ENEE381

Lecture 2

Time Dependent Fields

Faraday’s Law

Electromotive Force

Statics

Integrals over closed surfaces

Poisson: $$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q / \varepsilon_0$$

Gauss' Law: $$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

Integrals around closed loops

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0$$

Ampere's Law:

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} \right]$$

Statics to Dynamics

Integrals over closed surfaces

Poisson: $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q / \varepsilon_0$

Gauss' Law: $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$

Integrals around closed loops

Faraday's Law:

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\boxed{\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}}$$

Ampere's Law:

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} + \boxed{\varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}} \right]$$

Dynamic Fields

Faraday's Law

Maxwell's Displacement Current

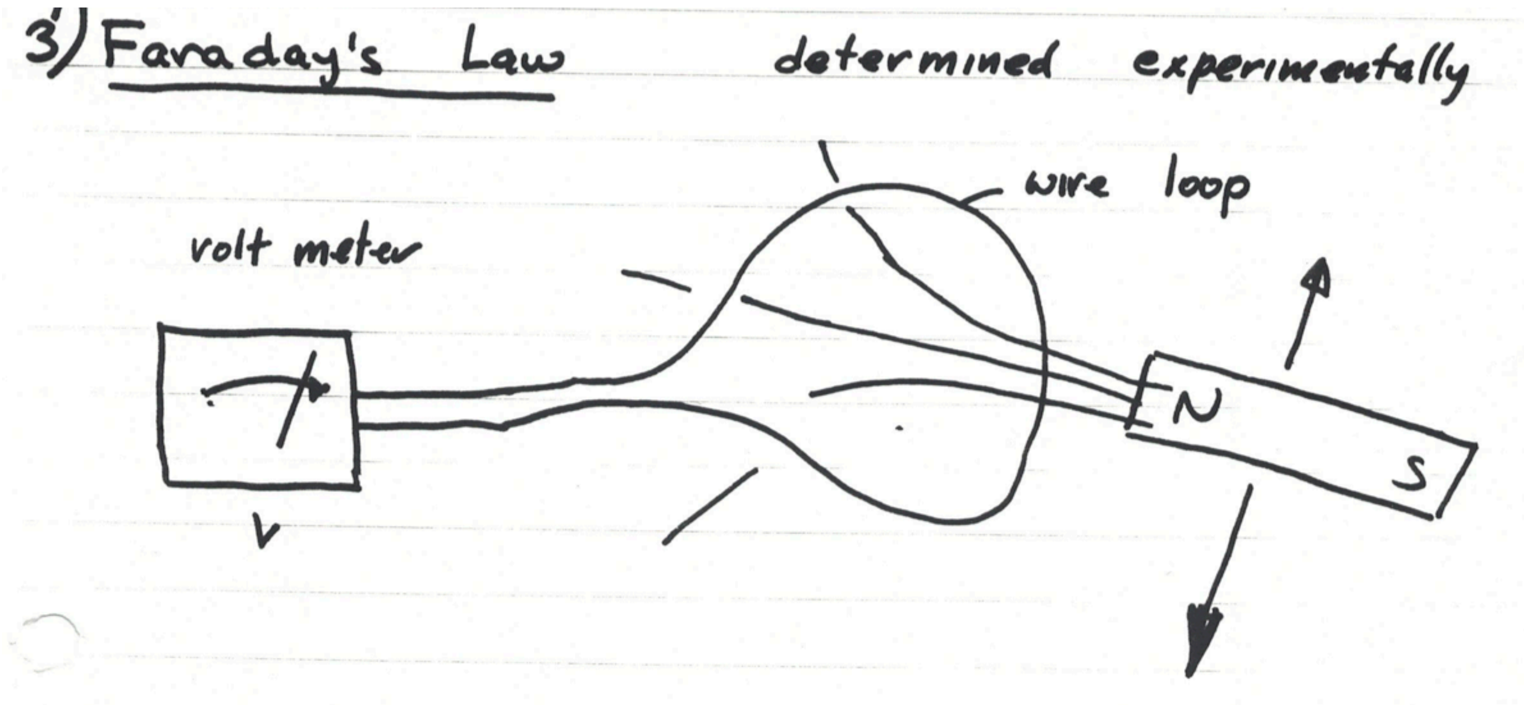
Integrals around closed loops

Faraday's Law:

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Ampere's Law:

$$\oint_{Loop} \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{l}} = \mu_0 \int_S d\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right]$$



As the magnet was moved a voltage appeared on the meter.

The polarity of the voltage depended on whether the magnetic flux threading the loop was increasing or decreasing

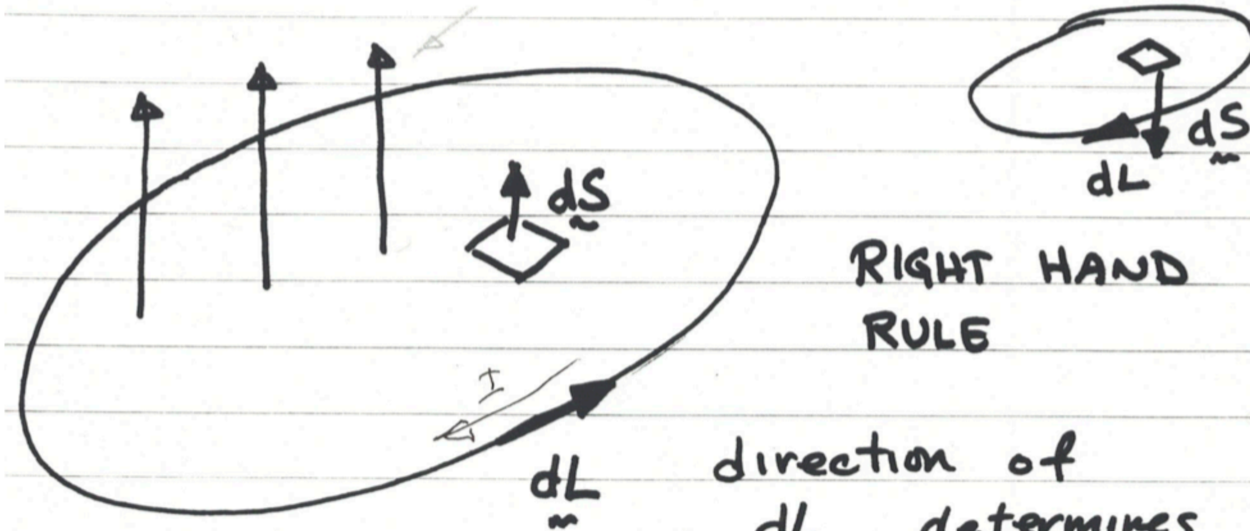
Experimentally deduced relation

$$V = -\oint_C \underline{E} \cdot d\underline{L} = \frac{d\psi}{dt}$$

where $$\psi = \int_S \underline{B} \cdot d\underline{S}$$

For stationary loops .

Sign determined by right hand rule



direction of $d\underline{L}$ determines direction of $d\underline{S}$

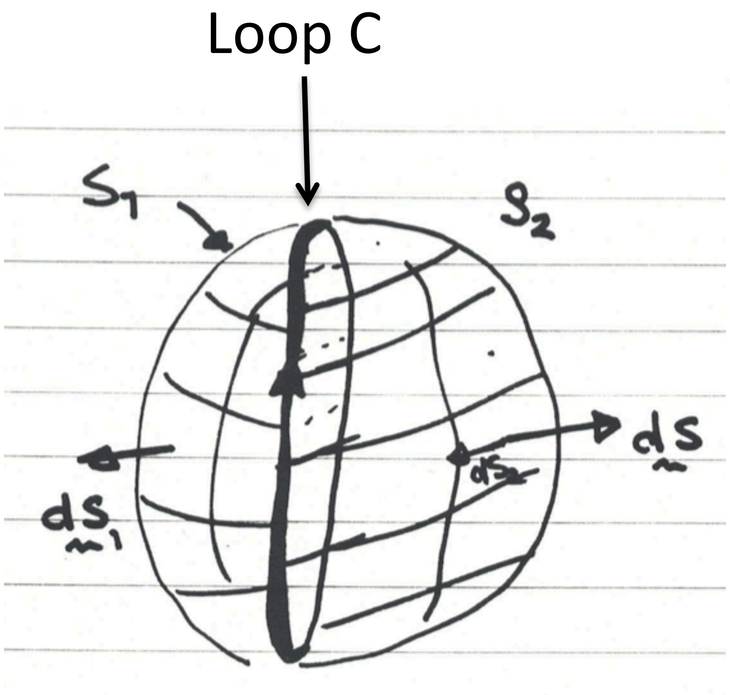
$$V = -\oint_C \underline{E} \cdot d\underline{L} = \frac{d\psi}{dt}$$

where $\psi = \int_S \underline{B} \cdot d\underline{S}$

Lenz Law
E would induce I to cancel change in B

$$V = -\int_C \underline{E} \cdot d\underline{L} = \frac{d\psi}{dt}$$

where $\psi = \int_S \underline{B} \cdot d\underline{S}$



Which surface S_1 or S_2?

Answer: Either one

From Gauss' Law

$$\int_{S_1+S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$$

$$d\vec{\mathbf{S}}_1 = d\vec{\mathbf{S}}$$

$$d\vec{\mathbf{S}}_2 = -d\vec{\mathbf{S}}$$

$$\int_{S_1+S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0 \Rightarrow \int_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}_1 = \int_{S_2} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}_2$$

Using Stokes' Theorem

$$\int_C \underset{\sim}{E} \cdot dL = \int_S d\underset{\wedge}{S} \cdot \nabla \times \underset{\sim}{E} = -\frac{\partial}{\partial t} \int_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

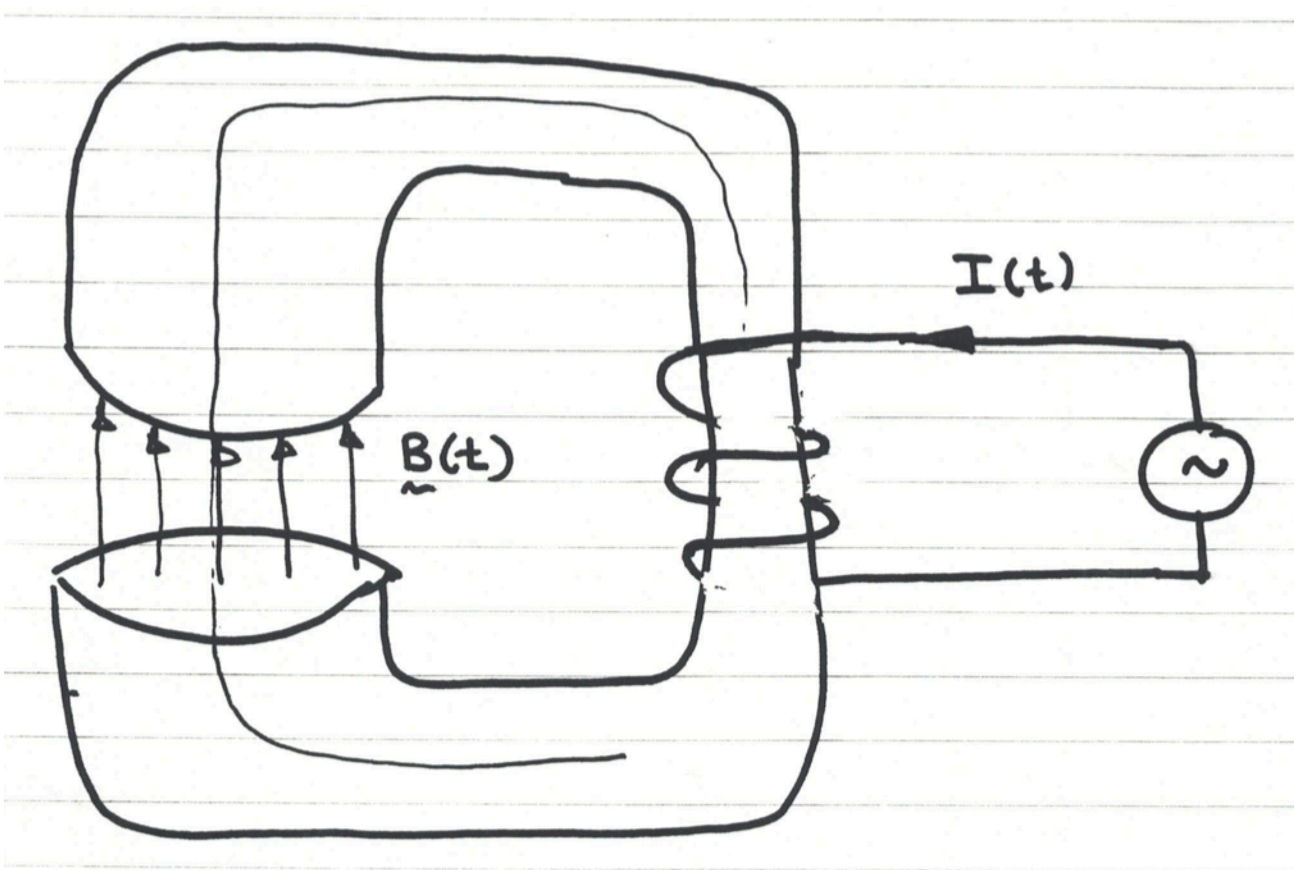
True for any loop and any surface

THUS

$$\nabla \times \underset{\sim}{E} = -\frac{\partial}{\partial t} \underset{\sim}{B}$$

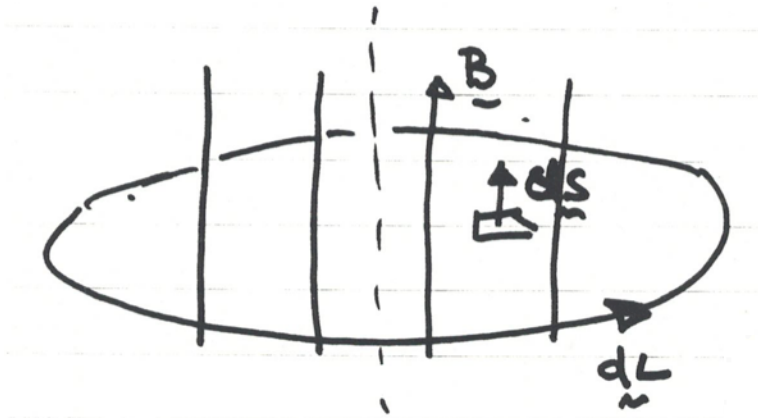
Faraday's Law in differential form

Time varying B induces E



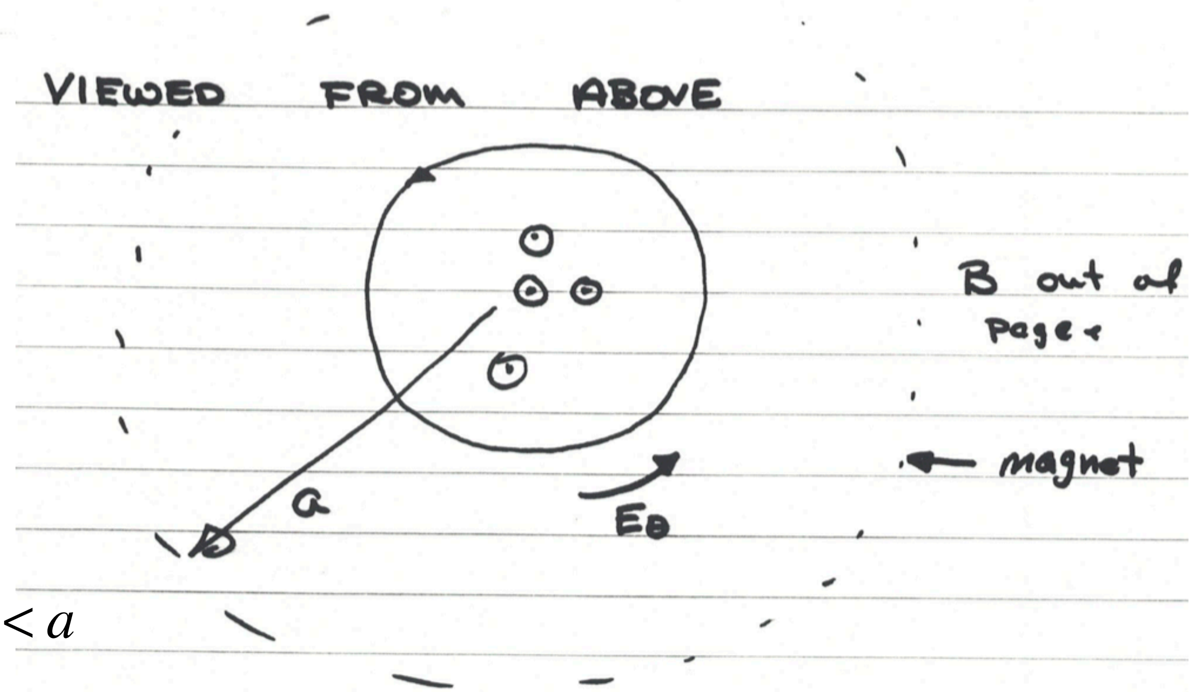
Find E in the gap

Gap Field



Evaluate on a loop of radius *r*.

Assume: $$\frac{\partial E_\theta}{\partial \theta} = 0$$

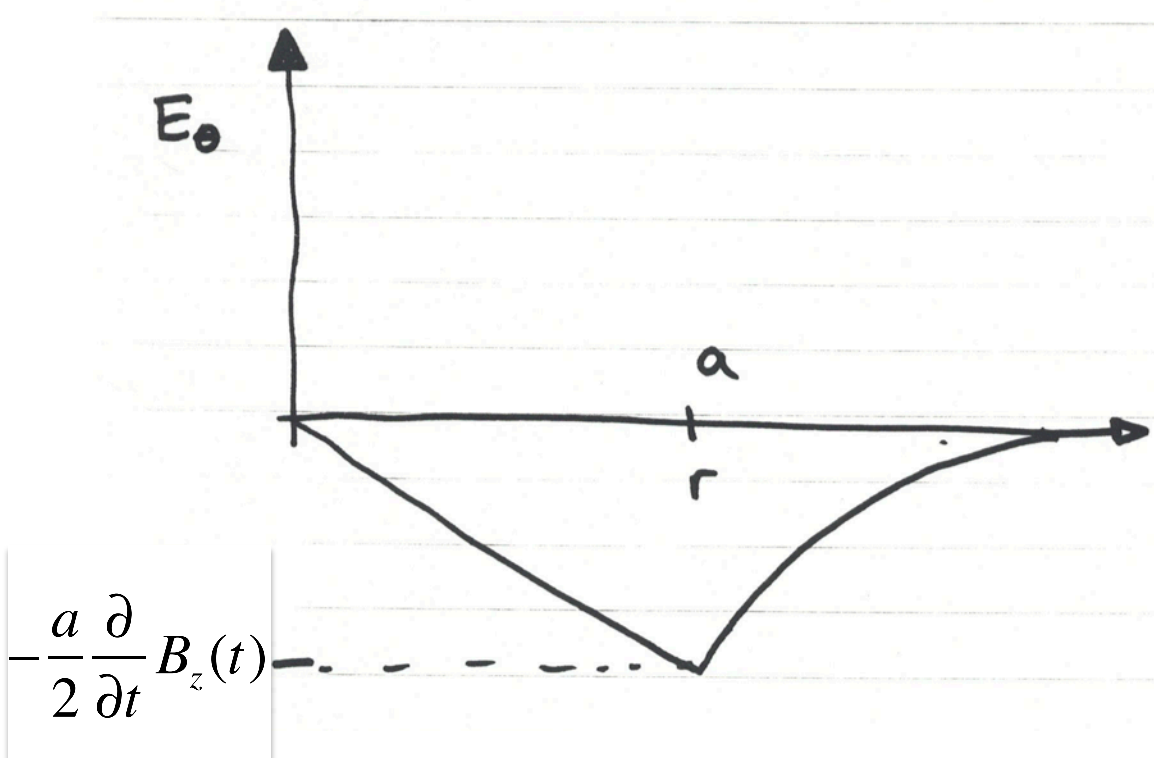


$$\int_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 2\pi r E_\theta(r)$$

$$\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \begin{cases} \pi r^2 B_z, & r < a \\ \pi a^2 B_z, & r > a \end{cases}$$

$$2\pi r E_\theta(r) = -\frac{\partial}{\partial t} \begin{cases} \pi r^2 B_z(t), & r < a \\ \pi a^2 B_z(t), & r > a \end{cases}$$

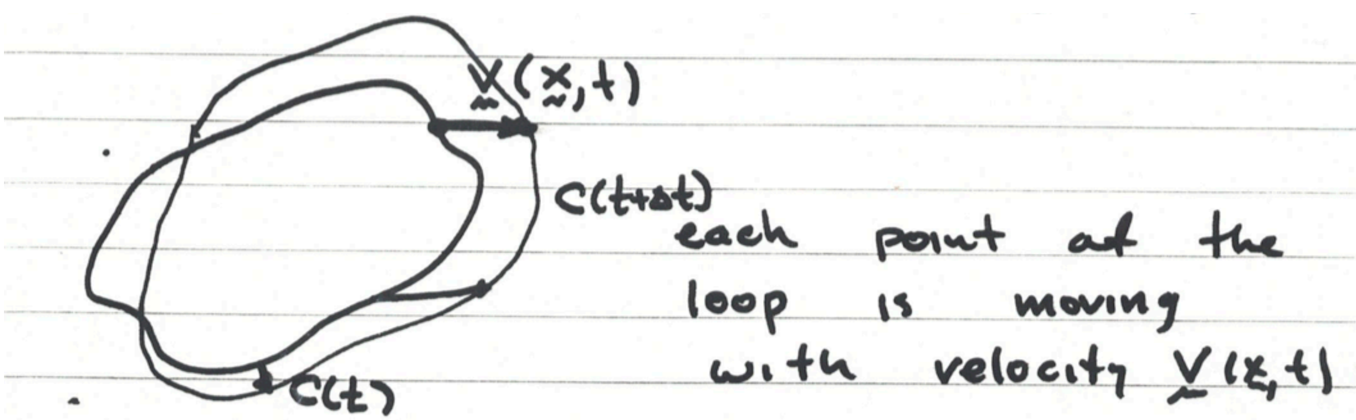
$$E_\theta(r) = -\begin{cases} \dfrac{r}{2}\dfrac{\partial}{\partial t}B_z(t), & r < a \\ \dfrac{a^2}{2r}\dfrac{\partial}{\partial t}B_z(t), & r > a \end{cases}$$



Moving Loops

So far we have considered stationary loops.

What is the rate of change of flux through a moving loop?

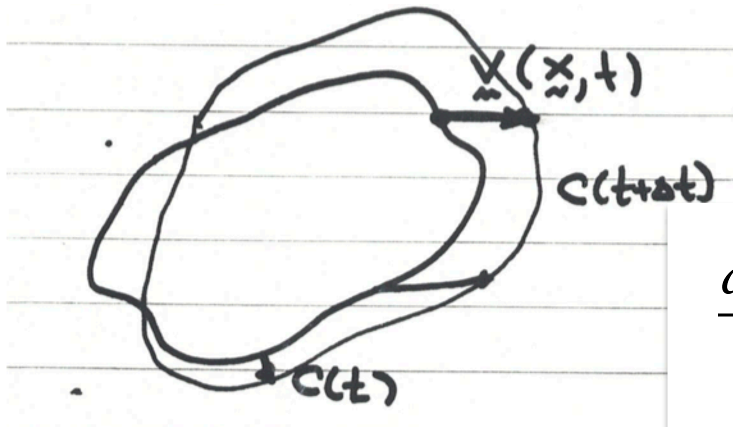


It can be shown

$$\frac{d\psi}{dt} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \oint_{C(t)} d\mathbf{l} \cdot \mathbf{V} \times \mathbf{B}$$

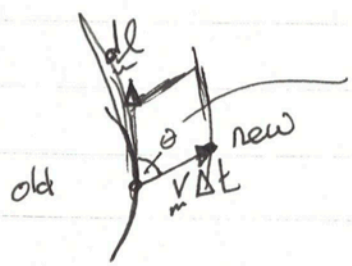
Contribution from time changing B, Contribution from moving loop.

Rate of change of flux



$$\frac{d\psi}{dt} = \lim_{\Delta t \to 0} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t}$$

Derivation



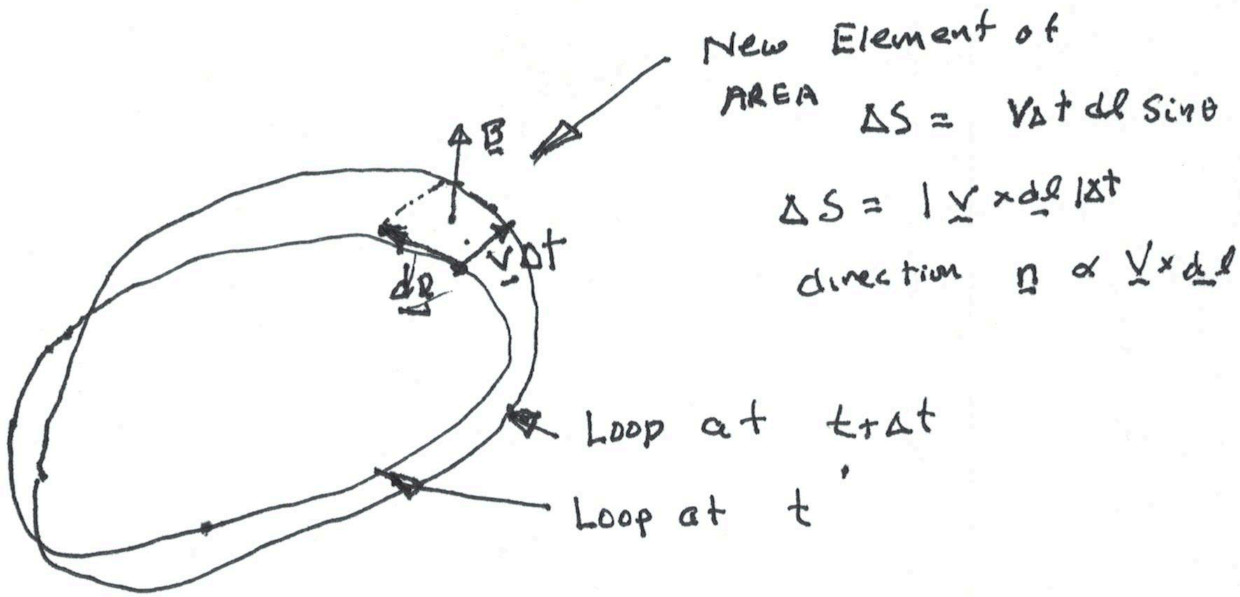
new area

$|\Delta S| = |\underset{\sim}{V}\Delta t||\underset{\sim}{dl}|\sin\theta = |\underset{\sim}{V}\times\underset{\sim}{dl}|\Delta t$

direction $\underset{\sim}{V}\times\underset{\sim}{dl}$

$$\int_{C(t)} \underset{\sim}{B}\cdot\underset{\sim}{V}\times\underset{\sim}{dl} = -\int \underset{\sim}{dl}\cdot\underset{\sim}{V}\times B$$

Contribution from moving loop



$$\Delta\Psi = \Delta t \int \underline{B} \cdot (\underline{V} \times d\underline{l}) = -\Delta t \int d\underline{l} \cdot \underline{V} \times \underline{B}$$

EMF – electromotive force

$$\frac{d\psi}{dt} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} - \int_{C(t)} d\mathbf{l}\cdot \mathbf{v}\times\mathbf{B}$$

Convert surface integral to line integral

$$\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} = -\int_S d\mathbf{S}\cdot\nabla\times\mathbf{E} = -\int_{C(t)} d\mathbf{l}\cdot\mathbf{E}$$

$$\frac{d\psi}{dt} = -\int_{C(t)} d\mathbf{l}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

Two ways to compute EMF

$$EMF = -\frac{d}{dt}\psi = -\frac{d}{dt}\int_{S(t)} d\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$$

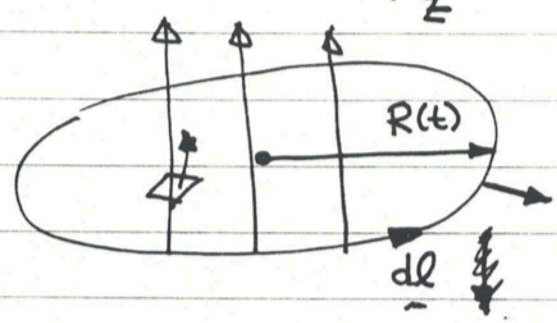
$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

Both are always true. One may be easier to determine than the other.

Note: same combination of E, B, and v appears in force

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

Example



$$EMF = -\frac{d}{dt}\psi(t) = -\frac{d}{dt}\pi R^2(t) B_z(t)$$

$$= -\pi R^2(t)\frac{d}{dt}B_z(t) - 2\pi R(t) B_z(t)\frac{d}{dt}R(t)$$

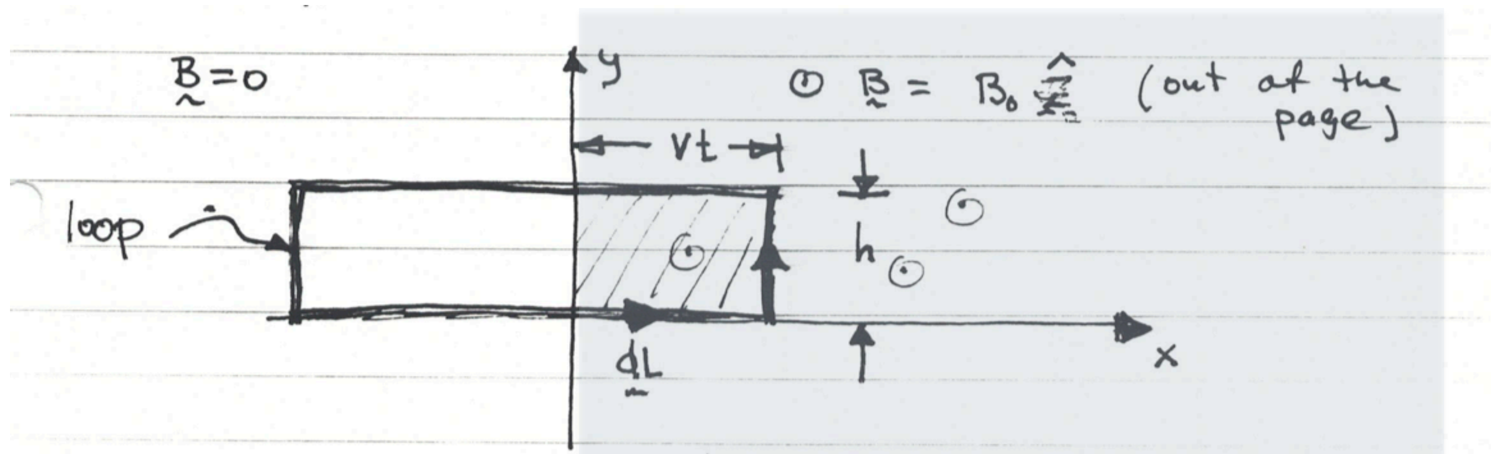
$$EMF = -\int_S d\vec{\mathbf{A}} \cdot \frac{\partial \vec{\mathbf{B}}}{\partial t} + \int_C d\vec{\mathbf{l}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

$$+\int_C d\vec{\mathbf{l}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{B}} = -2\pi R \frac{dR}{dt} B_z$$

$$\hat{\theta} \cdot \hat{r} \times \hat{z} = -1$$

Calculate the EMF

$$B(x,y,z,t)=\begin{cases} B_0\hat{z}, & x>0 \\ 0, & x<0 \end{cases}$$

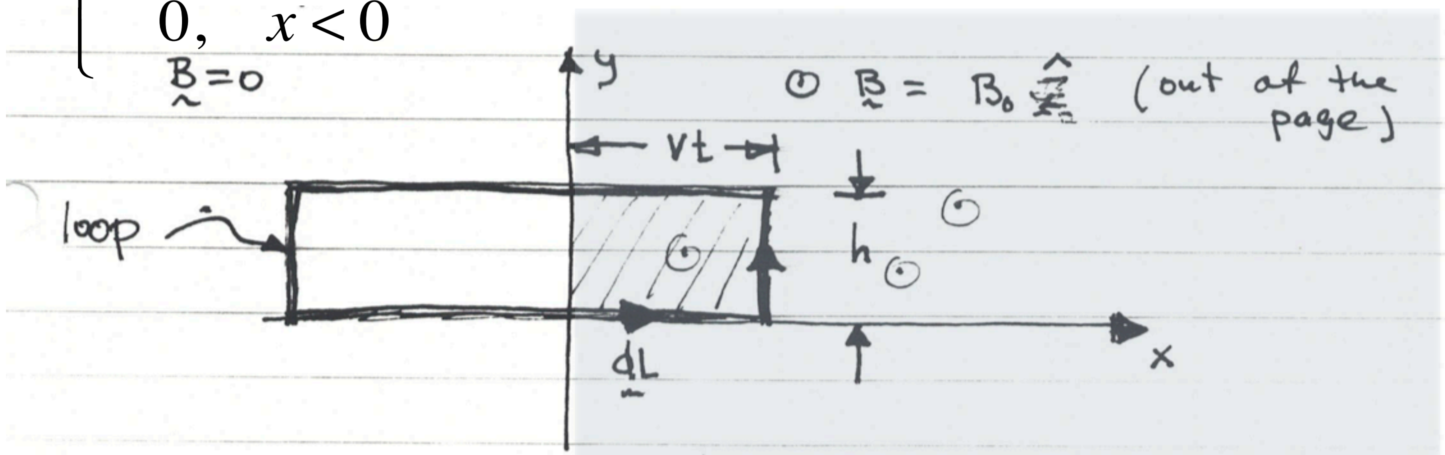


Three cases:

1. Loop is nonconducting
2. Loop is partially conducting
3. Loop is fully conducting

Case #1 non-conducting

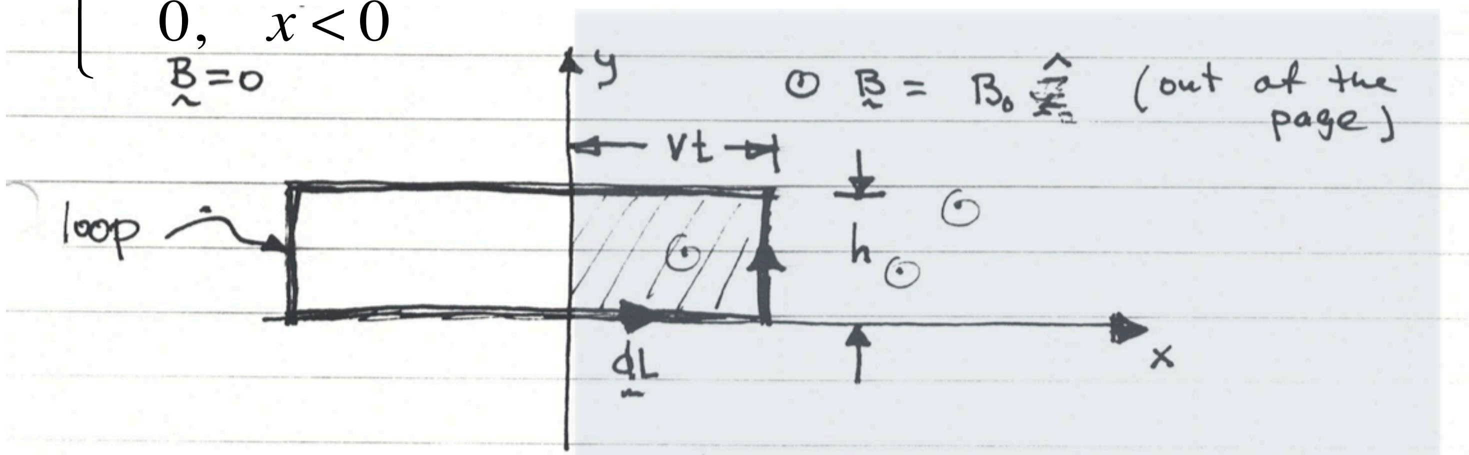
$$B(x,y,z,t) = \begin{cases} B_0\hat{z}, & x > 0 \\ 0, & x < 0 \end{cases}$$



$$EMF = -\frac{d}{dt}\psi = -\frac{d}{dt}\left[h(vt)B_0\right] = -hvB_0$$

Case #1 non-conducting

$$B(x,y,z,t) = \begin{cases} B_0 \hat{z}, & x > 0 \\ 0, & x < 0 \end{cases}$$



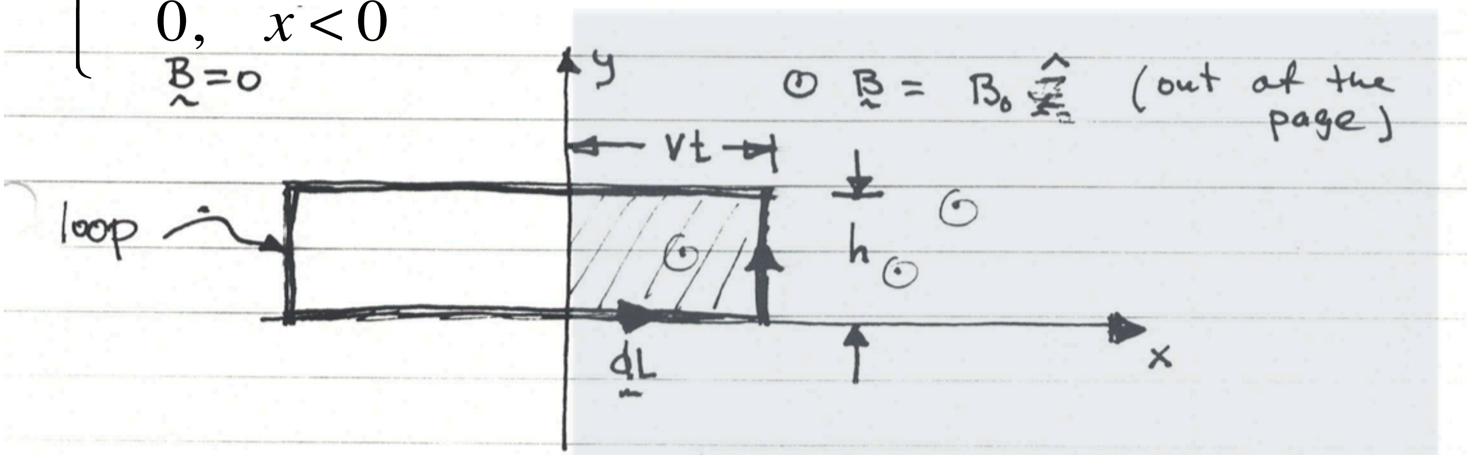
$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \rightarrow \nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla\phi \rightarrow \oint_{loop} d\vec{\mathbf{l}} \cdot \mathbf{E} = 0$$

$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\mathbf{E} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

$$= \int dy v B_0 (\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} \times \hat{\mathbf{z}}) = -hvB_0$$

Case #1 non-conducting

$$B(x,y,z,t) = \begin{cases} B_0\hat{z}, & x > 0 \\ 0, & x < 0 \end{cases}$$

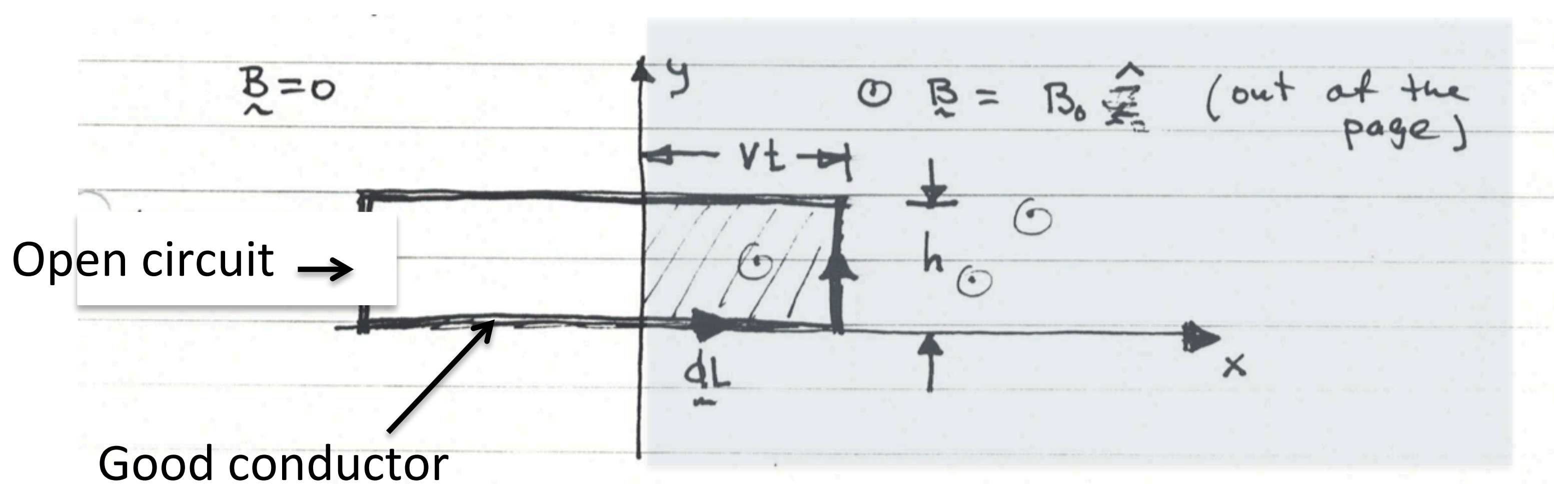


$$EMF = -\frac{d}{dt}\psi = -\frac{d}{dt}\left[h(vt)B_0\right] = -hvB_0$$

$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) =$$

$$\oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{0}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = \int dy v B_0 (\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} \times \hat{\mathbf{z}}) = -hvB_0$$

Case #2: partially conducting

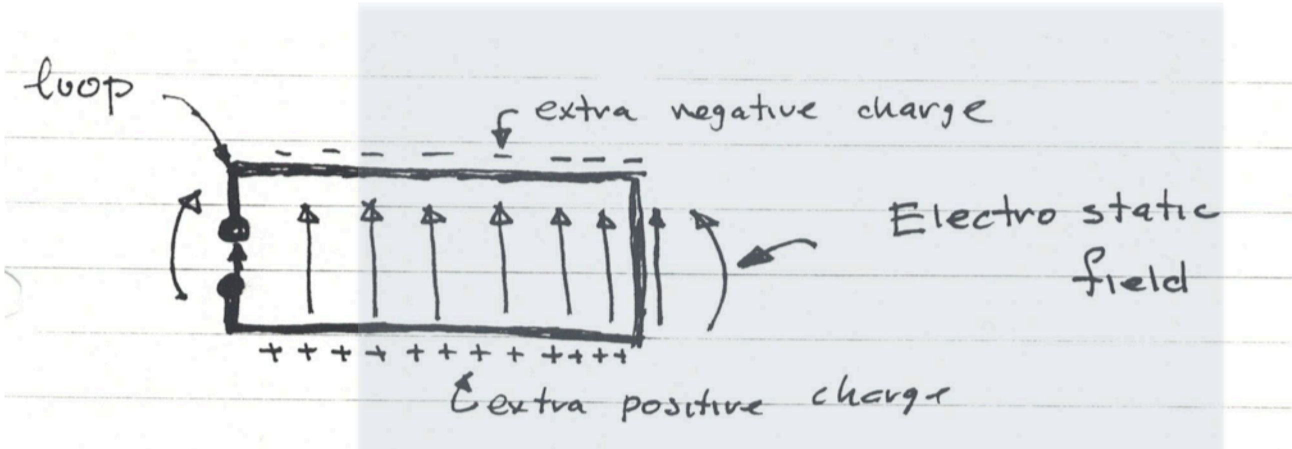


No current flows in conductor, B is unchanged,

$$EMF = -\frac{d}{dt}\psi = -hvB_0$$

$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \right)$$

In conductor $\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \right) = 0$



On right end of moving loop

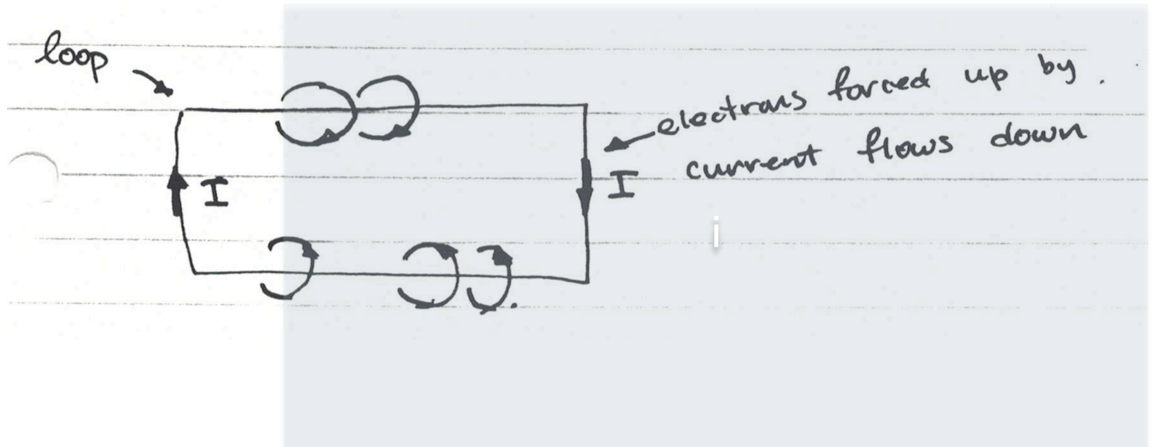
$$\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = 0$$

$$\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)_y = E_y - v_x B_0 = 0$$

Electrostatic field $\vec{\mathbf{E}} = -\nabla\phi, \quad \oint_{loop} d\vec{\mathbf{l}} \cdot \vec{\mathbf{E}} = 0$

$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = -hvB_0$$

Case #3: Conducting Loop



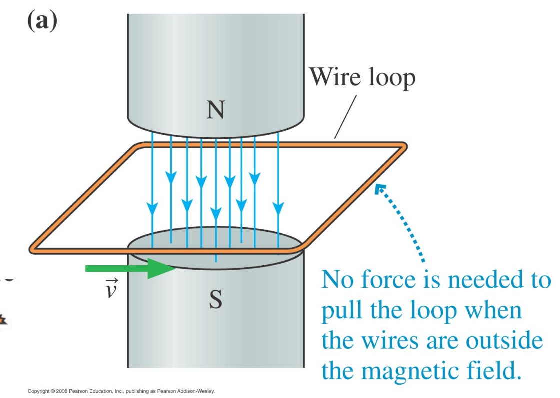
$$EMF = \oint_{loop} d\vec{\mathbf{l}} \cdot \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \right) = 0$$

Induced currents keep flux constant

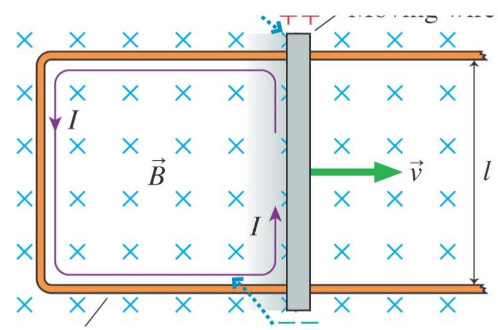
Reasons Flux Through a Loop Can Change

$$\frac{d}{dt}\psi = \frac{d}{dt}\int_{Area} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

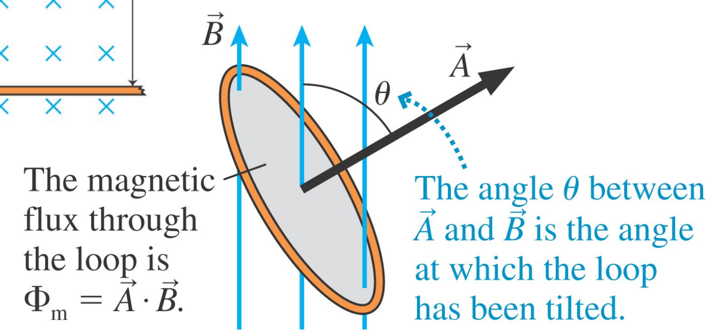
A. Location of loop can change



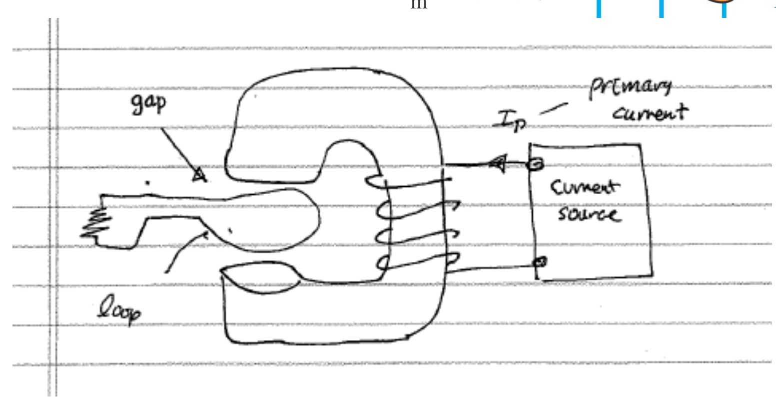
B. Shape of loop can change



C. Orientation of loop can change



D. Magnetic field can change



Faraday's Law for Moving Loops

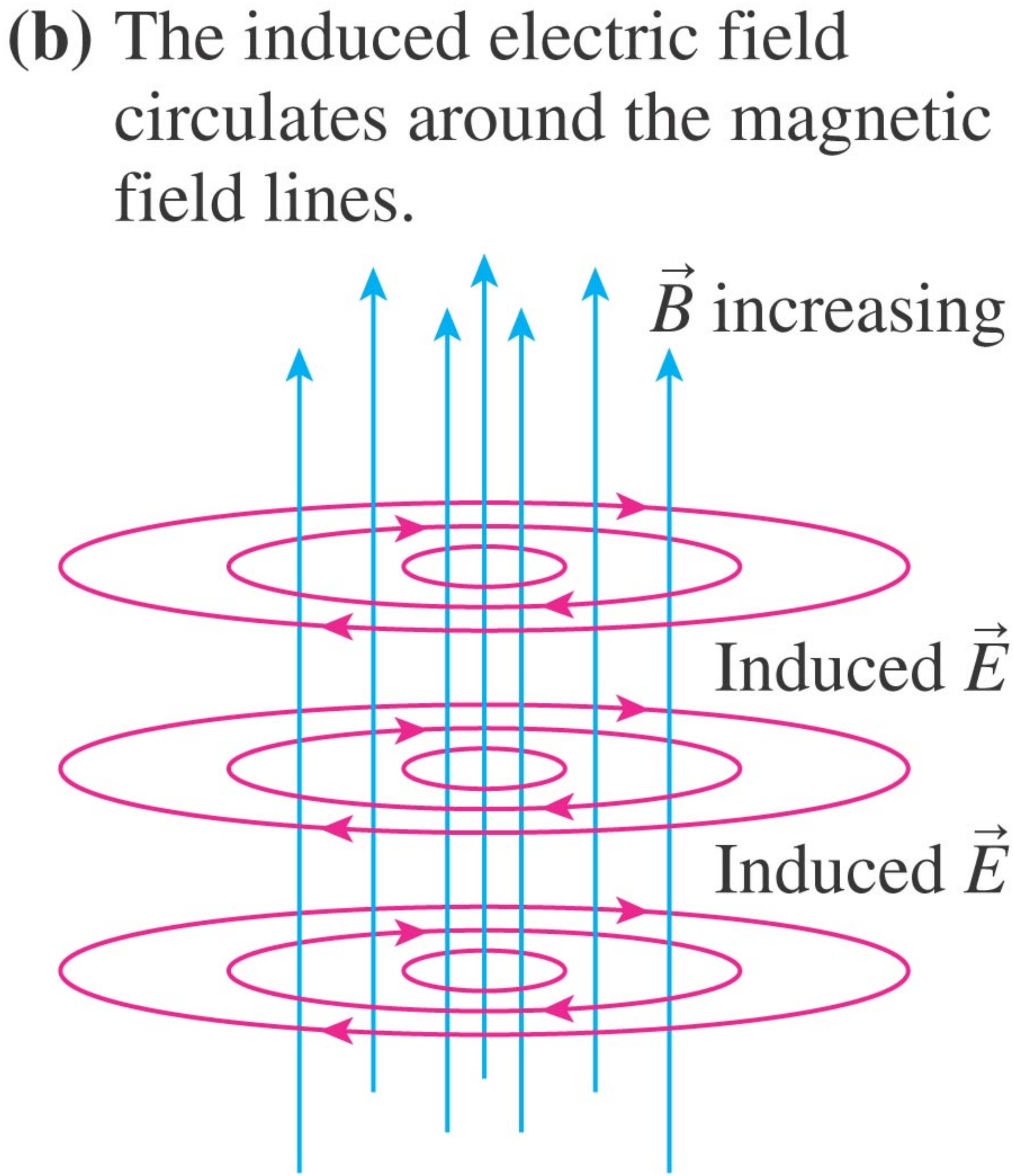
$$EMF = \oint_{loop} \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \right) \cdot d\vec{\mathbf{l}} = -\frac{d}{dt}\psi = -\frac{d}{dt} \int_{Area} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

Faraday's Law for Stationary Loops

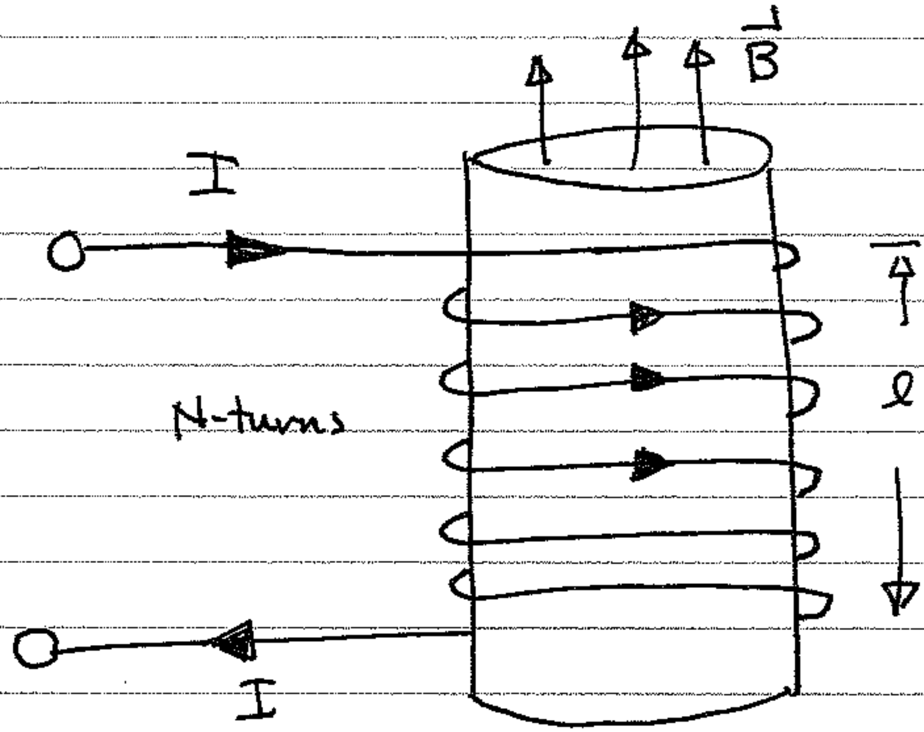
$$\oint_{loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} == -\int_{Area} \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}}$$

Only time derivative of B enters

There is an electric field even with no wire.



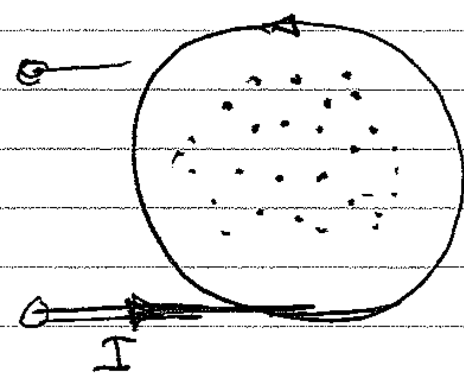
Consider a solenoid with N turns



Put your right thumb in direction of I. Fingers give direction of $\vec{B}$ (up inside)

$$|\vec{B}| = \frac{\mu_0 I N}{l}$$

VIEW FROM ABOVE



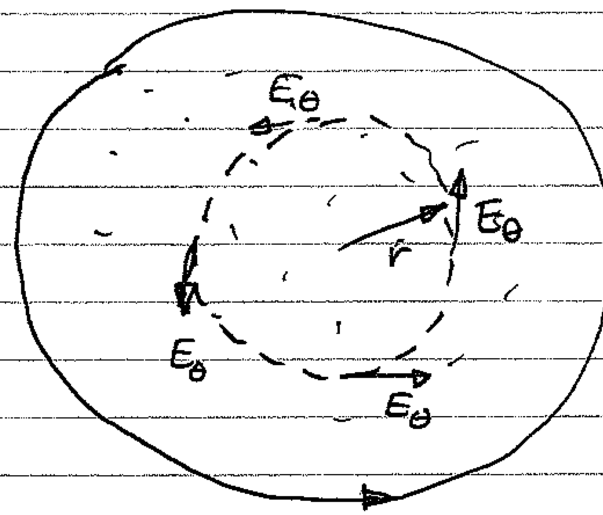
$\vec{B}$ out of page

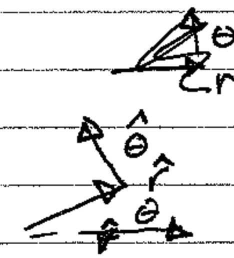
Calculate induced $\vec{E}$-field as a function of r

$\vec{B}$ - out of page

$\vec{B} = B_z \hat{k}$

Consider a loop of radius r





Q: Which direction is $\vec{E}$? $+\hat{\theta}$ or $-\hat{\theta}$

Ans: We don't know. Is B increasing or decreasing?

$$E_\theta = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

Faraday's Law for Stationary Loops

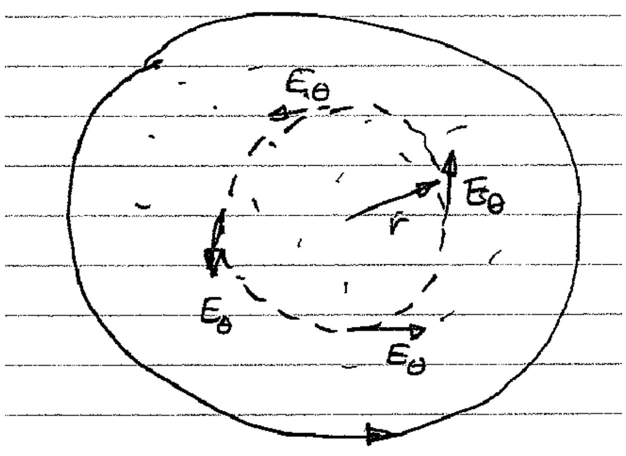
ccw

Out of page (+z)

$$\oint_{loop} \vec{\mathbf{E}} \cdot d\mathbf{l} == -\int_{Area} \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}}$$

Only time derivative of B enters

Call component of E in theta direction $E_q(r,t)$



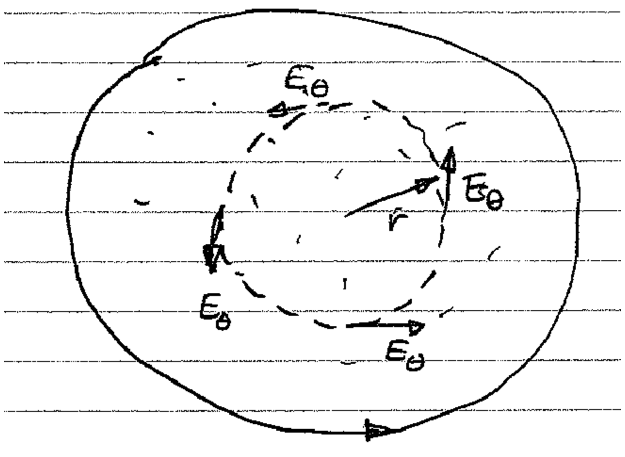
$$\oint_{loop} \vec{\mathbf{E}} \cdot d\mathbf{l} = 2\pi r E_\theta(r,t)$$

$$\int_{Area} \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} = \pi r^2 \frac{\partial B_z}{\partial t}$$

Therefore:

$$E_\theta(r,t) = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

Is Lenz's law satisfied ????

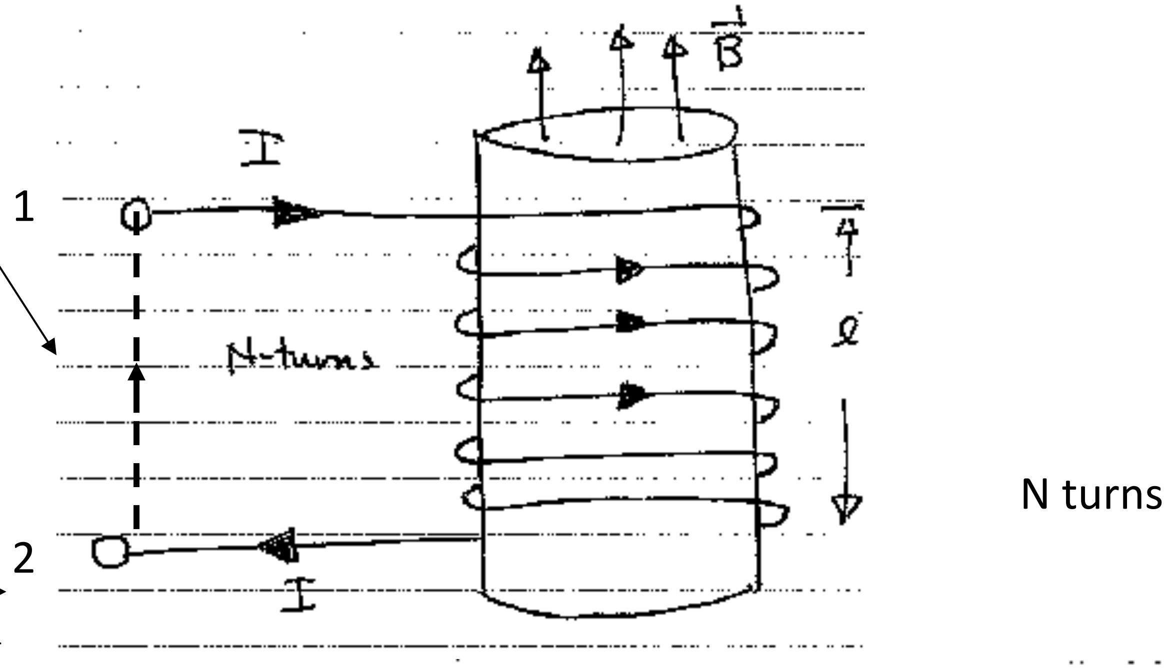


$$E_\theta(r,t) = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

B_z - out of page and increasing

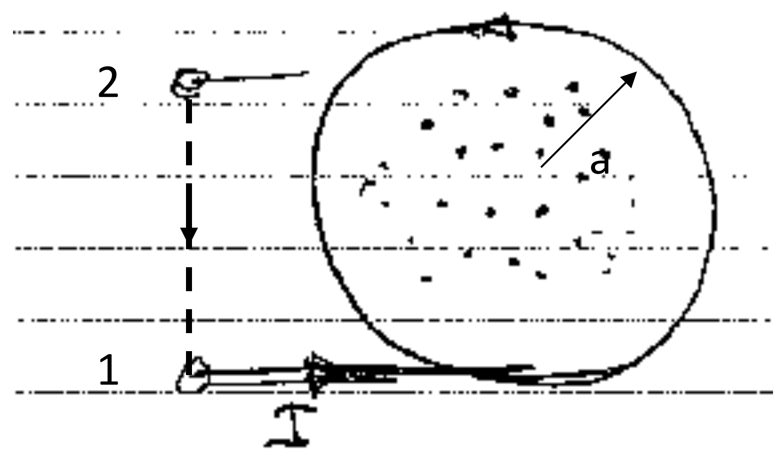
An induced current would flow Counterclockwise

What is $\int_{2}^{1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$



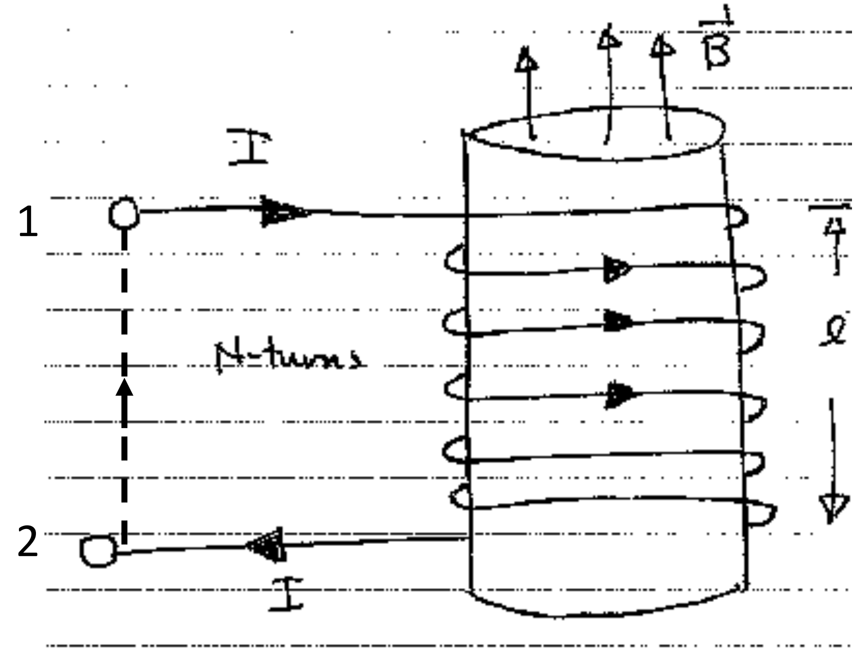
$$\int_{2}^{1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} + \int_{1,\ \mathbf{wire}}^{2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

$$= \oint_{Loop} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -N\pi a^2 \frac{\partial B_z}{\partial t}$$



Top view

Inductance



$$\int_2^1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -N\pi a^2 \frac{\partial B_z}{\partial t}$$

$$B_z = \frac{\mu_o N I}{l}$$

$$V_1 - V_2 = -\int_2^1 \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \frac{\mu_0 N^2 \pi a^2}{l} \frac{dI}{dt} = L \frac{dI}{dt}$$

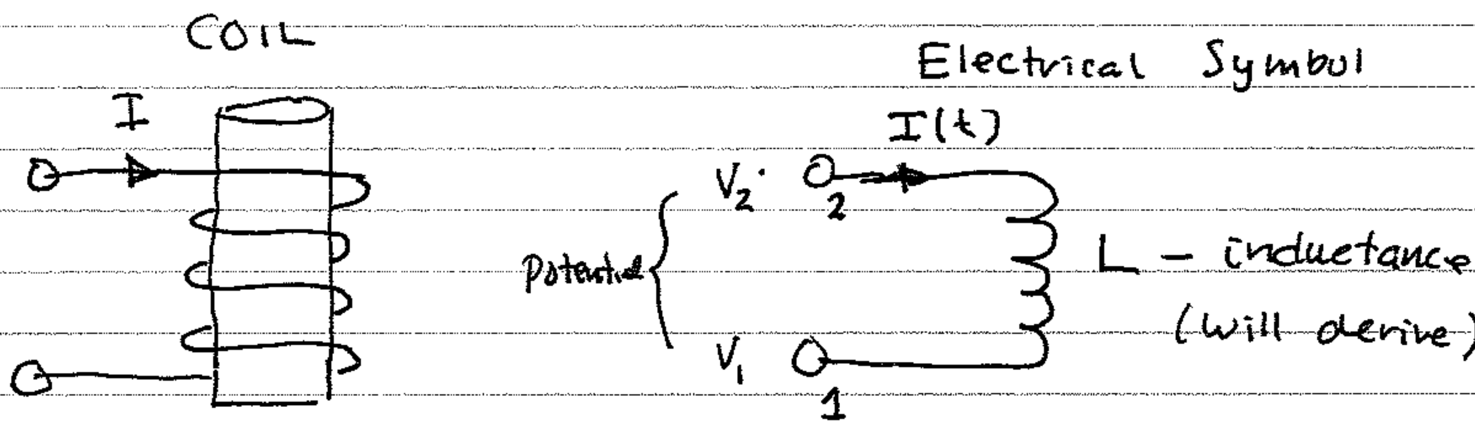
$$L = \frac{\mu_0 N^2 \pi a^2}{l}$$

Depends in geometry of coil, not I

Inductors

An inductor is a coil of wire

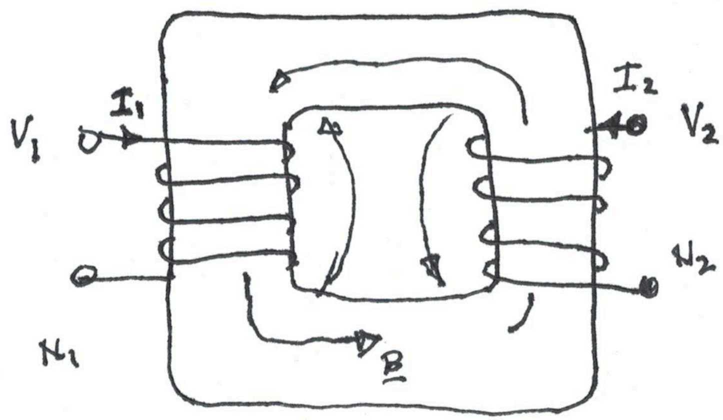
Any length of wire has inductance: but it's usually negligible



Engineering sign convention for labeling voltage and current

$$V_L = V(2) - V(1) = L\, dI/dt$$

Transformer



$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{pmatrix} dI_1/dt \\ dI_2/dt \end{pmatrix}$$

Inductance Matrix

$$\det\left[\mathbf{L}\right] = L_{11}L_{22} - L_{12}L_{21} > 0$$

Reciprocal

$$L_{12} = L_{21}$$

Coupling coefficient, $-1 < k < 1$

$$L_{12} = k\sqrt{L_{11}L_{22}}$$