HW Problem 1

Consider the complex numbers

$$z_1 = 2 - 3j \qquad \text{and} \qquad z_2 = -1 + 8j$$

(i) (2 pts.) Plot both numbers on the complex plane.

(ii) (2 pts.) Evaluate $|z_i|$ and $\angle z_i$ for both values of i ($i = 1, 2$).

(iii) (4 pts.) Express each of $2z_1 - z_2$ and z_1/z_2 in both Cartesian and polar form.

(iv) (3 pts.) If $v = z_1^* \cdot z_2^{-3}$, determine $|v|$ and $\angle v$. Also, obtain v in Cartesian form.

(v) (3 pts.) If $w = z_1^{4000}$, determine $\angle w$ in the range $[0, 2\pi)$. Your answer should be correct to five decimal places.

(vi) (3 pts.) Determine the only *real* values of a and b such that

$$z^2 + az + b = 0$$

has $z = 2z_1 - z_2$ as a root.

(vii) (3 pts.) Without using a calculator, show that $\angle z_1$ and $\angle(z_1 + z_2)$ differ by an integer multiple of $\pi/4$. (*Hint: Use coordinate geometry; specifically, the dot product between two vectors.*)

HW Problem 2

Do not use a calculator for this problem. Express your answers using square roots and/or fractional multiples of π.

Throughout this problem, let

$$u = 1 - j \qquad \text{and} \qquad v = -3 + j\sqrt{3}$$

(i) (2 pts.) Express each number in the form $re^{j\theta}$. Plot both numbers on the complex plane.

(ii) (4 pts.) Let a be a real scaling factor. Determine the *exact* value(s) of a such that

$$|u + av| \;=\; \sqrt{2}$$

(iii) (5 pts.) Express v^4 in polar form. Determine all the roots z of the equation

$$z^4 - v^4 \;=\; 0$$

and plot them on the complex plane.

(iv) (3 pts.) Sketch the circle described by the equation

$$|z - u| \;=\; 1$$

Determine the points of intersection (if any) of this circle and the two axes, real and imaginary.

(v) (4 pts.) Sketch the line described by the equation

$$|z - v| \;=\; |z|$$

and determine the point at which it intersects the real axis.

(vi) (2 pts.) Sketch the line or curve described by the equation

$$\Re e\{z\} \;=\; \Im m\{2z^*\}$$

HW Problem 3

Use your calculator for algebraic calculations only. Solutions based on trial and error, inspection of numerical plots, etc., are not acceptable.

Consider the sinusoid $x(t) = A\cos(\Omega t + \phi)$, where $A > 0$ and $\phi \in (-\pi, \pi]$. Time t is in seconds.

It is known that

- $x(t) \geq 2.4$ for exactly 18% of each period;
- it takes 0.123 seconds for the value of the sinusoid to drop from 2.4 to the next minimum ("valley");
- the first zero of the sinusoid in positive time occurs at $t = 0.040$ (seconds).

(i) (5 pts.) Determine the amplitude A.

(ii) (5 pts.) Determine the period and angular frequency Ω of $x(t)$.

(iii) (5 pts.) Determine the initial phase ϕ of $x(t)$ as a fraction of π. (*Two values are possible here.*)

(iv) (5 pts.) Write simple MATLAB code which computes and plots two periods of $|x(t)|$ (i.e., the absolute value of $x(t)$) starting at $t = 0$, using 150 uniformly spaced samples per period (i.e., a total of 300 samples). Attach a printout of the code and a plot of the result.

HW Problem 4

Consider the discrete-time sinusoids

$$x[n] \;=\; \cos\left(\frac{5\pi n}{8} + \frac{\pi}{4}\right) \qquad \text{and} \qquad y[n] \;=\; \cos\left(\frac{10\pi n}{7} - \frac{2\pi}{3}\right)$$

(i) (3 pts.) What is the fundamental period of each sinusoid?

(ii) (4 pts.) Use MATLAB to generate separate plots of $x[n]$ and $y[n]$ for $n = 0, \ldots, 111$.

(iii) (3 pts.) If N_x and N_y are the two fundamental periods found in part (i), show that $u[n] = x[n] + y[n]$ is also periodic with period $N_u = N_x N_y$.

(iv) (3 pts.) An equivalent form for $y[\cdot]$ is

$$y[n] \;=\; \cos(\omega n + \phi)$$

where ω is between 0 and π. What are the values of ω and ϕ?

(v) (2 pts.) The sequence $v[\cdot]$ is formed by taking every other sample in $x[\cdot]$, i.e.,

$$v[n] \;=\; x[2n]$$

Write an equation for $v[n]$. What is the period of $v[\cdot]$?

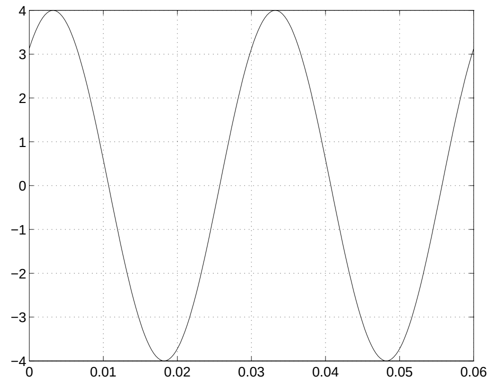
(vi) (5 pts.) Using phasors, express

$$x[n] - 2x[n-1] + 3x[n-2]$$

as a single real-valued sinusoid.

HW Problem 5

Two periods of the sinusoid $x(t) = A\cos(\Omega t + \phi)$ are plotted below. The value of $x(0)$ equals $4\sin(2\pi/7)$.



(i) (4 pts.) Determine A, Ω and ϕ. Express ϕ as an exact rational multiple of π in the range $[0, 2\pi)$. *(You may want to verify your answers by plotting the resulting sinusoid—it should be identical to the one in the figure.)*

In what follows: A, Ω and ϕ are as found in part (i) and $x[n] = x(nT_s)$, where T_s is a suitably chosen sampling period.

(ii) (4 pts.) Using phasors, express

$$y(t) \;=\; x(t) + 2x(t - (\pi/4\Omega))$$

as a single sinusoid.

(iii) (3 pts.) Determine all values of T_s such that $x[n]$ is constant for all n.

(iv) (3 pts.) Determine all values of T_s such that $x[n] = A\cos((\pi/12)n + \phi)$.

(v) (3 pts.) Determine all values of T_s such that $x[n] = x[n+4]$ for all n (*Note: The fundamental period of $x[\,\cdot\,]$ will equal 1, 2 or 4.*)

(vi) (3 pts.) Determine all values of T_s such that $x[n] = A\cos((5\pi/6)n - \phi)$.

HW Problem 6

(i) (4 pts.) Consider the frequency $f_0 = 180$ Hz and the sampling rate $f_s = 640$ samples/sec. List all the aliases of f_0 with respect to f_s in the frequency range 0.0 to 2.5 kHz.

(ii) (2 pts.) If we sample, at rate $f_s = 640$ samples/sec, a continuous-time real-valued sinusoid whose frequency is one of the values found in part (i) above, what is the resulting frequency ω of the sample sequence? Give your answer in the range $[0, \pi]$ (radians/sample).

(iii) (5 pts.) The discrete-time sinusoid $x[n] = 5.9\cos(0.625\pi n + 2.1)$ is obtained by sampling a continuous-time sinusoid $x(t)$ at a rate of 640 samples per second. If it is known that the frequency of $x(t)$ is in the range 1,280 to 1,600 Hz, write an equation for $x(t)$.

(iv) (4 pts.) Let $x[n]$ be as in (iii), with the sampling rate unchanged. If, instead, it is known that the frequency of $x(t)$ is in the range 960 to 1,280 Hz, write a new equation for $x(t)$.

(v) (5 pts.) The continuous-time signals

$$x_1(t) = \cos(144\pi t - 1.4) , \qquad \text{and} \qquad x_3(t) = \cos(256\pi t + 1.4)$$

are sampled at the same rate $f_a = 1/T_a$ to produce the sequences $x_1[n] = x_1(nT_a)$ and $x_2[n] = x_2(nT_a)$. Determine the highest value of f_a such that the two sequences are identical, i.e.,

$$x_1[n] = x_2[n] \qquad (\text{all } n)$$

For that value of f_a, are the two (identical) sequences periodic, and if so, what is the fundamental period?

HW Problem 7

Let

$$\mathbf{A}\begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -5 \end{bmatrix}, \qquad \mathbf{A}\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$$

(i) (1 pt.) What are the dimensions of $\mathbf{A}$?

(ii) (8 pts.) Determine the three columns $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ of $\mathbf{A}$ (and thus $\mathbf{A}$ itself). *Hint: Sum both sides of the last two equalities.*

(iii) (4 pts.) In the usual three-dimensional Cartesian space (with orthogonal axes), consider the plane $\mathcal{S}$ which contains the origin and has normal vector $[\,1\ 1\ 1\,]^T$. Determine the reflection of the point $[\,1\ 0\ 0\,]^T$ about $\mathcal{S}$. (*Hint: Express* $[\,1\ 0\ 0\,]^T$ *as a sum of two vectors, one parallel to* $[\,1\ 1\ 1\,]^T$ *and one orthogonal to it.*)

(iv) (4 pts.) By symmetry, determine the reflections of $[\,0\ 1\ 0\,]^T$ and $[\,0\ 0\ 1\,]^T$ about $\mathcal{S}$. Explain why the matrix $\mathbf{A}$ represents a reflection of an arbitrary point (in three-dimensional space) about $\mathcal{S}$.

(v) (3 pts.) Study the function `TOEPLITZ` in MATLAB documentation. Give a single row vector `c` such that

```
A = toeplitz(c)
```

produces the matrix $\mathbf{A}$ of parts (i)–(iv) above.

HW Problem 8

(i) (5 pts.) Review the concept of a rotation matrix. Show that any matrix of the form

$$\begin{bmatrix} r & -s \\ s & r \end{bmatrix} \qquad (r, s \in \mathbf{R})$$

represents a counterclockwise rotation on the plane, preceded or followed by scaling (the scaling factor being nonnegative). Express the angle of rotation and the scaling factor in terms of r and s.

(ii) (5 pts.) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \cos(5\pi/24) & -\sin(5\pi/24) \\ \sin(5\pi/24) & \cos(5\pi/24) \end{bmatrix}$$

Does there exist a positive integer n such that $\mathbf{A}^n = \mathbf{I}$ (where $\mathbf{I}$ is the 2×2 identity)? If so, what is the smallest such integer? *Explain.*

(iii) (5 pts.) Now let

$$\mathbf{B} = \begin{bmatrix} \cos(3\pi/16) & \sin(3\pi/16) \\ -\sin(3\pi/16) & \cos(3\pi/16) \end{bmatrix}$$

Without explicitly computing matrix products, inverses, etc., determine the matrix $\mathbf{C}$ such that $\mathbf{A}^2\mathbf{C}\mathbf{B}^2$ equals the identity matrix $\mathbf{I}$.

(iv) (5 pts.) Consider the matrices

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \qquad \text{and} \qquad \mathbf{F} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

What does $\mathbf{E}$ represent, geometrically? (The matrix $\mathbf{F}$ was encountered in Problem 7). Is it true that

$$\mathbf{EF} = \mathbf{FE}\ ?$$

HW Problem 9

Let

$$\mathbf{A} = \begin{bmatrix} a & b & 0 & c \\ d & e & f & 0 \\ 0 & r & s & t \\ u & 0 & v & w \end{bmatrix}$$

In each of the following cases, find matrices **P** and **Q** such that **PAQ** equals the matrix shown. (Note that if either **P** or **Q** is not needed, it can be set equal to **I**.)

(i) $\begin{bmatrix} v & w & u & 0 \\ s & t & 0 & r \\ f & 0 & d & e \\ 0 & c & a & b \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & r & s & t \\ u & 0 & v & w \\ a & b & 0 & c \\ d & e & f & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} e & d & d & e \\ b & a & a & b \end{bmatrix}$

(iv) $\begin{bmatrix} a & b-r & -s \end{bmatrix}$

(v) $[e-r-f+s]$ (1 × 1, i.e., a scalar)

HW Problem 10

Solve by hand without using your calculator. Show all intermediate steps.

Let

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1/2 & -3/2 & 2 & 0 \\ 1 & 4 & 1 & 1 \end{bmatrix}$$

(i) (6 pts.) Solve $\mathbf{Lx} = \mathbf{b}$ for an arbitrary vector $\mathbf{b}$. Display $\mathbf{L}^{-1}$.

(ii) (4 pts.) Express the matrix

$$\mathbf{G} = \begin{bmatrix} 2 & -8 & 0 & 0 \\ 2 & -16 & 3 & -1 \\ 1 & 6 & 6 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

in the form $\mathbf{G} = \mathbf{PLD}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{D}$ is a diagonal matrix and $\mathbf{L}$ is as given above. Determine $\mathbf{G}^{-1}$.

(iii) (6 pts.) If

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 2 & -5 & 1 & 7 \\ 1 & 5 & 2 & -2 \\ 2 & -11 & 5 & 13 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 12 & 8 & 1 & 6c \end{bmatrix}^T$, solve the equation $\mathbf{Ax} = \mathbf{b}$ using Gaussian elimination (without row interchanges). Here c is an arbitrary real constant.

(iv) (4 pts.) For $\mathbf{A}$ and $\mathbf{L}$ as above, show that $\mathbf{A} = \mathbf{LU}$, where $\mathbf{U}$ is an upper-triangular matrix. (*Hint*: You know $\mathbf{L}^{-1}$ from part **(i)**.)

HW Problem 11

Solve by hand without using calculator matrix functions. Show all intermediate steps.

Let $\mathbf{v}^{(1)} = [\,3 \quad 1 \quad 1 \quad -1\,]^T$, $\mathbf{v}^{(2)} = [\,1 \quad -2 \quad 0 \quad 1\,]^T$ and $\mathbf{v}^{(3)} = [\,-1 \quad 1 \quad 5 \quad 3\,]^T$.

(i) (3 pts.) Show that $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$ are (pairwise) orthogonal, and compute their norms.

In what follows, let $\mathbf{s} = [\,1 \quad 6 \quad 2 \quad 7\,]^T$.

(ii) (3 pts.) Determine the projection $\mathbf{f}^{(i)}$ of $\mathbf{s}$ onto each $\mathbf{v}^{(i)}$ (where $i = 1, 2, 3$).

(iii) (2 pts.) Determine the angle between $\mathbf{s}$ and $\mathbf{v}^{(1)}$.

(iv) (3 pts.) Determine the angle between $\mathbf{s}$ and the plane defined by $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$.

(v) (3 pts.) Determine the projection $\mathbf{g}$ of $\mathbf{s}$ onto the three-dimensional subspace defined by $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$.

(vi) (2 pts.) Verify that error vector $\mathbf{s} - \mathbf{g}$ is orthogonal to each of $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$.

(vii) (4 pts.) Solve the system

$$\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 \\ 1 & 0 & 5 & -2 \\ -1 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 4 \\ 9 \end{bmatrix}$$

without using Gaussian elimination. (*Hint: What is the relationship between* $\mathbf{s} - \mathbf{g}$ *and the fourth column of the matrix above?*)

HW Problem 12

Consider the complex-valued matrix

$$\mathbf{V} = \left[\begin{array}{cccc} \mathbf{v}^{(1)} & \mathbf{v}^{(2)} & \mathbf{v}^{(3)} & \mathbf{v}^{(4)} \end{array} \right] = \left[\begin{array}{cccc} 2 & 3+jc & -8 & a+jb \\ a+jb & 2 & 3+jc & -8 \\ -8 & a+jb & 2 & 3+jc \\ 3+jc & -8 & a+jb & 2 \end{array} \right]$$

(i) (6 pts.) Show that there exists only one choice of constants $a \in \mathbf{R}$, $b \in \mathbf{R}$ and $c > 0$ such that the columns of $\mathbf{V}$ are pairwise orthogonal. For that choice of a, b and c, what are the resulting column norms? (*You will need to set two column inner products equal to zero. Check your answers in MATLAB using* `V'*V` *before proceeding further.*)

From now on, assume that a, b, c and d are as found in part (i) above.

(ii) (6 pts.) Determine $\mathbf{d}$ such that the real-valued vector

$$\mathbf{s} = \left[\begin{array}{cccc} 27 & 45 & 41 & 23 \end{array} \right]^T$$

equals $\mathbf{Vd}$. (*Gaussian elimination is not needed here. Again, verify your answers in MATLAB.*)

(iii) (5 pts.) Determine the projection $\hat{\mathbf{s}}$ of $\mathbf{s}$ onto the subspace generated by the vectors $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(4)}$. What is the value of $\|\mathbf{s} - \hat{\mathbf{s}}\|^2$?

(iv) (3 pts.) If

$$\begin{aligned} \mathbf{x} &= \mathbf{v}^{(1)} + 2j\mathbf{v}^{(2)} + \mathbf{v}^{(3)} + 2j\mathbf{v}^{(4)} \\ \mathbf{y} &= j\mathbf{v}^{(1)} - 3\mathbf{v}^{(2)} - 3\mathbf{v}^{(3)} + j\mathbf{v}^{(4)} \end{aligned}$$

determine $\|\mathbf{x} - \mathbf{y}\|^2$ without using any of the numerical entries of the vectors $\mathbf{v}^{(i)}$.

HW Problem 13

Let

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}^{(0)} & \mathbf{v}^{(1)} & \mathbf{v}^{(2)} & \mathbf{v}^{(3)} & \mathbf{v}^{(4)} & \mathbf{v}^{(5)} & \mathbf{v}^{(6)} & \mathbf{v}^{(7)} \end{bmatrix}$$

be the matrix of Fourier sinusoids of length $N = 8$.

(i) (6 pts.) If

$$\mathbf{x} = \begin{bmatrix} 3 & 1 & -5 & 3 & 3 & 1 & -5 & 3 \end{bmatrix}^T ,$$

use projections to represent $\mathbf{x}$ in the form $\mathbf{x} = \mathbf{V}\mathbf{c}$. Verify that $\mathbf{x}$ is a linear combination of four columns of $\mathbf{V}$ (only).

(ii) (6 pts.) Repeat for

$$\mathbf{y} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \end{bmatrix}^T ,$$

expressing it as $\mathbf{y} = \mathbf{V}\mathbf{d}$. Verify that $\mathbf{y}$ is a linear combination of four columns of $\mathbf{V}$.

(iii) (2 pts.) Verify your results in (i) and (ii) using the `FFT` command in MATLAB (which will generate the vectors $8\mathbf{c}$ and $8\mathbf{d}$).

(iv) (2 pts.) Verify that the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal by computing their inner product. Do your answers to (i) and (ii) above also support this conclusion?

(v) (4 pts.) If $\mathbf{s} = \mathbf{x} + \mathbf{y}$, use your results from (i) and (ii) above to obtain the least squares approximation $\hat{\mathbf{s}}$ of $\mathbf{s}$ in terms of $\mathbf{v}^{(0)}$, $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(7)}$. Display the entries of $\hat{\mathbf{s}}$. Also, compute the squared error norm $\|\mathbf{s} - \hat{\mathbf{s}}\|^2$.

HW Problem 14

All vectors have length $N = 9$.

(i) (4 pts.) The entries of the time-domain vector

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \end{bmatrix}^T$$

are given by $2\cos\omega n$, where $n = 0:8$. What is the value of ω? Express $\mathbf{x}^{(1)}$ as the sum of two Fourier sinusoids. By considering the appropriate column of the Fourier matrix $\mathbf{V}$, determine and display the DFT $\mathbf{X}^{(1)}$.

(ii) (4 pts.) Similarly, express the time-domain vector

$$\mathbf{x}^{(2)} = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix}^T$$

as a linear combination of the same two Fourier sinusoids as in part (i). Hence determine and display the DFT $\mathbf{X}^{(2)}$.

(iii) (4 pts.) Determine and display the DFT $\mathbf{X}^{(3)}$ of

$$x^{(3)}[n] = \cos(2\pi n/9) + 3\cos(4\pi n/9) , \qquad n = 0, \dots, 8$$

(iv) (4 pts.) Determine and display the DFT $\mathbf{X}^{(4)}$ of

$$x^{(4)}[n] = \sin(8\pi n/9) , \qquad n = 0, \dots, 8$$

(v) (4 pts.) If the time-domain vector $\mathbf{x}^{(5)}$ has DFT

$$\mathbf{X}^{(5)} = \begin{bmatrix} 18 & 0 & 9j & 27 & 0 & 0 & 27 & -9j & 0 \end{bmatrix}^T$$

write an equation for $x^{(5)}[n]$, where $n = 0:8$. Your equation should be in terms of cosines and sines.

HW Problem 15

A real-valued signal vector $\mathbf{s}$ of length $N = 8$ has DFT

$$\mathbf{S} = \begin{bmatrix} 16 & z_1 & z_2 & z_3 & -4 & 7+j & 2j & -4+j5 \end{bmatrix}^T$$

(i) (2 pts.) What are the values of z_1, z_2 and z_3?

(ii) (3 pts.) Without inverting the DFT $\mathbf{S}$, evaluate the sum of the entries of the time domain vector $\mathbf{s}$.

(iii) (3 pts.) Without inverting the DFT $\mathbf{S}$, evaluate the sum

$$s[0] + s[2] + s[4] + s[6]$$

(iv) (4 pts.) Display the amplitude ($|S[k]|$) and phase ($\angle S[k]$) spectra as vectors.

(v) (6 pts.) Write an equation for $s[n]$ (where $n = 0 : 7$) in the form of a constant plus four real-valued sinusoids with frequencies in the range $(0, \pi]$.

(vi) (2 pts.) In MATLAB, generate the vector $\mathbf{s}$ given by the equation found in part (iv) and verify that `fft(s)` agrees with $\mathbf{S}$.

HW Problem 16

The vector $\mathbf{s}$ is given by

$$\mathbf{s} = \begin{bmatrix} a & b & c & d & e & f & g & h \end{bmatrix}^T$$

(i) (8 pts.) Display the following vectors:

- $\mathbf{s}^{(1)} = \mathbf{s} + \mathbf{R}\mathbf{s}$
- $\mathbf{s}^{(2)} = \mathbf{P}^4\mathbf{R}\mathbf{s} - \mathbf{s}$
- $\mathbf{s}^{(3)} = \mathbf{F}^2\mathbf{s} + \mathbf{F}^{-2}\mathbf{s}$
- $\mathbf{s}^{(4)} = \mathbf{P}\mathbf{s} + \mathbf{P}^5\mathbf{s}$

(ii) (12 pts.) Express the following vectors in terms of $\mathbf{P}$, $\mathbf{R}$, $\mathbf{F}$ and $\mathbf{s}$:

- $\mathbf{s}^{(5)} = \begin{bmatrix} h-b & a-c & b-d & c-e & d-f & e-g & f-h & g-a \end{bmatrix}^T$
- $\mathbf{s}^{(6)} = \begin{bmatrix} a+e & b-f & c+g & d-h & e+a & f-b & g+c & h-d \end{bmatrix}^T$
- $\mathbf{s}^{(7)} = \begin{bmatrix} 0 & \sqrt{2}b & 2c & \sqrt{2}d & 0 & -\sqrt{2}f & -2g & -\sqrt{2}h \end{bmatrix}^T$

HW Problem 17

MATLAB functions `OSHIFT`*,* `OFLIP` *and* `FDIAG` *can be used for circular shifts, circular reversals and modulation (by means of* **F***). You may use these functions to verify your results below. Copy these M-files to a local directory and include that directory in the MATLAB path.*

Suppose the signal

$$\mathbf{s} = \begin{bmatrix} a & b & c & d & e & f & g & h \end{bmatrix}^T$$

of HW Problem 16 has DFT

$$\mathbf{S} = \begin{bmatrix} 4 & 1+2j & 5 & 3j & -7 & -3j & 5 & 1-2j \end{bmatrix}^T$$

Without computing any DFTs or inverse DFTs, determine (in numerical form) the DFTs of the signal vectors $\mathbf{s}^{(1)}$ through $\mathbf{s}^{(7)}$ constructed in that problem.

HW Problem 18

Let $\mathbf{x}$ be a *real-valued* vector of length $N = 64$ whose DFT $\mathbf{X}$ satisfies

$$X[k] \neq 0 \text{ for } k = 0:13 \qquad \text{and} \qquad X[k] = 0 \text{ for } k = 14:32$$

Explain your answers to **(i)–(v)** below.

(i) (4 pts.) For what values of k between 33 and 63 does $X[k]$ equal zero?

(ii) (4 pts.) Let

$$x^{(1)}[n] = x[n]\cos(3\pi n/8)\ , \qquad n = 0:63$$

For what values of k does $\mathbf{X}^{(1)}[k]$ equal zero?

(iii) (4 pts.) Let

$$\mathbf{x}^{(2)} = \mathbf{P}^4\mathbf{x} + \mathbf{P}^{-4}\mathbf{x}$$

For what values of k does $\mathbf{X}^{(2)}[k]$ equal zero?

(iv) (4 pts.) Let

$$\mathbf{y} = \mathbf{x} + \mathbf{R}\mathbf{x}$$

What is the imaginary part of the vector $\mathbf{Y}$? Is it true that $\mathbf{Y} = \mathbf{R}\mathbf{Y}$?

(v) (4 pts.) Determine all the values of ω such that the vector $\mathbf{y}^{(1)}$ defined by

$$y^{(1)}[n] = y[n]\cos(\omega n)\ , \qquad n = 0:63$$

is certain to have a real-valued DFT $\mathbf{Y}^{(1)}$.

HW Problem 19

(i) (6 pts.) Determine the circular convolution $\mathbf{s} = \mathbf{x} \circledast \mathbf{y}$ of

$$\mathbf{x} = \begin{bmatrix} 4 & -3 & 1 & -1 & 2 \end{bmatrix}^T$$

and

$$\mathbf{y} = \begin{bmatrix} 5 & 2 & 3 & 0 & -2 \end{bmatrix}^T$$

Using the `FFT` function in MATLAB, verify that the DFTs $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{S}$ satisfy $\mathbf{X} \diamond \mathbf{Y} = \mathbf{S}$.

(ii) (3 pts.) Without any numerical computation, determine the circular convolution of

$$\mathbf{a} = \begin{bmatrix} 1 & -1 & 2 & 4 & -3 \end{bmatrix}^T$$

and

$$\mathbf{b} = \begin{bmatrix} 2 & 3 & 0 & -2 & 5 \end{bmatrix}^T$$

(iii) (6 pts.) If

$$\begin{bmatrix} 2 & -2 & 3 & -3 & 5 & -1 \\ -1 & 2 & -2 & 3 & -3 & 5 \\ 5 & -1 & 2 & -2 & 3 & -3 \\ -3 & 5 & -1 & 2 & -2 & 3 \\ 3 & -3 & 5 & -1 & 2 & -2 \\ -2 & 3 & -3 & 5 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ 9 \\ 2 \\ 12 \\ -7 \end{bmatrix},$$

explain how the vector $\mathbf{c}$ can be obtained using DFTs (as opposed to a conventional solution of a 6×6 system of equations). Implement this solution in MATLAB to obtain $\mathbf{c}$.

(iv) (5 pts.) Let $\mathbf{x}$ and $\mathbf{y}$ be time domain vectors of length N such that for every $n = 0, \ldots, N-1$,

$$x[n] = \frac{1}{y[n]} \neq 0$$

If $\mathbf{X}$ and $\mathbf{Y}$ are the respective DFTs, determine the circular convolution vector $\mathbf{X} \circledast \mathbf{Y}$.

HW Problem 20

The signal

$$\mathbf{x} = \begin{bmatrix} a & b & c & 0 & 0 & 0 & a & b & c & 0 & 0 & 0 \end{bmatrix}^T$$

has DFT $\mathbf{X}$ given by

$$\mathbf{X} = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 & D_9 & D_{10} & D_{11} \end{bmatrix}^T$$

(i) (4 pts.) For which indices k is D_k necessarily equal to zero?

(ii) (4 pts.) If

$$\mathbf{x}^{(1)} = \begin{bmatrix} a & b & c \end{bmatrix}^T ,$$

express the DFT $\mathbf{X}^{(1)}$ in terms of nonzero D_k's.

(iii) (4 pts.) If

$$\mathbf{x}^{(2)} = \begin{bmatrix} a & b & c & a & b & c & a & b & c \end{bmatrix}^T ,$$

express the DFT $\mathbf{X}^{(2)}$ in terms of nonzero D_k's.

(iv) (4 pts.)

$$\mathbf{x}^{(3)} = \begin{bmatrix} a & jb & -c & 0 & 0 & 0 & -a & -jb & c & 0 & 0 & 0 \end{bmatrix}^T$$

has DFT $\mathbf{X}^{(3)}$ given by

$$\mathbf{X}^{(3)} = \begin{bmatrix} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} & E_{11} \end{bmatrix}^T$$

express each E_ℓ in terms of nonzero D_k's.

(v) (4 pts.) If the *time-domain* signal $\mathbf{x}^{(4)}$ has DFT

$$\mathbf{X}^{(4)} = \begin{bmatrix} 0 & 0 & 0 & a & b & c \end{bmatrix}^T ,$$

express the entries of $\mathbf{x}^{(4)}$ in terms of nonzero D_k's.

HW Problem 21

The continuous-time signal

$$s(t) \;=\; A_1 \cos(2\pi(248)t + \phi_1) + A_2 \cos(2\pi(378)t + \phi_2)$$

is sampled at rate $f_s = 224$ samples/second.

(i) (4 pts.) Write an equation for the sample sequence $s[n]$, using frequency parameters ω_1 and ω_2 in the range $[0, \pi]$ rad/sample.

(ii) (4 pts.) Let $\mathbf{s} = s[0 : L-1]$. What is the smallest L such that the DFT $\mathbf{S}$ has exactly four nonzero entries?

(iii) (6 pts.) For the value of L found in **(ii)**, sketch the DFT magnitude $|S[k]|$ and phase $\angle S[k]$ as functions of the frequency index k. Label the plots using the (yet to be determined) parameters A_1, A_2, ϕ_1 and ϕ_2.

Figures `HW21mag.fig` and `HW21phase.fig` were generated using $\mathbf{s}$ and L as in **(ii)** and **(iii)** above, with numerical values for A_1, A_2, ϕ_1 and ϕ_2. Specifically,

```
N = 3000 ;
X = fft(s, N) ;            % zero-padded DFT
plot(0:N-1, abs(X)) ;      % saved as HW21mag.fig
plot(0:N-1, angle(X)) ;    % saved as HW21phase.fig
```

(iv) (3 pts.) Use the magnitude plot to determine A_1 and A_2.

(v) (3 pts.) Use the phase plot to determine ϕ_1 and ϕ_2 (both values are exact tenths of a radian).

HW Problem 22

Let

$$s(t) \;=\; -9.1 \;+\; 3.8\cos(165\pi t + 2.3) \;+\; 1.4\cos(330\pi t - 0.4) \;+\; 6.2\cos(440\pi t + 1.8)\ ,$$

where t is in seconds.

(i) (4 pts.) Is $s(t)$ periodic? If so, what is its fundamental period T_0 and angular frequency Ω_0?

(ii) (6 pts.) If

$$s(t) \;=\; \sum_{k=-\infty}^{\infty} S_k e^{jk\Omega_0 t}\ ,$$

determine the value of each coefficient S_k (it suffices to leave it in polar form).

(iii) (4 pts.) Suppose that $s(t)$ is sampled every $T_s = T_0/N$ seconds, where N is an integer, to produce

$$s[n] \;=\; s(nT_s)$$

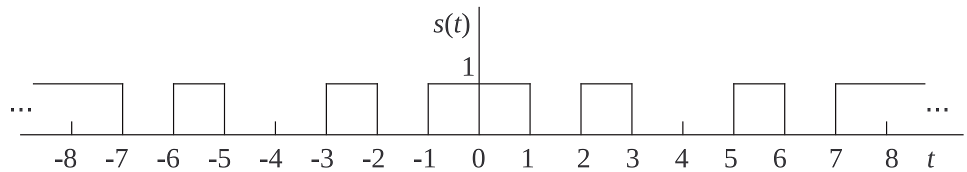
Write an equation for $s[n]$ in terms of real sinusoids. What are the frequencies of these sinusoids? Are they Fourier frequencies for an N-point vector?

(iv) (6 pts.) Let $N = 300$. Use the `IFFT` function in MATLAB to generate the vector $s[0:299]$, which consists of N uniform samples of $s(t)$ over its first period $[0, T_0)$. *Submit the commands used and the resulting plot; do not include a printout of the vector.*

HW Problem 23

(i) (2 pts.) In the lecture notes, you will find the Fourier series for the symmetric (even) rectangular pulse train of unit height and duty factor α. Write down *both* the complex and real (cosines-only) form of the series for a fundamental period $T_0 = 8$.

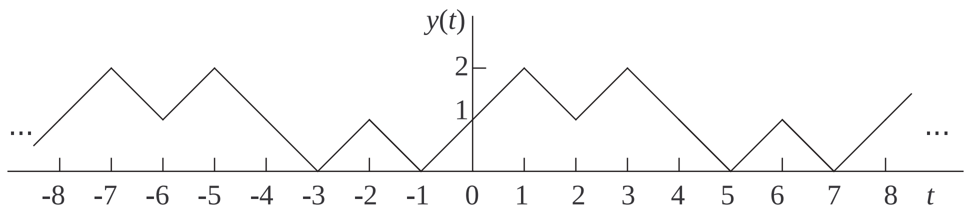
(ii) (5 pts.) Express the periodic signal $s(t)$ (of period $T_0 = 8$) shown below as a sum of three symmetric rectangular pulse trains of the same period. Using the result of (i), derive the complex Fourier series coefficients $\{S_k\}$. Also, write down the real (cosines-only) form of the series for $s(t)$.



(iii) (3 pts.) Sketch the periodic signal $x(t)$ which has period $T_0 = 8$ (i.e., same as $s(t)$) and complex Fourier series coefficients given by

$$X_k = \begin{cases} 0, & k = 0; \\ 2S_k, & k \neq 0. \end{cases}$$

For (iv)–(vi), consider the periodic signal $y(t)$ shown below.



(iv) (1 pt.) Determine the mean value (or DC offset) of $y(t)$.

(v) (4 pts.) Determine the values taken by the derivative $dy(t)/dt$ over one period, e.g., for $t \in (-3, 5]$. What is the relationship between $dy(t)/dt$ and the signal $x(t)$ of part (iii)? (*It is a simple relationship.*)

(vi) (5 pts.) The real form of the Fourier series expansion of $y(t)$ is

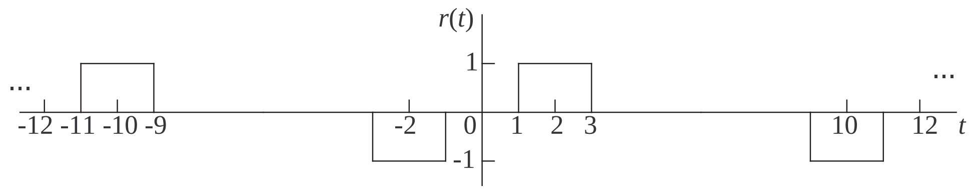
$$y(t) = Y_0 + 2\sum_{k=1}^{\infty} B_k \sin(k\Omega_0 t)$$

Using the results of (ii), (iv) and (v), together with the well-known identity $(d/dt)\sin(at) = a\cos(at)$, determine Y_0 and B_k for $k \geq 1$.

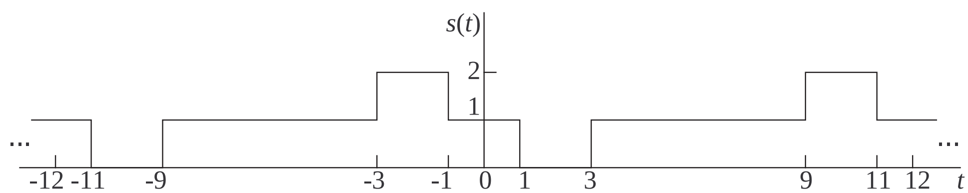
(*Note:* The signal $y(t) - Y_0$ is antisymmetric, or, "odd", about $t = 0$. The *complex* Fourier series coefficients Y_k are related to B_k by $Y_k = jB_{-k}$ for $k < 0$; and $Y_k = -jB_k$ for $k > 0$.)

HW Problem 24

(i) (5 pts.) Using the time delay property, derive the Fourier series coefficients $\{R_k\}$ of the periodic signal $r(t)$ of period $T_0 = 12$ (seconds) shown below. Verify that R_k is purely imaginary and such that $R_{-k} = -R_k$.

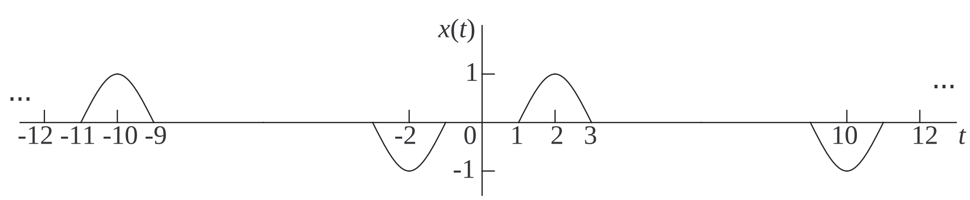


(ii) (4 pts.) Using the result of (i), determine the complex Fourier series coefficients $\{S_k\}$ of the periodic signal $s(t)$ shown below.



(iii) (4 pts.) Use MATLAB to graph the output of an ideal lowpass filter with cutoff frequency 3.4 Hz when the input is $s(t)$. Display two periods corresponding to the time interval $[0, 24)$ with 256 samples per period. Submit commands and graphs.

(iv) (7 pts.) Using the result of (i) and the modulation property, *carefully* derive the Fourier series coefficients $\{X_k\}$ of the periodic signal $x(t)$ shown below (curved segments are sinusoidal). Simplify your answer as much as possible. Also, verify that X_k is purely imaginary and such that $X_{-k} = -X_k$.



HW Problem 25

Consider the FIR filter given by the following input-output relationship (note the missing coefficient):

$$y[n] = x[n] + \sqrt{3}x[n-1] - \sqrt{3}x[n-3] - x[n-4] \ , \qquad n \in \mathbf{Z}$$

(i) (3 pts.) Show that the input sequences defined for all n by $x^{(1)}[n] = 1$ and $x^{(2)}[n] = (-1)^n$ both result in output sequences which are identically equal to zero.

(ii) (3 pts.) Write MATLAB code which computes and plots the amplitude and phase response of the filter at 1024 equally spaced frequencies in $[0, 2\pi)$. Submit the plots, properly labeled.

(iii) (4 pts.) Express the filter's frequency response in the form

$$H(e^{j\omega}) = je^{-j(\omega M/2)}F(\omega)$$

where $F(\omega)$ is a real-valued sum of sines.

(iv) (5 pts.) The amplitude response plotted in (ii) above has six zeros at frequencies other than $\omega = 0$ and $\omega = \pi$. Determine the values of these frequencies analytically, using the result of part (iii) and the identity $\sin 2\theta = 2 \sin\theta \cos\theta$.

(v) (5 pts.) Determine analytically the exact locations of the local maxima of the amplitude response $|H(e^{j\omega})|$ in the frequency range $[0, \pi/2]$. Do so by differentiating $F(\omega)$ and using the identity $\cos 2\theta = 2\cos^2\theta - 1$. Express your answers using $\cos^{-1}(\rho_1)$ and $\cos^{-1}(\rho_2)$, where ρ_1 and ρ_2 are exact (sums of rational numbers and square roots thereof).

HW Problem 26

Consider the FIR filter with the following input-output relationship (note the missing coefficient):

$$y[n] \;=\; x[n] + 3x[n-1] + 3x[n-3] + x[n-4]\ , \qquad n \in \mathbf{Z}$$

(i) Determine the response $y^{(i)}[\,\cdot\,]$ of the filter to each of the input signals given by the equations below (valid for all $n \in \mathbf{Z}$).

$$\begin{aligned}
x^{(1)}[n] &= (3/4)^n && \textbf{(2 pts.)}\\
x^{(2)}[n] &= (-4/3)^n && \textbf{(2 pts.)}\\
x^{(3)}[n] &= 1 + 3^{-n} && \textbf{(2 pts.)}\\
x^{(4)}[n] &= \cos(n(\pi/6) + 2.5) && \textbf{(3 pts.)}\\
x^{(5)}[n] &= 2^{-n}\cdot\, \cos(n\pi/6) && \textbf{(4 pts.)}
\end{aligned}$$

(ii) (5 pts.) The filter above is connected in series (cascade) with a filter having input-output relationship

$$y[n] \;=\; x[n] - 2x[n-1] + x[n-2]\ , \qquad n \in \mathbf{Z}$$

Determine the system function $H(z)$ of the two-filter cascade.

(iii) (2 pts.) Write out the input-output relationship of the two-filter cascade.

HW Problem 27

Consider the FIR filter with coefficient vector $\mathbf{b} = \begin{bmatrix} 1 & 3 & 0 & -3 & -1 \end{bmatrix}^T$.

The following MATLAB script computes a segment of the filter output sequence $y^{(1)}[\,\cdot\,]$ for a periodic input sequence $x^{(1)}[\,\cdot\,]$ of period $L = 6$. Specifically, the vector `y1` below equals $y^{(1)}[-1:4]$.

```
b  = [1 3 0 -3 -1].' ;
H  = fft(b,6) ;
x1 = [1 2 4 -1 -2 -4].' ;
X1 = fft(x1) ;
Y1 = H.*X1 ;
y1 = ifft(Y1)
```

(i) (3 pts.) Display the vector $x^{(1)}[0:5]$. (*It is not the same as* `x1`, *introduced above.*)

(ii) (5 pts.) Compute $y^{(1)}[-1:4]$ by hand using a circular convolution, or, equivalently, using the filter's input-output relationship.

For parts (iii) and (iv) below, consider the following MATLAB script, which computes a segment of the filter output sequence $y^{(2)}[\,\cdot\,]$ for a periodic input sequence $x^{(2)}[\,\cdot\,]$ of period $L = 3$. Specifically, the vector `y2` equals $y^{(2)}[-1:4]$.

```
b  = [1 3 0 -3 -1].' ;
H  = fft(b,6) ;
x2 = [2 -1 5 2 -1 5].' ;
X2 = fft(x2) ;
Y2 = H.*X2 ;
y2 = ifft(Y2)
```

(iii) (2 pts.) Display the vector $x^{(2)}[0:2]$.

(iv) (5 pts.) Compute $y^{(2)}[-1:4]$ by hand using a circular convolution, based on the filter's input-output relationship.

(v) (5 pts.) Fully explain the relationship between `y3` computed below and `y2` computed earlier.

```
b  = [1 3 0 -3 -1].' ;
H  = fft(b,6) ;
H  = H(1:2:6) ;
x3 = [2 -1 5].' ;
X3 = fft(x3) ;
Y3 = H.*X3 ;
y3 = ifft(Y3)
```

HW Problem 28

Consider the FIR filter with impulse response given by

$$h[n] \;=\; b_0\delta[n] + b_1\delta[n-1] + b_2\delta[n-2] + b_3\delta[n-3] \;+ b_4\delta[n-4] + b_5\delta[n-5]$$

It is known that the response of the filter to the input sequence

$$x[0:3] \;=\; \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix}^T \;; \qquad x[n] = 0 \text{ for } n < 0 \text{ and } n > 3$$

is the output sequence given by

$$y[0:8] \;=\; \begin{bmatrix} 1 & -3 & 8 & -16 & 20 & -24 & 17 & -7 & 4 \end{bmatrix}^T$$

and $y[n] = 0$ for $n < 0$ and $n > 8$.

(i) (5 pts.) Use the `FFT` command in MATLAB to determine the filter coefficient vector $\mathbf{b}$. (Alternatively, $\mathbf{b}$ can be determined by solving a triangular system of linear equations.)

(ii) (3 pts.) Using the system function $H(z)$ or otherwise, determine the response of the filter to the exponential input

$$x^{(2)}[n] \;=\; (-2)^{-n} \;, \qquad n \in \mathbf{Z}$$

(iii) (7 pts.) Using convolution, determine the response of the filter to the finite-duration sequence

$$x^{(3)}[n] \;=\; 16\delta[n] - 8\delta[n-1] + 4\delta[n-2] - 2\delta[n-3] + \delta[n-4]$$

(Make sure to specify the output sequence completely.)

(iv) (5 pts.) Explain how some of the values computed in (iii) above can be combined with the answer to part (ii) to yield the response of the filter to the one-sided exponential input

$$x^{(4)}[n] \;=\; \begin{cases} (-2)^{-n}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Determine that response.
