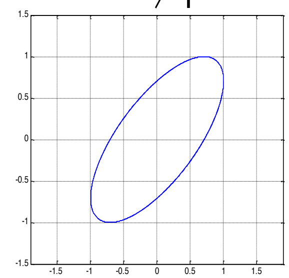
linear

$\Delta\varphi = \pi/8$



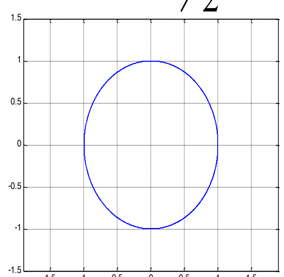
elliptical

$\Delta\varphi = \pi/4$



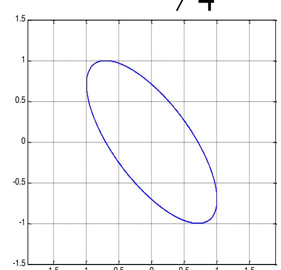
elliptical

$\Delta\varphi = \pi/2$



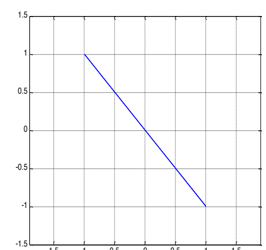
circular

$\Delta\varphi = 3\pi/4$



elliptical

$\Delta\varphi = \pi$



linear

Problem

An electromagnetic wave travelling in vacuum in the +z direction has the real electric field at z=0,

$$\mathbf{E}(z=0,t)=E_{0x}\cos(\omega t+\pi/4)\hat{\mathbf{x}}+E_{0y}\cos(\omega t-\pi/4)\hat{\mathbf{y}}$$

Represent this wave in phasor form:

$$\mathbf{E}(z,t)=\mathrm{Re}\left\{\hat{\mathbf{E}}\exp\left[i\left(kz-\omega t\right)\right]\right\}$$

$$\mathbf{H}(z,t)=\mathrm{Re}\left\{\hat{\mathbf{H}}\exp\left[i\left(kz-\omega t\right)\right]\right\}$$

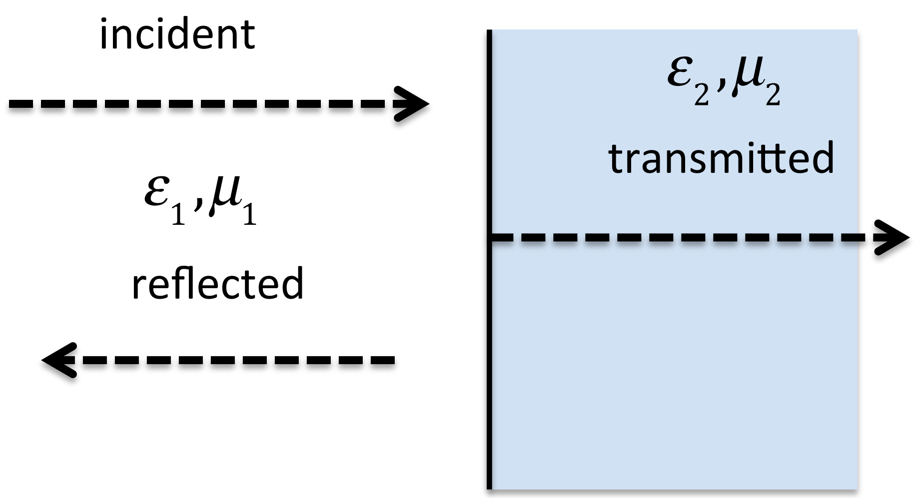
What is the polarization of the wave? Plane, circular, elliptical?

$$\mathbf{E}(z=0,t)=E_{0x}\cos(\omega t+\pi/4)\hat{\mathbf{x}}+E_{0y}\cos(\omega t-\pi/4)\hat{\mathbf{y}}$$

Reflection at an interface

Impedance

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = Z_1$$



Impedance

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = Z_2$$

Generic Rules

$$\frac{E_{reflected}}{E_{incident}} \equiv \rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\frac{E_{transmitted}}{E_{incident}} \equiv \tau = 1 + \rho = \frac{2Z_2}{Z_2 + Z_1}$$

ρ Voltage Reflection Coefficient

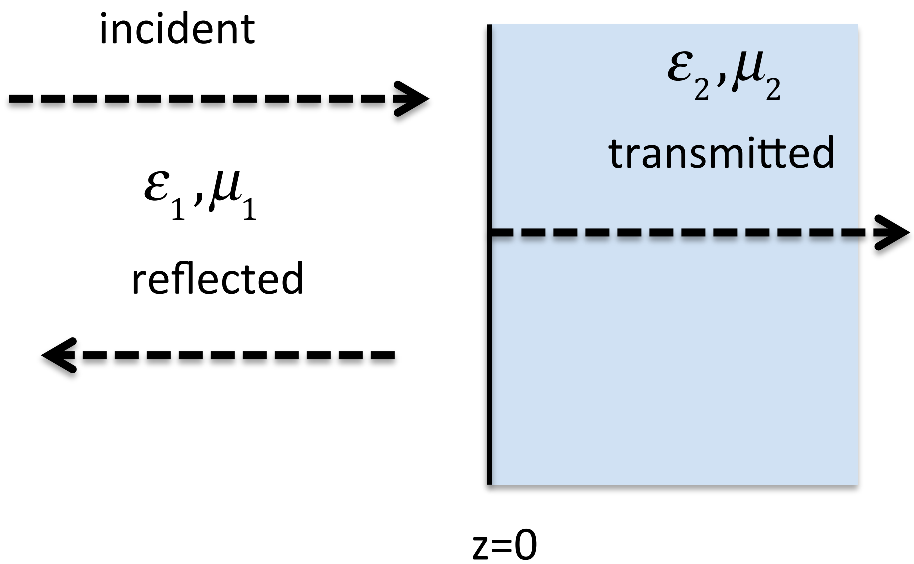
τ Voltage Transmission Coefficient

Note, if $Z_2 = Z_1$ $\rho = 0, \tau = 1$

Normal Incidence Linear Polarization

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$k_1 = \omega\sqrt{\varepsilon_1\mu_1}$$



$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$k_2 = \omega\sqrt{\varepsilon_2\mu_2}$$

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc}e^{ik_1 z} + \hat{E}_{ref}e^{-ik_1 z}\right)e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_1}\left(\hat{E}_{inc}e^{ik_1 z} - \hat{E}_{ref}e^{-ik_1 z}\right)e^{-i\omega t}\right\}$$

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{trans}e^{ik_2 z}\right)e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_2}\left(\hat{E}_{trans}e^{ik_2 z}\right)e^{-i\omega t}\right\}$$

At z=0   tangential E and tangential H are continuous

At z=0 tangential E and tangential H are continuous

Region 1

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{inc} e^{ik_1 z} + \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_1}\left(\hat{E}_{inc} e^{ik_1 z} - \hat{E}_{ref} e^{-ik_1 z}\right) e^{-i\omega t}\right\}$$

Region 2

$$E_x = \mathrm{Re}\left\{\left(\hat{E}_{trans} e^{ik_2 z}\right) e^{-i\omega t}\right\}$$

$$H_y = \mathrm{Re}\left\{\frac{1}{\eta_2}\left(\hat{E}_{trans} e^{ik_2 z}\right) e^{-i\omega t}\right\}$$

At z=0

Tangential E $\quad \hat{E}_{inc} + \hat{E}_{ref} = \hat{E}_{trans}$

Tangential H $\quad \dfrac{\hat{E}_{inc} - \hat{E}_{ref}}{\eta_1} = \dfrac{\hat{E}_{trans}}{\eta_2}$

solve $\longrightarrow$

$$\frac{\hat{E}_{ref}}{\hat{E}_{inc}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \equiv \rho$$

$$\frac{\hat{E}_{trans}}{\hat{E}_{inc}} = \frac{2\eta_2}{\eta_2 + \eta_1} \equiv \tau = 1 + \rho$$

Reflected and Transmitted Power

$$S_z = E_x H_y = \frac{1}{2}\mathrm{Re}\left\{\hat{E}_x^* \hat{H}_y\right\}$$

Region 1:

$$S_z = E_x H_y = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_1}\left(\hat{E}_{inc}^* + \hat{E}_{ref}^*\right)\left(\hat{E}_{inc} - \hat{E}_{ref}\right)\right\} = \frac{1}{2\eta_1}\left(\left|\hat{E}_{inc}\right|^2 - \left|\hat{E}_{ref}\right|^2\right)$$

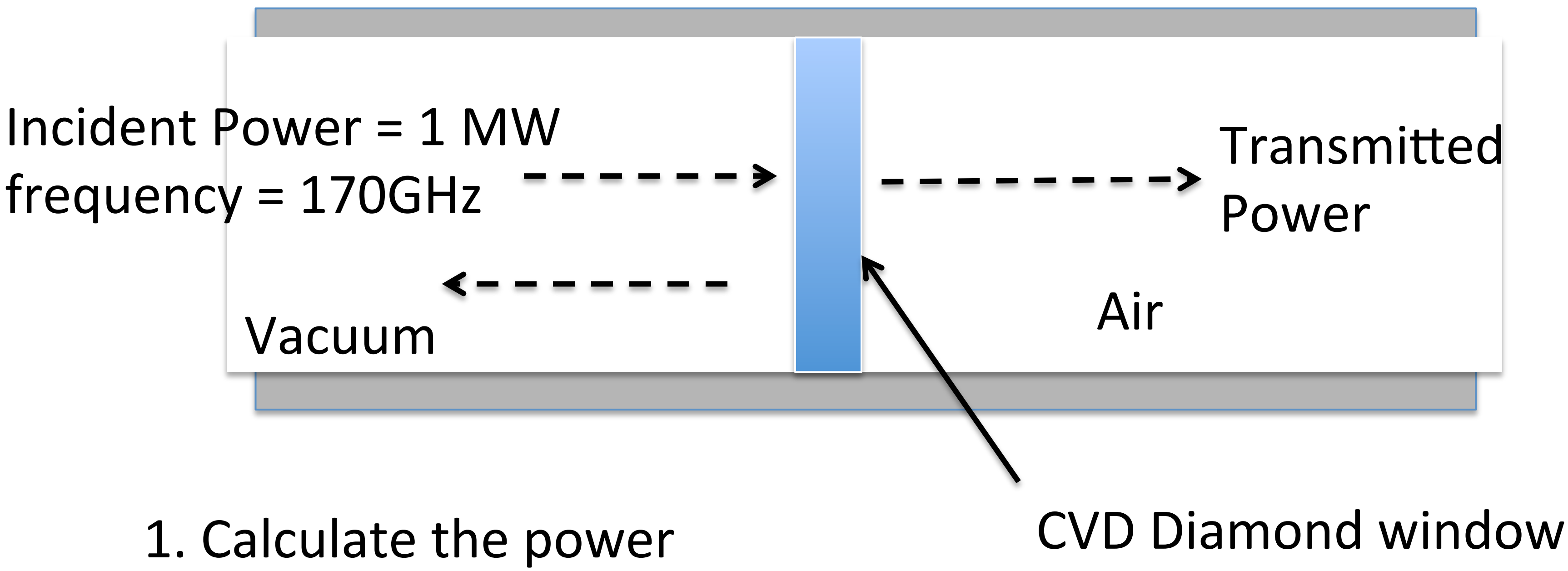
$$P_{inc}\left(1 - |\rho|^2\right)$$

Region2:

$$S_z = E_x H_y = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\eta_2}\left|\hat{E}_{trans}\right|^2\right\} = \frac{1}{2\eta_2}\left|\tau \hat{E}_{inc}\right|^2$$

$$= P_{inc}\frac{\eta_1}{\eta_2}\left|\frac{2\eta_2}{\eta_2 + \eta_1}\right|^2 = P_{inc}\left[1 - \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2\right] = P_{inc}\left(1 - |\rho|^2\right)$$

Diamond Window



$$\varepsilon = 5.7\varepsilon_0$$

1. Calculate the power reflection coefficient at the first surface.
2. Is that acceptable?
3. What are we missing?