# Lecture 17

# Lecture 17

- Periodic and zero-padded extensions of a vector

# Lecture 17

- Periodic and zero-padded extensions of a vector
- The spectrum of a sinusoidal vector

# Lecture 17

- Periodic and zero-padded extensions of a vector

- The spectrum of a sinusoidal vector: Fourier frequencies as special cases

# Lecture 17

- Periodic and zero-padded extensions of a vector
- The spectrum of a sinusoidal vector: Fourier frequencies as special cases
- Frequency estimation

# Lecture 17

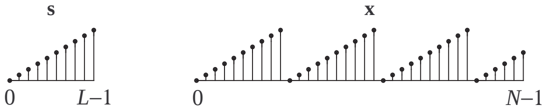
- Periodic and zero-padded extensions of a vector

- The spectrum of a sinusoidal vector: Fourier frequencies as special cases

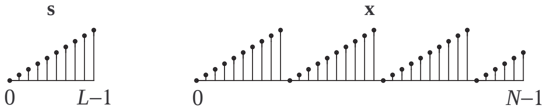
- Frequency estimation: application of zero-padding

# Periodic Extension

# Periodic Extension

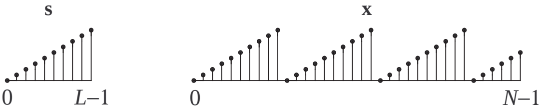


# Periodic Extension



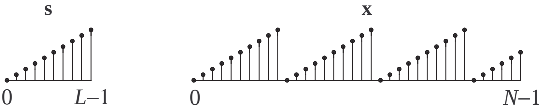

If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$,

# Periodic Extension



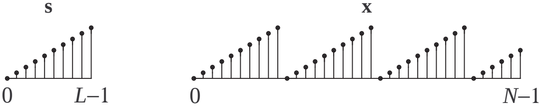

If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$,

# Periodic Extension




If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$

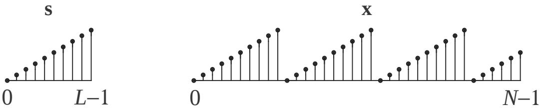
# Periodic Extension




If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$:

$$\mathbf{X} = M$$

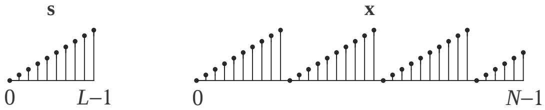
# Periodic Extension




If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$:

$$\mathbf{X} \;=\; M \cdot \left[\, S[0] \quad \mathbf{0}^T_{M-1}\right.$$

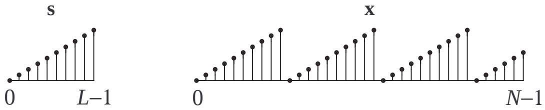
# Periodic Extension




If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$:

$$\mathbf{X} \quad = \quad M \cdot \left[\, S[0] \quad \mathbf{0}^T_{M-1} \quad S[1] \quad \mathbf{0}^T_{M-1} \right.$$

# Periodic Extension

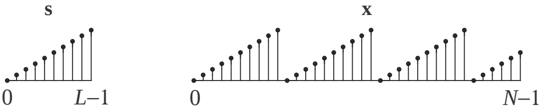



If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$:

$$\mathbf{X} \;=\; M \cdot \big[\, S[0] \quad \mathbf{0}_{M-1}^T \quad S[1] \quad \mathbf{0}_{M-1}^T \quad \ldots$$

$$\ldots$$

# Periodic Extension

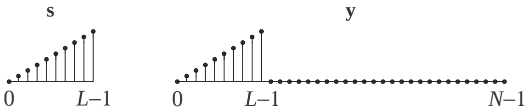



If $\mathbf{x}$ consists of a whole number of copies of $\mathbf{s}$, i.e., $N = ML$, then the DFT $\mathbf{X}$ is easily obtained from $\mathbf{S}$:

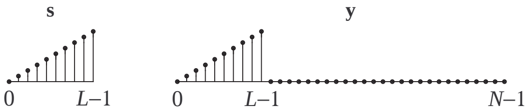
$$\mathbf{X} = M \cdot \left[\, S[0] \quad \mathbf{0}_{M-1}^T \quad S[1] \quad \mathbf{0}_{M-1}^T \quad \ldots \quad \ldots \quad S[L-1] \quad \mathbf{0}_{M-1}^T \,\right]^T$$

# Zero-Padded Extension

# Zero-Padded Extension

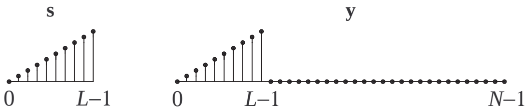


# Zero-Padded Extension



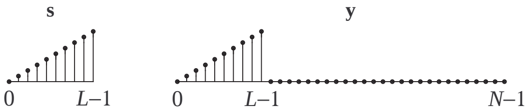

- If $N = ML$,

# Zero-Padded Extension



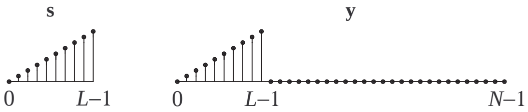

- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\text{th}}$ entry of $\mathbf{Y}$

# Zero-Padded Extension




- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\mathrm{th}}$ entry of $\mathbf{Y}$
- Zero-padding allows us to interpolate a spectrum.

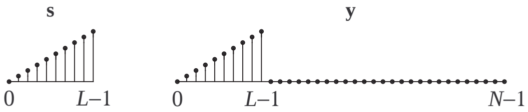
# Zero-Padded Extension




- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\text{th}}$ entry of $\mathbf{Y}$
- Zero-padding allows us to interpolate a spectrum. In effect, we are computing the inner product

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle$$

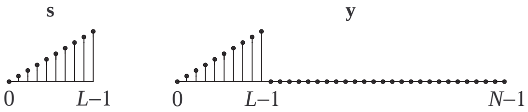
# Zero-Padded Extension




- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\text{th}}$ entry of $\mathbf{Y}$
- Zero-padding allows us to interpolate a spectrum. In effect, we are computing the inner product

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle \;=\; \sum_{n=0}^{L-1} s[n] e^{-j\omega n}$$

# Zero-Padded Extension

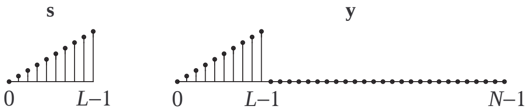



- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\mathrm{th}}$ entry of $\mathbf{Y}$
- Zero-padding allows us to interpolate a spectrum. In effect, we are computing the inner product

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle \;=\; \sum_{n=0}^{L-1} s[n] e^{-j\omega n}$$

for an arbitrary number $N$ of frequencies in $[0, 2\pi)$

# Zero-Padded Extension




- If $N = ML$, then the DFT $\mathbf{S}$ is obtained by sampling every $M^{\text{th}}$ entry of $\mathbf{Y}$
- Zero-padding allows us to interpolate a spectrum. In effect, we are computing the inner product

$$\langle \mathbf{v}^{(\omega)}, \mathbf{s} \rangle \; = \; \sum_{n=0}^{L-1} s[n] e^{-j\omega n}$$

for an arbitrary number $N$ of frequencies in $[0, 2\pi)$. (Here, $\mathbf{v}^{(\omega)}$ is a complex sinusoid of frequency $\omega$)

# Estimating the Frequency of a Sinusoid

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \;=\; A\cos(\Omega_0 t + \phi)$$

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \;=\; A\cos(\Omega_0 t + \phi)$$

is sampled at times $0, T_s, \ldots, (L-1)T_s$.

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \;=\; A\cos(\Omega_0 t + \phi)$$

is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \ = \ A\cos(\Omega_0 t + \phi)$$

is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

- Under what conditions does $\Omega_0$ correspond to a Fourier frequency for $\mathbf{s}$?

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \;=\; A\cos(\Omega_0 t + \phi)$$

  is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

- Under what conditions does $\Omega_0$ correspond to a Fourier frequency for $\mathbf{s}$?

- If $\Omega_0$ does not correspond to a Fourier frequency, what does the DFT $\mathbf{S}$ look like?

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \ = \ A\cos(\Omega_0 t + \phi)$$

  is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

- Under what conditions does $\Omega_0$ correspond to a Fourier frequency for $\mathbf{s}$?
- If $\Omega_0$ does not correspond to a Fourier frequency, what does the DFT $\mathbf{S}$ look like?
- How to estimate $\Omega_0$ with arbitrary accuracy

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \ = \ A\cos(\Omega_0 t + \phi)$$

  is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

- Under what conditions does $\Omega_0$ correspond to a Fourier frequency for $\mathbf{s}$?

- If $\Omega_0$ does not correspond to a Fourier frequency, what does the DFT $\mathbf{S}$ look like?

- How to estimate $\Omega_0$ with arbitrary accuracy using zero-padding

# Estimating the Frequency of a Sinusoid

- The continuous-time sinusoid

$$s(t) \;=\; A\cos(\Omega_0 t + \phi)$$

  is sampled at times $0, T_s, \ldots, (L-1)T_s$. Sample vector: $\mathbf{s}$

- Under what conditions does $\Omega_0$ correspond to a Fourier frequency for $\mathbf{s}$?
- If $\Omega_0$ does not correspond to a Fourier frequency, what does the DFT $\mathbf{S}$ look like?
- How to estimate $\Omega_0$ with arbitrary accuracy using zero-padding (complex sinusoid, for simplicity)