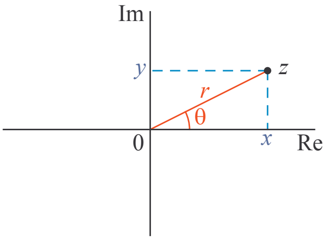
Lecture 1

Introduction to Complex Numbers

- Cartesian and polar forms
- Scaling and addition
- The imaginary unit j

Cartesian and Polar Forms

A complex number z is represented by a point on the Cartesian plane.



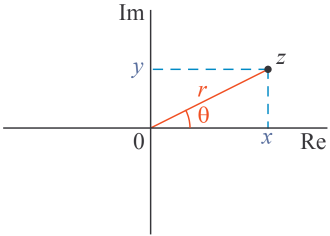
Cartesian coordinates:

- $x = \Re e\{z\}$: real part of z
- $y = \Im m\{z\}$: imaginary part of z

Polar coordinates:

- $r = |z|$: modulus (or magnitude) of z
- $\theta = \angle z$: angle of z

Coordinate Conversions



Polar to Cartesian:

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

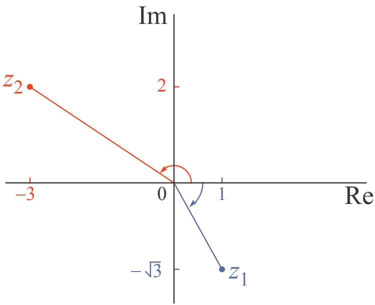
Cartesian to polar:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right) + \begin{cases} 0 & \text{if } x \geq 0 \\ \pi & \text{if } x < 0 \end{cases}$$

- π radians equals 180^0
- θ and $\theta + 2\pi$ (rad) are the same angle

Examples



- If $|z_1| = 2$ and $\angle z_1 = -\pi/3$ (same as $5\pi/3$), then

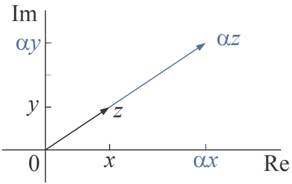
$$\Re e\{z_1\} = 2(1/2) = 1\,, \qquad \Im m\{z_1\} = 2(-\sqrt{3}/2) = -\sqrt{3}$$

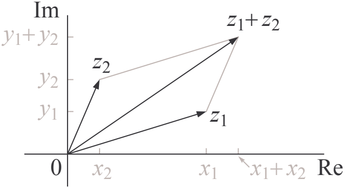
- If $\Re e\{z_2\} = -3$ and $\Im m\{z_2\} = 2$, then

$$\begin{aligned} |z_2| &= \sqrt{(-3)^2 + 2^2} = 3.606 \\ \angle z_2 &= \arctan(-2/3) + \pi = 2.553 \text{ rad} \end{aligned}$$

Scaling and Addition

Treat each z as a vector (from origin to z) and apply usual rules.





- Scaling by $\alpha \in \mathbb{R}$:

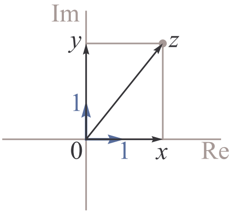
$$\Re e\{\alpha z\} = \alpha \cdot \Re e\{z\}, \qquad \Im m\{\alpha z\} = \alpha \cdot \Im m\{z\}$$

- Addition:

$$\Re e\{z_1 + z_2\} = \Re e\{z_1\} + \Re e\{z_2\}$$
$$\Im m\{z_1 + z_2\} = \Im m\{z_1\} + \Im m\{z_2\}$$

(Parallelogram law)

The Imaginary Unit j



For any z on the complex plane $\mathbb{C}$, we have the Cartesian form

$$\begin{aligned} z &= (x, y) \\ &= (x, 0) + (0, y) \\ &= x(1, 0) + y(0, 1) \\ &= x + jy \end{aligned}$$

where

- $j = (0, 1)$, i.e., unit vector along the imaginary axis
- unit vector $(1, 0)$ along the real axis is implied (scaled by x)