Lecture 2, Part 2

Product of Complex Numbers

- Definition using polar coordinates
- Computation using Cartesian coordinates

The Complex Product $z_1 z_2$

Completely different from product of two vectors (dot or cross).

$z_1 z_2$ is also a complex number

Two equivalent definitions, in terms of polar and Cartesian coordinates.

In polar coordinates, the product $z_1 z_2$ is defined by

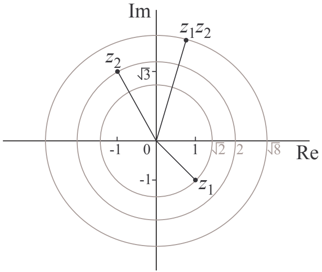
$$|z_1 z_2| = |z_1| \cdot |z_2| \qquad \text{and} \qquad \angle z_1 z_2 = \angle z_1 + \angle z_2$$

- Clearly, $z_1 z_2 \;=\; z_2 z_1$
- If $z_1 = a + j0$, then $z_1 z_2 \;=\; a z_2$ (i.e., scaling by real a):

$$|z_1 z_2| \;=\; |a| \cdot |z_2| \qquad \text{and} \qquad \angle z_1 z_2 \;=\; \angle z_2 + \;(0 \text{ or } \pi)$$

$$(\,0 \text{ if } a > 0\,; \;\text{ or } \pi \text{ if } a < 0\,)$$

Example



$$z_1 = 1 - j \quad \Rightarrow \quad |z_1| = \sqrt{2}\ , \quad \angle z_1 = -\pi/4$$

$$z_2 = -1 + j\sqrt{3} \quad \Rightarrow \quad |z_2| = 2\ , \quad \angle z_2 = 2\pi/3$$

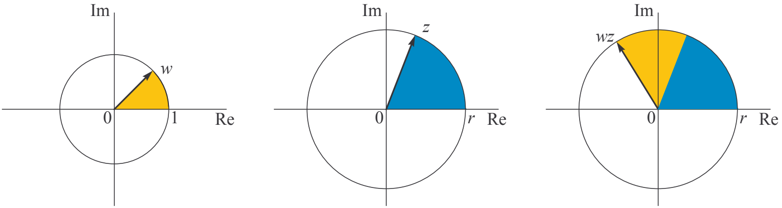
Thus

$$|z_1 z_2| \ = \ 2\sqrt{2}\ , \qquad \angle z_1 z_2 \ = \ -\frac{\pi}{4} + \frac{2\pi}{3} = \ \frac{5\pi}{12}$$

In Cartesian form:

$$z_1 z_2 \ = \ 2\sqrt{2}\cdot\Big(\cos(5\pi/12) + j\sin(5\pi/12)\Big)$$

Multiplication by a Number on the Unit Circle

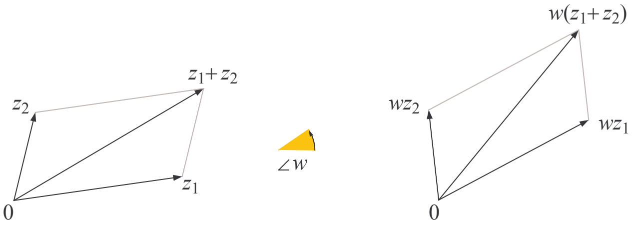


If $|w| = 1$ and z is arbitrary, then

$$|wz| = |z| \qquad \text{and} \qquad \angle wz = \angle w + \angle z$$

Conclusion: Multiplying z by a complex number w on the unit circle is equivalent to rotating the vector of z by $\angle w$.

Distributive Property



If $|w| = 1$, then

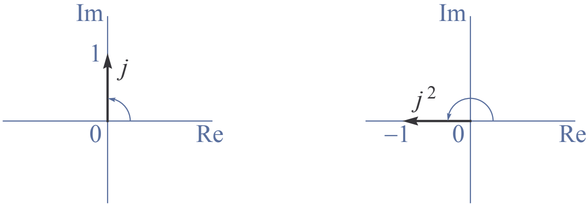
$$w(z_1 + z_2) = wz_1 + wz_2$$

Any $z_0 \in \mathbb{C}$ can be written as aw, where $a = |z_0| > 0$. Scale above figures by a to obtain

$$z_0(z_1 + z_2) = z_0 z_1 + z_0 z_2$$

for any z_0, z_1 and z_2 in $\mathbb{C}$.

Product in Cartesian Coordinates (and $j^2 = -1$)



$$|j^2| = 1 \cdot 1 = 1 \quad \text{and} \quad \angle j^2 = 2(\pi/2) = \pi$$

Thus $j^2 = -1$.

The distributive property can now be used to compute $z_1 z_2$ directly in Cartesian form. For example:

$$\begin{aligned}(3+4j)(2-5j) &= 3(2-5j) + 4j(2-5j) \\ &= 6 - 15j + 8j - 20j^2 \\ &= 26 - 7j\end{aligned}$$