Lecture 3, Part 1

The n^{th} Root of a Complex Number

Problem: Solve $z^n = v$

Here, v is a given complex number. Equivalent equation:

$$z^n - v = 0$$

LHS is a polynomial of degree n. It therefore has n roots, i.e., it can be rewritten as

$$(z - z_1)(z - z_2) \cdots (z - z_n)$$

In this case, the roots z_1, z_2, ... , z_n will be distinct.

Two familiar examples:

- $z^2 = 1 \quad \Leftrightarrow \quad z^2 - 1 = 0 \quad \Leftrightarrow \quad (z-1)(z+1) = 0$
- $z^2 = -1 \quad \Leftrightarrow \quad z^2 + 1 = 0 \quad \Leftrightarrow \quad (z-j)(z+j) = 0$

Solution of $z^n = v$

$$z^n = v \quad \Leftrightarrow \quad |z^n| = |v| \;\text{ and }\; \angle z^n = \angle v$$

Moduli:

$$|z^n| = |z|^n = |v| \quad \Rightarrow \quad \boxed{|z| = |v|^{1/n}}$$

Angles:

$$\angle z^n = n(\angle z) = \angle v$$

Angles are unchanged by adding a multiple of 2π, therefore

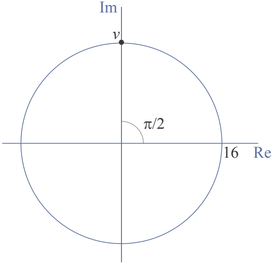
$$n(\angle z) = \angle v + k(2\pi)$$

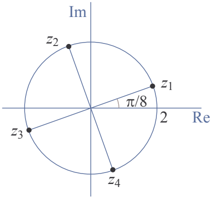
Divide both sides by n:

$$\boxed{\angle z = \frac{\angle v}{n} + k\frac{2\pi}{n}}$$

Letting $k = 0, 1, \ldots, n-1$, we obtain n distinct angles and thus also n distinct roots.

Example: $z^4 = 16j$





Moduli:

$$|v| = 16 \qquad \Rightarrow \qquad |z| = 16^{1/4} = 2$$

Angles:

$$\angle v = \frac{\pi}{2} \qquad \Rightarrow \qquad \angle z = \frac{\pi}{8} + k\frac{2\pi}{4}$$

for $k = 0, 1, 2$ and 3. Thus the four roots have angles

$$\frac{\pi}{8}, \quad \frac{5\pi}{8}, \quad \frac{9\pi}{8} \quad \text{and} \quad \frac{13\pi}{8}$$