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- Finite-length periodic extension: special cases

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- More on this at the end of the semester.

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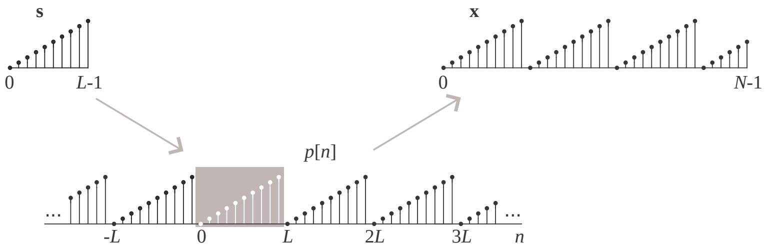

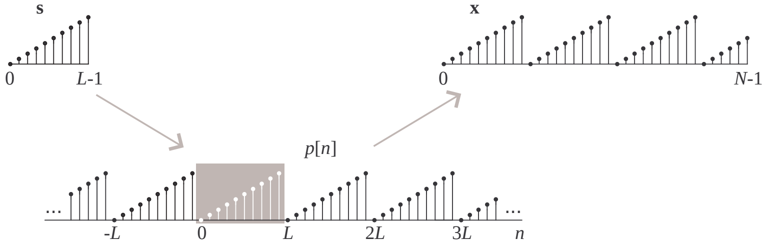

s
0
L-1
x
0
N-1
p[n]
...
...
-L
0
L
2L
3L
n