ENEE 222 EXAM 2: SAMPLE PROBLEMS

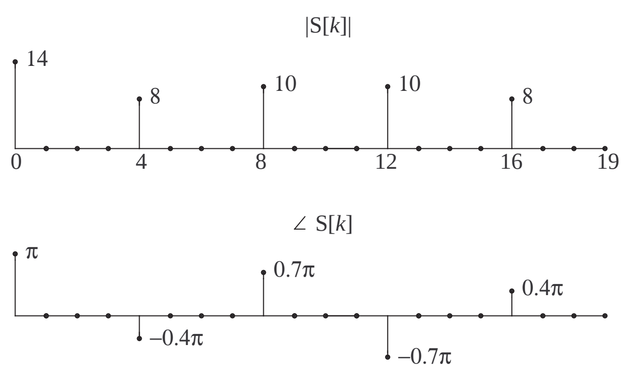
Calculators will not be allowed.

The following formulas will be given on the exam sheet:

Time Domain		*Frequency Domain*
$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$	$\longleftrightarrow$	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$
$\mathbf{x} = \frac{1}{N}\mathbf{V}\mathbf{X}$	$\longleftrightarrow$	$\mathbf{X} = \mathbf{W}\mathbf{x}$
$\alpha\mathbf{x} + \beta\mathbf{y}$	$\longleftrightarrow$	$\alpha\mathbf{X} + \beta\mathbf{Y}$
$\mathbf{x}^*$	$\longleftrightarrow$	$\mathbf{R}\mathbf{X}^*$
$\mathbf{R}\mathbf{x}$	$\longleftrightarrow$	$\mathbf{R}\mathbf{X}$
$\mathbf{P}^m\mathbf{x}$	$\longleftrightarrow$	$\mathbf{F}^{-m}\mathbf{X}$
$\mathbf{F}^m\mathbf{x}$	$\longleftrightarrow$	$\mathbf{P}^m\mathbf{X}$
$\mathbf{X}$	$\longleftrightarrow$	$N\mathbf{R}\mathbf{x}$
$\mathbf{x} \diamond \mathbf{y}$	$\longleftrightarrow$	$\frac{1}{N}(\mathbf{X} \circledast \mathbf{Y})[k] = \frac{1}{N}\mathbf{X}^T\mathbf{P}^k\mathbf{R}\mathbf{Y}$
$(\mathbf{x} \circledast \mathbf{y})[n] = \mathbf{x}^T\mathbf{P}^n\mathbf{R}\mathbf{y}$	$\longleftrightarrow$	$\mathbf{X} \diamond \mathbf{Y}$

PROBLEM 1

The continuous-time real-valued signal $s(t)$ is a sum of two real-valued sinusoids of frequency no higher than 25 Hz, plus a DC offset (i.e., a constant). It is sampled at times $t = nT_s$, where $n = 0, \ldots, 19$ and $T_s = 0.02$ seconds. The amplitude and phase spectra of the sampled signal are shown in the figure below.



Use the information given above to write an equation for $s(t)$, valid for all $t \in \mathbf{R}$.

PROBLEM 2

The time-domain signal

$$\mathbf{s} = \begin{bmatrix} a & b & c & d & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

has DFT given by

$$\mathbf{S} = \begin{bmatrix} 4 & 4-j(2-\sqrt{2}) & 2 & 4+j(2+\sqrt{2}) & 8 & 4-j(2+\sqrt{2}) & 2 & 4+j(2-\sqrt{2}) \end{bmatrix}^T$$

(i) Compute the numerical value of $a - b + c - d$.

(ii) Compute the DFT $\mathbf{X}$ of the time-domain signal

$$\mathbf{x} = \begin{bmatrix} a & b & c & d & a & b & c & d \end{bmatrix}^T$$

(iii) Compute the DFT $\mathbf{Y}$ of the time-domain signal

$$\mathbf{y} = \begin{bmatrix} a & b & c & d & -a & -b & -c & -d \end{bmatrix}^T$$

PROBLEM 3

The time-domain signal

$$\mathbf{s} = \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T$$

has DFT given by

$$\mathbf{S} = \begin{bmatrix} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 \end{bmatrix}^T$$

(i) Determine (in terms of $a, \ldots, f$) the time-domain signal $\mathbf{x}$ whose DFT is given by

$$\mathbf{X} = \begin{bmatrix} S_0 - S_3 & S_1 - S_4 & S_2 - S_5 & S_3 - S_0 & S_4 - S_1 & S_5 - S_2 \end{bmatrix}^T$$

(ii) Determine (in terms of $a, \ldots, f$) the time-domain signal $\mathbf{y}$ whose DFT is given by

$$Y[k] = S_k \cos(2\pi k/3) \ , \qquad k = 0, \ldots, 5$$

PROBLEM 4

The time-domain signals

$$\mathbf{x} = \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}^T \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} 3 & -1 & 4 & -2 \end{bmatrix}^T$$

have DFT's given by

$$\mathbf{X} = \begin{bmatrix} A & B & C & D \end{bmatrix}^T \qquad \text{and} \qquad \mathbf{Y} = \begin{bmatrix} E & F & G & H \end{bmatrix}^T$$

(i) Compute the time-domain signal $\mathbf{s}$ whose DFT is given by

$$\mathbf{S} = \begin{bmatrix} AE & BH & CG & DF \end{bmatrix}^T$$

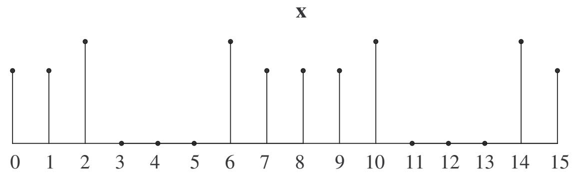
(ii) Let $\mathbf{u}$ be the time-domain signal defined by the circular convolution

$$\mathbf{u} = \begin{bmatrix} A & B & C & D \end{bmatrix}^T \circledast \begin{bmatrix} 3 & -1 & 4 & -2 \end{bmatrix}^T$$

Write an expression for the DFT $\mathbf{U}$ of $\mathbf{u}$ in terms of E, F, G and H.

PROBLEM 5

The time-domain signal $\mathbf{x}$ shown below takes three distinct values (zero being one of them) and has DFT $\mathbf{X}$. For simplicity, let $X[k] = X_k$.

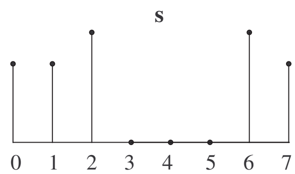


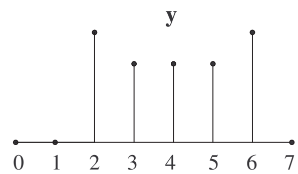
(i) For which values of k (if any) does X_k equal 0?

(ii) For which pairs (k, k') (if any) does X_k equal $X_{k'}$?

(iii) Express the DFT $\mathbf{S}$ of the time-domain signal $\mathbf{s}$ shown below (left) in terms of X_k's.

(iv) Express the DFT $\mathbf{Y}$ of the time-domain signal $\mathbf{y}$ shown below (right) in terms of X_k's.





PROBLEM 6

The DFT of the signal

$$\mathbf{x} = \begin{bmatrix} a & b & c & d & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

is given by

$$\mathbf{X} = \begin{bmatrix} X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \end{bmatrix}^T$$

(i) Determine the DFT $\mathbf{S}$ of the time-domain signal

$$\mathbf{s} = \begin{bmatrix} c & d & a & b \end{bmatrix}^T$$

(ii) Compute the value of $\cos(n\pi/2)$ for $n = 0, \ldots, 7$.

(iii) Using your answer to part (ii) above, determine the time-domain signal $\mathbf{y}$ whose DFT is given by

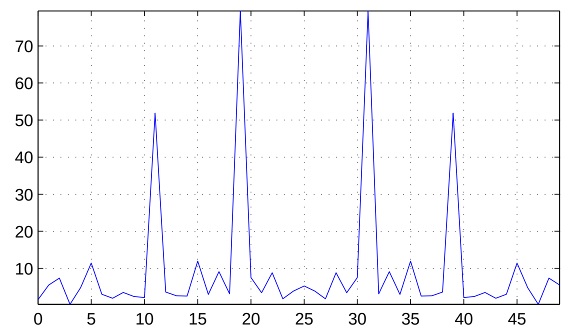
$$\mathbf{Y} = \begin{bmatrix} X_0 & 0 & -X_2 & 0 & X_4 & 0 & -X_6 & 0 \end{bmatrix}^T$$

PROBLEM 7

A continuous-time signal is given by

$$x(t) = A_1 \cos(2\pi f_1 t + \phi_1) + A_2 \cos(2\pi f_2 t + \phi_2) + z(t)$$

where $z(t)$ is noise. The signal $x(t)$ is sampled every 2.0 ms for a total of 50 samples. The figure shows the graph of the magnitude (or amplitude) spectrum of the 50-point signal as a function of the frequency index $k = 0, \ldots, 49$.



Based on the given graph, what are your estimates of f_1 and f_2?

(You may assume that no aliasing has occurred, i.e., the sampling rate of 500 samples/sec is no less than twice each of f_1 and f_2.)

PROBLEM 8

The continuous-time signal

$$x(t) = -4 + 5\cos(700\pi t + 0.2) + 2\cos(450\pi t - 0.8)$$

is sampled every $T_s = 0.001$ seconds starting at $t = 0$. The first $N = 40$ samples are stored in the vector $\mathbf{x}$.

(i) Verify that the frequencies of the sinusoidal components of $x(t)$ become (after sampling) Fourier frequencies for the N-point vector $\mathbf{x}$.

(ii) Compute the nonzero entries in the DFT $\mathbf{X}$ of $\mathbf{x}$. Give your answers in either Cartesian or polar form, clearly showing the corresponding indices k.

(iii) Sketch the amplitude and phase spectra $|X[k]|$ and $\angle X[k]$ as functions of the frequency index $k = 0, \ldots, 39$.

PROBLEM 9

Let

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 3 & 9 \\ 9 & -3 & 1 & 3 \\ 3 & 9 & -3 & 1 \\ 1 & 3 & 9 & -3 \end{bmatrix}$$

(i) Carefully compute $\mathbf{A}^T\mathbf{A}$. What special property do the columns of $\mathbf{A}$ have?

(ii) The vector $\mathbf{b} = [\ 2\ \ -1\ \ 8\ \ -9\ ]^T$ can be written as a linear combination of three columns of $\mathbf{A}$. Determine that linear combination (i.e., its coefficients) *without solving a* 4×4 *system.*

PROBLEM 10

Consider the matrix of Fourier sinusoids

$$\mathbf{V} \;=\; \left[\ \mathbf{v}^{(0)} \quad \mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \mathbf{v}^{(3)}\ \right] \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

and the vector $\mathbf{s} = [\ 6 \quad 2 \quad -1 \quad 4\ ]^T$.

(i) Determine the coefficient vector $\mathbf{c}$ such that $\mathbf{s} = \mathbf{V}\mathbf{c}$.

(ii) Display the vector $\hat{\mathbf{s}}$, which is the least-squares approximation of $\mathbf{s}$ in terms of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(3)}$ only.

(iii) Consider the vector

$$\mathbf{x} \;=\; \mathbf{s} + [\ z+u \quad z-u \quad z+u \quad z-u\ ]^T\ ,$$

where z and u are arbitrary complex constants. Let $\hat{\mathbf{x}}$ be its least-squares approximation in terms of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(3)}$. Explain why $\hat{\mathbf{x}} = \hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ was defined in part **(ii)**.